Modeling Manakov Soliton Trains: Effects of External Potentials and Inter-channel Interactions

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1 Introduction

By now it is well known that the Complex Toda chain (CTC) models adequately the N-soliton train dynamics for the perturbed (scalar) NLS equations for various choices of the perturbation $iR[u]$ [1], including the case $iR[u] = V(x)u$, $V(x)$ being an external potentials, see [3] and the references therein.

First we prove that the CTC models also the interactions of the Manakov soliton trains [2]. In addition the perturbed CTC models Manakov soliton trains also in external potentials or the one-dimensional Gross-Pitaevsky eq.:

$$i\vec{u}_t + \frac{1}{2}\vec{u}_{xx} + (\vec{u}^\dagger, \vec{u})\vec{u} = V(x)\vec{u}(x,t) + c_1\sigma_1\vec{u}(x,t), \quad \sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}. \quad (1)$$

and in the presence of inter-channel interaction $c_1 \neq 0$, see [3, 4, 5].

- Check the validity of the CTC as a model for the N-soliton interactions of the Manakov model;
- Analyze the effects of three types of external potentials: anharmonic, periodic and wide-well:

$$V_{\rm anh}(x) = \sum_{s=0}^{4} V_s x^s, \qquad V_{\rm per}(x) = A\cos(\Omega x + \Omega_0),$$
$$V_{\rm ww}(x) = \sum_s c_s(\tanh(2\nu_0 x + y_f) - \tanh(2\nu_0 x - y_i)). \quad (2)$$

- Test a criteria under which given the effect of the potential on the soliton train can be viewed as an adiabatic perturbation, namely $|H_V| \ll |H_0|$

$$H_V = \int_{-\infty}^{\infty} dx\, V(x)(\vec{u}^\dagger, \vec{u})(x,t), \quad H_0 = \int_{-\infty}^{\infty} dx \left((\vec{u}_x^\dagger, \vec{u}_x) - \frac{1}{2}(\vec{u}^\dagger, \vec{u})^2 \right). \quad (3)$$

2 Manakov solitons and the CTC model

Start with the (unperturbed) Manakov model, i.e. $V(x) = 0$, $c_1 = 0$. The N-soliton Manakov train is a solution of (1) determined by the initial condition:

$$\vec{u}(x, t=0) = \sum_{k=1}^{N} \vec{u}_k(x, t=0), \quad \vec{u}_k(x,t) = u_k(x,t)\vec{n}_k, \quad u_k(x,t) = \frac{2\nu_k e^{i\phi_k}}{\cosh(z_k)}$$
$$z_k = 2\nu_k(x - \xi_k(t)), \qquad \xi_k(t) = 2\mu_k t + \xi_{k,0},$$
$$\phi_k = \frac{\mu_k}{\nu_k} z_k + \delta_k(t), \qquad \delta_k(t) = 2(\mu_k^2 + \nu_k^2)t + \delta_{k,0}, \quad (4)$$

where the 2-component polarization vector is $\vec{n}_k = \left(n_{k,1}e^{i\beta_k}, n_{k,2}e^{-i\beta_k}\right)^T$ with real $n_{k,\alpha}$ and $\beta_{k,\alpha}$, $\langle \vec{n}_k^\dagger, \vec{n}_k \rangle = 1$. The adiabatic approximation holds true if:

$$|\nu_k - \nu_0| \ll \nu_0, \qquad |\mu_k - \mu_0| \ll \mu_0, \qquad |\nu_k - \nu_0||\xi_{k+1,0} - \xi_{k,0}| \gg 1,$$
$$\nu_0 = \frac{1}{N}\sum_{k=1}^{N} \nu_k, \qquad \mu_0 = \frac{1}{N}\sum_{k=1}^{N} \mu_k, \quad (5)$$

In fact we have two different scales:

$$|\nu_k - \nu_0| \simeq \varepsilon_0^{1/2}, \qquad |\mu_k - \mu_0| \simeq \varepsilon_0^{1/2}, \qquad |\xi_{k+1,0} - \xi_{k,0}| \simeq \varepsilon_0^{-1}.$$

The dynamical system that describes the evolution of the Manakov soliton trains is [2]

$$\frac{d(\mu_k + i\nu_k)}{dt} = 4\nu_0 \left[\langle \vec{n}_k, \vec{n}_{k-1} \rangle e^{q_k - q_{k-1}} - \langle \vec{n}_{k+1}, \vec{n}_k \rangle e^{q_{k+1} - q_k} \right],$$
$$\frac{dq_k}{dt} = -4\nu_0(\mu_k + i\nu_k), \quad (6)$$

where

$$q_k = -2\nu_0\xi_k + k\ln 4\nu_0^2 - i(\delta_k + \delta_0 + k\pi - 2\mu_0\xi_k),$$
$$\nu_0 = \frac{1}{N}\sum_{s=1}^{N}\nu_s, \qquad \mu_0 = \frac{1}{N}\sum_{s=1}^{N}\mu_s, \qquad \delta_0 = \frac{1}{N}\sum_{s=1}^{N}\delta_s. \quad (7)$$

