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Обобщен метод на Левин и приложения за нелинейни дисперсни уравнения

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Abstract

A new ordinary differential inequality without global solutions is proposed. Comparison with similar differential inequalities in the well-known concavity method is performed. As an application, finite time blow up of the solutions to nonlinear Klein–Gordon equation is proved. The initial energy is arbitrary high positive. The structural conditions on the initial data generalize the assumptions used in the literature for the time being.

1 Introduction

In a series of papers [5, 6, 7] Levine proves nonexistence of global solutions to the nonlinear wave equation by means of the reduction of the wave equation to a differential inequality for a functional of the solution. His results are improved by Straughan in [8] for positive initial energy. In this way the finite time blow up of the solutions to many nonlinear dispersive equations can be obtained as a consequence of the blow up results of the solutions to the differential inequality

$$\Psi''(t)\Psi(t) - \gamma\Psi'^2(t) \geq 0, \quad t \geq 0, \tag{1.1}$$

and the more general one

$$\Psi''(t)\Psi(t) - \gamma\Psi'^2(t) \geq -\beta\Psi(t), \quad t \geq 0. \tag{1.2}$$

Here $\gamma > 1$, $\beta > 0$ are constants.

Later on, Kalantarov and Ladyzhenskaya [2] extend (1.1) to

$$\Psi''(t)\Psi(t) - \gamma\Psi'^2(t) \geq -2\delta\Psi(t)\Psi'(t) - \mu\Psi^2(t), \quad t \geq 0, \tag{1.3}$$

while Korpusov [4] generalizes (1.2) to

$$\Psi''(t)\Psi(t) - \gamma\Psi'^2(t) \geq -2\delta\Psi(t)\Psi'(t) - \beta\Psi(t), \quad t \geq 0, \tag{1.4}$$

for some constants $\delta \geq 0$, $\mu \geq 0$.

Applying inequalities (1.1)-(1.4) for special choice of the initial data $\Psi(0)$ and $\Psi'(0)$, finite time blow up results for the solutions of Klein–Gordon and other dispersive equations are obtained. A careful analysis of these applications shows that the stronger inequality

$$\Psi''(t)\Psi(t) - \gamma\Psi'^2(t) \geq \alpha\Psi^2(t) - \beta\Psi(t), \quad t \geq 0, \tag{1.5}$$

with $\alpha > 0$ naturally appears instead of inequalities (1.1)-(1.4). In this study we are focusing on an extension of the concavity method of Levine, based on inequality (1.5). We prove nonexistence of global solutions to inequality (1.5) for a suitable class of initial data $\Psi(0)$ and $\Psi'(0)$. As an application, we obtain finite time blow up results for the solutions to the Cauchy problem of Klein–Gordon equation

$$u_{tt} - \Delta u + u = f(u), \qquad (t,x) \in \mathbb{R}^+ \times \mathbb{R}^n, \tag{1.6}$$

$$u(0,x) = u_0(x), \quad u_t(0,x) = u_1(x), \qquad x \in \mathbb{R}^n. \tag{1.7}$$

We consider very general nonlinear term $f(u)$, satisfying either condition $(H1)$ or $(H2)$:

$$(H1) \qquad f(u) = \sum_{k=1}^{l} a_k |u|^{p_k-1}u - \sum_{j=1}^{s} b_j |u|^{q_j-1}u,$$

$$(H2) \qquad f(u) = a_1|u|^{p_1} + \sum_{k=2}^{l} a_k |u|^{p_k-1}u - \sum_{j=1}^{s} b_j |u|^{q_j-1}u,$$

where the constants a_k, p_k $(k = 1, 2, \dots, l)$ and b_j, q_j $(j = 1, 2, \dots, s)$ fulfil the conditions

$$\begin{aligned} &a_1 > 0, \ a_k \geq 0, \ b_j \geq 0 \ \text{ for } \ k = 2, \dots, l, \quad j = 1, \dots, s, \\ &1 < q_s < q_{s-1} < \dots < q_1 < p_1 < p_2 < \dots < p_l, \\ &p_l < \infty \quad \text{for} \quad n = 1, 2; \quad p_l < \frac{n+2}{n-2} \quad \text{for} \quad n \geq 3. \end{aligned}$$

Throughout this paper for functions, depending on t and x, we use the following short notations:

$$\|u\| = \|u(t,\cdot)\|_{\mathrm{L}^2(\mathbb{R}^n)}, \quad \|u\|_1 = \|u(t,\cdot)\|_{\mathrm{H}^1(\mathbb{R}^n)}, \quad (u,v) = (u(t,\cdot), v(t,\cdot)) = \int_{\mathbb{R}^n} u(t,x)v(t,x)\,dx.$$

For Klein–Gordon equation with arbitrary high positive energy there are a few results concerning finite time blow up of the weak solutions to (1.6), (1.7) with nonlinearities:

$$f(u) = u^2 + u^3, \tag{1.8}$$

$$f(u) = a|u|^{p-1}u, \quad a > 0, \quad p > 1 \tag{1.9}$$

and more general nonlinearity $f(u)$ satisfying

$$uf(u) \geq (2+\varepsilon)\int_0^u f(s)\,ds, \quad \forall u \in R, \quad \varepsilon > 0, \tag{1.10}$$

see [3, 9, 10]. In [9] the blow up result is proved for problem (1.6), (1.7) and (1.10) under the following assumptions on the initial data u_0, u_1:

$$(u_0, u_1) > 0, \quad I(0) < 0, \quad \frac{\varepsilon}{2(2+\varepsilon)}\|u_0\|^2 \geq E(0) > 0, \tag{1.11}$$

where $I(0)$ is the Nehari functional, defined by

$$I(t) = I(u(t,\cdot)) = \|u\|_1^2 - \int_{\mathbb{R}^n} f(u)u\,dx.$$

A similar blow up result is obtained in [10] for nonlinearity (1.9) provided

$$(u_0, u_1) \geq 0, \quad I(0) < 0, \quad \frac{p-1}{2(p+1)}\|u_0\|^2 > E(0) > 0. \tag{1.12}$$

The proofs in [9, 10] are based on the well-known concavity method of Levine (see [2], [5]) and the sign preserving property of the Nehari functional $I(t)$ under the flow of (1.6), (1.7). Let us note that the idea of the blow up results under conditions (1.11) and (1.12) and arbitrary high positive energy comes from the paper of Gazzola and Squassina [1] for nonlinear wave equation in a bounded domain.

A different technique for proving finite time blow up is suggested by Straughan in [8] for the solutions of nonlinear wave equation with nonlinearity (1.10). For Klein–Gordon equation a similar result is obtained in [3] for nonlinearity (1.8). In both papers, [3, 8], the main assumptions on the initial data are:

$$(u_0, u_1) > 0, \quad \frac{1}{2}\frac{(u_0, u_1)^2}{\|u_0\|^2} > E(0) > 0, \tag{1.13}$$

which allow arbitrary high positive energy.

The goal of this study is to prove the finite time blow up of the solutions to (1.5) for a suitable class of initial data $\Psi(0)$, $\Psi'(0)$. As an application, we improve the well-known results for nonexistence of global solutions to Klein–Gordon in case of arbitrary high positive energy and for a wide class of combined power–type nonlinearities $(H1)$ or $(H2)$. Under structural assumptions on the initial data, that are more general than the assumptions used in the literature for the time being, we obtain finite time blow up of the solutions to the Cauchy problem of Klein–Gordon equation.

This article is structured as follows. In Section 2 we formulate the precise statements for nonexistence of global solutions to ordinary differential inequalities, mainly used in the applications. Section 3 deals with the proof of the main result, Theorem 3.1. In Section 4, as an application of Theorem 3.1, finite time blow up of the solutions to Klein–Gordon equation is obtained.

2 Preliminaries

For convenience we reformulate three generalizations of the Levine's result [5], based on differential inequalities (1.3), (1.2) and (1.4), respectively. The statements are suitably altered to fit into the present context.

Lemma 2.1 (Kalantarov-Ladyzhenskaya [2]). *Suppose that a nonnegative, twice-differentiable function $\Psi(t)$ satisfies on $t \geq 0$ the inequality*

$$\Psi''(t)\Psi(t) - \gamma\Psi'^2(t) \geq -2\delta\Psi(t)\Psi'(t) - \mu\Psi^2(t),$$

where $\gamma > 1$, $\delta \geq 0$ and $\mu \geq 0$ are constants.

(i) If $\delta = \mu = 0$, $\Psi(0) > 0$ and $\Psi'(0) > 0$, then $\Psi(t) \to \infty$ for $t \to t_*$, where

$$t_* \leq \boldsymbol{t^1_{KL}} = \frac{\Psi(0)}{(\gamma - 1)\Psi'(0)} < \infty.$$

(ii) If $\delta + \mu > 0$, $\Psi(0) > 0$ and $\Psi'(0) > -\gamma_2(\gamma - 1)^{-1}\Psi(0)$, then $\Psi(t) \to \infty$ for $t \to t_*$, where

$$t_* \leq \boldsymbol{t^2_{KL}} = \frac{1}{2\sqrt{\delta^2 + (\gamma - 1)\mu}} \ln \frac{\gamma_1\Psi(0) + (\gamma - 1)\Psi'(0)}{\gamma_2\Psi(0) + (\gamma - 1)\Psi'(0)} < \infty$$

and $\gamma_{1,2} = -\delta \pm \sqrt{\delta^2 + (\gamma - 1)\mu}$.

Lemma 2.2 (Straughan [8], Korpusov [4])**.** *Suppose that a nonnegative, twice-differentiable function $\Psi(t)$ satisfies on $t \geq 0$ the inequality*

$$\Psi''(t)\Psi(t) - \gamma\Psi'^2(t) \geq -\beta\Psi(t),$$

where $\gamma > 1$ and $\beta > 0$ are constants. If $\Psi(0) > 0$, $\Psi'(0) > 0$ and

$$\beta < \frac{(2\gamma - 1)}{2} \frac{\Psi'^2(0)}{\Psi(0)},$$

then $\Psi(t) \to \infty$ for $t \to t_*$, where

$$t_* \leq \boldsymbol{t_{SK}} = \frac{\Psi(0)}{(\gamma - 1)\sqrt{\Psi'^2(0) - \frac{2\beta}{2\gamma-1}\Psi(0)}} < \infty.$$

Lemma 2.3 (Korpusov [3])**.** *Suppose that a nonnegative, twice-differentiable function $\Psi(t)$ satisfies on $t \geq 0$ the inequality*

$$\Psi''(t)\Psi(t) - \gamma\Psi'^2(t) \geq -2\delta\Psi(t)\Psi'(t) - \beta\Psi(t),$$

where $\gamma > 1$, $\delta \geq 0$ and $\beta \geq 0$ are constants. If $\Psi(0) > 0$ and

$$\Psi'(0) > \frac{2\delta}{\gamma - 1}\Psi(0), \quad \left(\Psi'(0) - \frac{2\delta}{\gamma - 1}\Psi(0)\right)^2 > \frac{2\beta}{2\gamma - 1}\Psi(0),$$

then $\Psi(t) \to \infty$ for $t \to t_*$, where

$$t_* \leq \boldsymbol{t_K} = \frac{\Psi(0)}{(\gamma - 1)\sqrt{\left(\Psi'(0) - \frac{2\delta}{\gamma-1}\Psi(0)\right)^2 - \frac{2\beta}{2\gamma-1}\Psi(0)}} < \infty.$$

3 Main result

In the following theorem we prove nonexistence of global solutions to the new differential inequality (1.5) under suitable conditions on the initial data. Let us emphasize that this inequality naturally appears in the analysis of the behaviour of the solutions to many nonlinear dispersive equations.

Theorem 3.1 (Main result)**.** *Suppose that a twice-differentiable function $\Psi(t)$ satisfies on $t \geq 0$ inequality (1.5), i.e.*

$$\Psi''(t)\Psi(t) - \gamma\Psi'^2(t) \geq \alpha\Psi^2(t) - \beta\Psi(t),$$

where $\gamma > 1$, $\alpha \geq 0$ and $\beta > 0$ are constants. If $\Psi(0) > 0$, $\Psi'(0) > 0$ and

$$\beta < \frac{(2\gamma-1)}{2}\frac{\Psi'^2(0)}{\Psi(0)} + \alpha\Psi(0), \tag{3.1}$$

then $\Psi(t) \to \infty$ for $t \to t_* < \infty$.

More precisely:

(i) if $\alpha\Psi(0) - \beta \geq 0$, *then* $t_* \leq \boldsymbol{t_1^*} = \dfrac{\Psi(0)}{(\gamma-1)\Psi'(0)} < \infty$;

(ii) if $\alpha\Psi(0) - \beta \leq 0$, *then* $t_* \leq \boldsymbol{t_2^*} = \dfrac{\Psi(0)}{(\gamma-1)\sqrt{\Psi'^2(0) - \frac{2\beta}{2\gamma-1}\Psi(0) + \frac{2\alpha}{2\gamma-1}\Psi^2(0)}} < \infty.$

Proof. Suppose that $\Psi(t)$ is defined for $t \in [0, \infty)$. First, we prove that $\Psi'(t) > 0$ for every $t > 0$. Since $\Psi'(0) > 0$, in order to obtain a contradiction, we suppose that $\Psi'(t) > 0$ for $t \in [0, t_0)$, $0 < t_0 < \infty$ and $\Psi'(t_0) = 0$. After the substitution $z(t) = \Psi^{1-\gamma}(t)$ we have

$$z'(t) = (1-\gamma)\Psi^{-\gamma}(t)\Psi'(t), \qquad \frac{1}{1-\gamma}\Psi^{1+\gamma}(t)z''(t) = \Psi''(t)\Psi(t) - \gamma\Psi'^2(t).$$

From (1.5) and the monotonicity of $\Psi(t)$ for $t \in (0, t_0)$ we get the inequalities:

$$\begin{aligned} \frac{1}{1-\gamma}\Psi^{1+\gamma}(t)z''(t) &\geq \alpha\Psi^2(t) - \beta\Psi(t) \geq \alpha\Psi(t)\Psi(0) - \beta\Psi(t) \\ &\geq \Psi(t)\left(\alpha\Psi(0) + \frac{2\gamma-1}{2}\frac{\Psi'^2(0)}{\Psi(0)} - \beta - \frac{2\gamma-1}{2}\frac{\Psi'^2(0)}{\Psi(0)}\right) \\ &= \Psi(t)\left(A - \frac{2\gamma-1}{2}\frac{\Psi'^2(0)}{\Psi(0)}\right), \end{aligned} \tag{3.2}$$

where

$$A = \alpha\Psi(0) + \frac{2\gamma-1}{2}\frac{\Psi'^2(0)}{\Psi(0)} - \beta.$$

From (3.1) we have $A > 0$.

Multiplying (3.2) with $z'(t) < 0$ we get for $t \in (0, t_0)$ the inequality

$$z''(t)z'(t) \geq -(\gamma-1)\left(A - \frac{2\gamma-1}{2}\frac{\Psi'^2(0)}{\Psi(0)}\right) z^{\frac{\gamma}{\gamma-1}}z'(t) \tag{3.3}$$

Integrating (3.3) from 0 to t_0 we obtain the following impossible chain of inequalities:

$$\begin{aligned}0=\frac{1}{2}z'^2(t_0)\geq&\frac{1}{2}z'^2(0)-\frac{(\gamma-1)^2}{2\gamma-1}\left(A-\frac{2\gamma-1}{2}\frac{\Psi'^2(0)}{\Psi(0)}\right)\left(z^{\frac{2\gamma-1}{\gamma-1}}(t_0)-z^{\frac{2\gamma-1}{\gamma-1}}(0)\right)\\=&\frac{1}{2}(\gamma-1)^2\Psi^{-2\gamma}(0)\Psi'^2(0)+\frac{(\gamma-1)^2}{2\gamma-1}A(\Psi^{1-2\gamma}(0)-\Psi^{1-2\gamma}(t_0))\\&-\frac{(\gamma-1)^2}{2}\frac{\Psi'^2(0)}{\Psi(0)}\Psi^{1-2\gamma}(0)+\frac{(\gamma-1)^2}{2}\frac{\Psi'^2(0)}{\Psi(0)}\Psi^{1-2\gamma}(t_0)\\>&\frac{(\gamma-1)^2}{2\gamma-1}A(\Psi^{1-2\gamma}(0)-\Psi^{1-2\gamma}(t_0))>0,\end{aligned}$$

which gives a contradiction. Thus we prove that $\Psi'(t)>0$ for every $t>0$. Therefore $\Psi(t)$ is a strictly increasing function and $\Psi(t)>0$ for for every $t\geq 0$

(i) If $\alpha\Psi(0)-\beta\geq 0$ then from the monotonicity of $\Psi(t)$ it follows that $\alpha\Psi(t)-\beta\geq 0$ for every $t>0$. Hence $\Psi(t)$ satisfies (1.1) and from Lemma 2.1(i) we get that $\Psi(t)\to\infty$ for $t\to t_*\leq t_1^*=t_{KL}^1=\frac{\Psi(0)}{(\gamma-1)\Psi'(0)}$.

(ii) If $\alpha\Psi(0)-\beta\leq 0$ holds then we have $A-\frac{2\gamma-1}{2}\frac{\Psi'^2(0)}{\Psi(0)}\leq 0$. Hence after integration of (3.3) we have for every t

$$\begin{aligned}\frac{1}{2}z'^2(t)&\geq\frac{1}{2}z'^2(0)-\frac{(\gamma-1)^2}{2\gamma-1}\left(A-\frac{2\gamma-1}{2}\frac{\Psi'^2(0)}{\Psi(0)}\right)\left(z^{\frac{2\gamma-1}{\gamma-1}}(t)-z^{\frac{2\gamma-1}{\gamma-1}}(0)\right)\\&\geq\frac{1}{2}(\gamma-1)^2\Psi^{-2\gamma}(0)\Psi'^2(0)+\frac{(\gamma-1)^2}{2\gamma-1}\left(A-\frac{2\gamma-1}{2}\frac{\Psi'^2(0)}{\Psi(0)}\right)z^{\frac{2\gamma-1}{\gamma-1}}(0)\\&=\frac{(\gamma-1)^2}{2\gamma-1}A\Psi^{1-2\gamma}(0)>0.\end{aligned}$$

Since $z'(t)<0$ it follows that

$$z'(t)\leq-\sqrt{\frac{2(\gamma-1)^2}{2\gamma-1}A\Psi^{1-2\gamma}(0)}$$

and

$$z(t)\leq -t\sqrt{\frac{2(\gamma-1)^2}{2\gamma-1}A\Psi^{1-2\gamma}(0)}+z(0).$$

Therefore, $z(t)\to 0$ for $t\to t_*$,

$$\begin{aligned}t_*\leq t_2^*=z(0)\left(\frac{2(\gamma-1)^2}{2\gamma-1}A\Psi^{1-2\gamma}(0)\right)^{-\frac{1}{2}}&=\frac{\sqrt{2\gamma-1}}{\gamma-1}\Psi(0)\big(2\alpha\Psi^2(0)+(2\gamma-1)\Psi'^2(0)-2\beta\Psi(0)\big)^{-\frac{1}{2}}\\=\frac{\Psi(0)}{(\gamma-1)\sqrt{\Psi'^2(0)-\frac{2\beta}{2\gamma-1}\Psi(0)+\frac{2\alpha}{2\gamma-1}\Psi^2(0)}},\end{aligned}$$

or equivalently, $\Psi(t)\to\infty$ for $t\to t_*$, $t_*\leq t_2^*$. Theorem 3.1 is proved. □

4 Applications

In this section we use Theorem 3.1 to prove finite time blow up of the solutions to Klein–Gordon equation with arbitrary high positive energy.

We consider the Cauchy problem to Klein–Gordon equation, (1.6), (1.7), under the following assumptions on the initial data:

$$u_0(x) \in \mathrm{H}^1(\mathbb{R}^n), \quad u_1(x) \in \mathrm{L}^2(\mathbb{R}^n). \tag{4.1}$$

Let us recall the well-known local existence result for problem (1.6), (1.7):

Theorem 4.1. *Suppose* (4.1) *holds and* $f(u)$ *satisfies either* $(H1)$ *or* $(H2)$. *Then there exists a unique local weak solution to* (1.6), (1.7) $u(t,x) \in \mathrm{C}([0,T_m);\mathrm{H}^1(\mathbb{R}^n)) \bigcap \mathrm{C}^1([0,T_m);\mathrm{L}^2(\mathbb{R}^n)) \cap \mathrm{C}^2([0,T_m);\mathrm{H}^{-1}(\mathbb{R}^n))$ *on a maximal existence time interval* $[0,T_m)$. *In addition for every* $t \in [0,T_m)$ *the solution* $u(t,x)$ *satisfies the conservation law*

$$E(0) = E(t), \tag{4.2}$$

where

$$E(t) := E(u(t,\cdot)) = \frac{1}{2}\left(\|u_t\|^2 + \|u\|^2 + \|\nabla u\|^2\right) - \int_{\mathbb{R}^n}\int_0^u f(y)\,dy\,dx. \tag{4.3}$$

Theorem 4.2. *Suppose* (4.1) *holds and* $f(u)$ *satisfies either* $(H1)$ *or* $(H2)$. *If*

$$\|u_0\| \neq 0, \quad (u_0,u_1) > 0, \tag{4.4}$$

$$\frac{1}{2}\frac{(p_1-1)}{(p_1+1)}\|u_0\|^2 + \frac{1}{2}\frac{(u_0,u_1)^2}{\|u_0\|^2} > E(0) > 0, \tag{4.5}$$

then the weak solution of (1.6), (1.7) *blows up for a finite time* $t_* < \infty$. *More precisely:*

(i) if $\frac{1}{2}\frac{(p_1-1)}{(p_1+1)}\|u_0\|^2 \geq E(0)$, *then* $t_* \leq t_1^* = \frac{2}{(p_1-1)}\frac{\|u_0\|^2}{(u_0,u_1)} < \infty$; (4.6)

(ii) if $\frac{1}{2}\frac{(p_1-1)}{(p_1+1)}\|u_0\|^2 \leq E(0)$, *then* (4.7)

$$t_* \leq t_2^* = \frac{\sqrt{2}}{(p_1-1)}\frac{\|u_0\|}{\sqrt{\frac{1}{2}\frac{(p_1-1)}{(p_1+1)}\|u_0\|^2 + \frac{1}{2}\frac{(u_0,u_1)^2}{\|u_0\|^2} - E(0)}} < \infty. \tag{4.8}$$

Proof. For function $\Psi(t) = \|u\|^2$ we have $\Psi'(t) = 2(u,u_t)$ and

$$\Psi''(t) = 2\|u_t\|^2 + 2(u,u_{tt}) = 2\|u_t\|^2 - 2\left(\|u\|^2 + \|\nabla u\|^2 - (u,f(u))\right).$$

By means of conservation law (4.2), the following equality holds

$$\begin{aligned}(\|u\|^2 + \|\nabla u\|^2 - (u,f(u))) =& (p_1+1)E(0) - \frac{(p_1+1)}{2}\|u_t\|^2 - \frac{(p_1-1)}{2}(\|u\|^2 + \|\nabla u\|^2) \\ &- (p_1+1)B(t),\end{aligned} \tag{4.9}$$

where $B(t)$ is given by the expression

$$B(t) := B(u(t)) = \sum_{k=2}^{l} \frac{a_k(p_k - p_1)}{(p_k+1)(p_1+1)} \int_{\mathbb{R}^n} |u|^{p_k+1}\, dx + \sum_{j=1}^{s} \frac{b_j(p_1 - q_j)}{(q_j+1)(p_1+1)} \int_{\mathbb{R}^n} |u|^{q_j+1}\, dx$$

for both cases $(H1)$ and $(H2)$. Since $p_k > p_1 > q_j$ for $k = 2, \ldots, l$ and $j = 1, \ldots, s$, it follows that for every $t \in [0, T_m)$

$$B(t) \geq 0 \tag{4.10}$$

and $B(t) \equiv 0$ for $a_k = 0$, $k = 2, \ldots, l$ and $b_j = 0$, $j = 1, \ldots, s$.

From (4.9), (4.10) and Hölder inequality we have for function u with $\|u\| \neq 0$ the following estimates:

$$\begin{aligned}
\Psi''(t) &= 2\|u_t\|^2 - 2(p_1+1)E(0) + (p_1+1)\|u_t\|^2 + (p_1-1)(\|u\|^2 + \|\nabla u\|^2) + 2(p_1+1)B(t) \\
&\geq (p_1+3)\|u_t\|^2 - 2(p_1+1)E(0) + (p_1-1)\|u\|^2 \\
&\geq (p_1+3)\frac{(u,u_t)^2}{\|u\|^2} - 2(p_1+1)E(0) + (p_1-1)\|u\|^2
\end{aligned}$$

Hence $\Psi(t)$ satisfies on $t \geq 0$ the differential inequality

$$\Psi''(t)\Psi(t) - \frac{(p_1+3)}{4}\Psi'^2(t) \geq (p_1-1)\Psi^2(t) - 2(p_1+1)E(0)\Psi(t). \tag{4.11}$$

If we set $\gamma = \frac{p_1+3}{4} > 1$, $\alpha = (p_1 - 1) > 0$, $\beta = 2(p_1+1)E(0) > 0$, then Theorem 3.1 can be applied since $\Psi(0) = \|u_0\|^2 > 0$, $\Psi'(0) = 2(u_0, u_1) > 0$ and condition (3.1) are fulfilled thanks to condition (4.5) in Theorem 4.2. According to Theorem 3.1 function $\Psi(t) = \|u\|^2$ blows up for a finite time t_* and the upper bounds t_1^* and t_2^* are given by formulas (4.6) and (4.8) depending on the sign of the expression $\alpha\Psi(0) - \beta = (p_1-1)\|u_0\|^2 - 2(p_1+1)E(0)$. Thus Theorem 3.1 is proved. □

Remark 4.1. *If $f(u)$ satisfies either $(H1)$ or $(H2)$ and*

$$\frac{1}{2}\frac{(p_1-1)}{(p_1+1)}\|u_0\|^2 + \frac{1}{2}\frac{(u_0,u_1)^2}{\|u_0\|^2} > E(0) > \max\left\{\frac{1}{2}\frac{(p_1-1)}{(p_1+1)}\|u_0\|^2, \frac{1}{2}\frac{(u_0,u_1)^2}{\|u_0\|^2}\right\},$$

then the result in Theorem 4.2 is completely new. Let us note that in this case the conditions of the blow up theorems in [3, 8, 9, 10] are not fulfilled and they can not be applied. In this way we get a wider class of initial data for which the corresponding weak solution blows up in a finite time.

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References

[1] Gazzola F, Squassina M. Global solutions and finite time blow up for damped semilinear wave equations. *Ann. Inst. Henri Poincaré Anal. Non Linéaire* 2006;**23**:185–207.

[2] Kalantarov VK, Ladyzhenskaya OA. The occurrence of collapse for quasilinear equations of parabolic and hyperbolic types. *J. Soviet Math.* 1978;**10(1)**:53–70.

[3] Korpusov MO. Blowup of a positive-energy solution of model wave equations in nonlinear dynamics. *Theoretical and Mathematical Physics* 2012;**171(1)**:421–434.

[4] Korpusov MO. Non-existence of global solutions to generalized dissipative Klein-Gordon equations with positive energy. *Electronic Journal of Differential Equations* 2012;**2012(119)**:1–10.

[5] Levine HA. Instability and nonexistence of global solutions to nonlinear wave equations of the form $Pu_{tt} = Au + F(u)$. *Trans. Amer. Math. Soc.* 1974;**192**:1–21.

[6] Levine HA. Some additional remarks on the nonexistence of global solutions to nonlinear wave equations. *SIAM J. Math. Anal.* 1974;**5**:138–146.

[7] Levine HA. A note on a nonexistence theorem for nonlinear wave equations *SIAM J. Math. Anal.* 1974;**5**:644–648.

[8] Straughan B. Further global nonexistence theorems for abstract nonlinear wave equations. *Proc. Amer. Math. Soc.* 1975;**48**:381–390.

[9] Wang Y. A sufficient condition for finite time blow up of the nonlinear Klein–Gordon equations with arbitrary positive initial energy. *Proc. Amer. Math. Soc.* 2008;**136**:3477–3482.

[10] Xu R, Ding Y. Global solutions and finite time blow up for damped Klein–Gordon equation, *Acta Math. Scienta* 2013;**33B(3)**:643–652.

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