Thus we get the CTC:

$$\frac{d^2 q_k}{dt^2} = 16\nu_0^2 \left[\langle \vec{n}_{k+1}, \vec{n}_k \rangle e^{q_{k+1} - q_k} - \langle \vec{n}_k, \vec{n}_{k-1} \rangle e^{q_k - q_{k-1}} \right], \quad (8)$$

All terms in the right hand sides of the evolution equations for $\vec{n}_k$ are of the order of ε, so we can neglect the evolution of $\vec{n}_k$ and to approximate them with their initial values. It is easy to see, that if all $\langle \vec{n}_{k+1}{}^\dagger, \vec{n}_k \rangle = \text{const} \neq 0$ then the CTC (8) is a completely integrable dynamical system, just like the real Toda chain.

- CTC models the soliton interactions for the VNLSE with *any number of components*.
- The effect of the polarization vectors on the interaction comes into CTC only through the scalar products $m_{0s} = \langle \vec{n}_{k+1}, \vec{n}_k \rangle$. Thus CTC is invariant under the transformations $\vec{u} \to g_0\vec{u}$ with g_0 constant 2×2 unitary matrix.

2.1 CTC and the Asymptotic Regimes of N-soliton Trains

CTC allows Lax representation $\dot{L} = [B, L]$, where

$$L = \sum_{k=1}^{N} (b_k E_{kk} + a_k(E_{k,k+1} + E_{k-1,k})), \qquad B = \sum_{k=1}^{N} a_k((E_{k,k+1} - E_{k-1,k}). \quad (9)$$

Here the matrices $(E_{kn})_{pq} = \delta_{kp}\delta_{nq}$, and $E_{kn} = 0$ whenever one of the indices becomes 0 or $N+1$ and

$$a_k = \frac{1}{2}\sqrt{\langle \vec{n}_{k+1}, \vec{n}_k \rangle}\, e^{(q_{k+1} - q_k)/2}, \qquad b_k = \frac{1}{2}(\mu_k + i\nu_k). \quad (10)$$

Consequence of the Lax representation:

- CTC has N complex-valued integrals of motion: the eigenvalues of L: $\zeta_k = \kappa_k + i\eta_k$, $k = 1, \dots, N$.
- one can write down explicitly the solutions of CTC in terms of $\{\zeta_k, r_k\}_{k=1}^N$ where r_k are the first components of the properly normalized eigenvectors of L_0
- The asymptotics of the solutions for $t \to \pm\infty$ is:

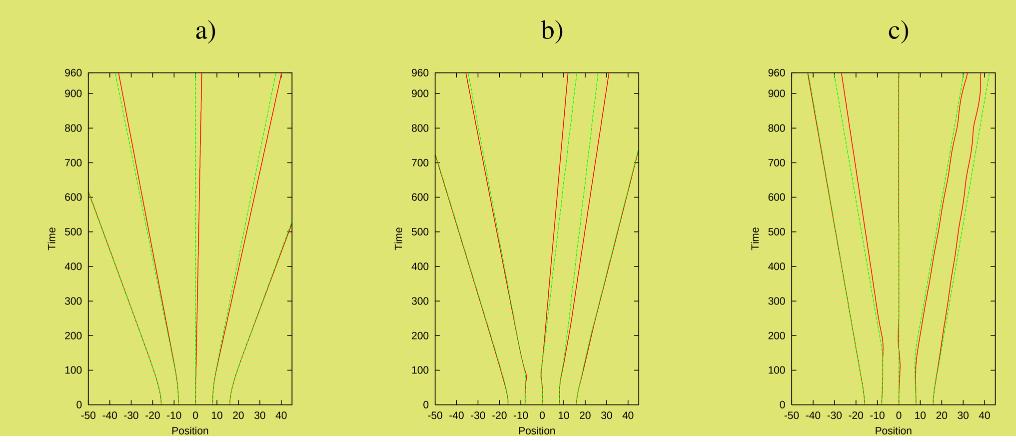
$$q_k(t) = -2\nu_0\zeta_k t - B_k + \mathcal{O}(e^{-Dt}), \quad (11)$$

i.e. $-2\kappa_k$ are the asymptotic velocities of the solitons.

AFR The asymptotically free regime takes place if $\kappa_k \neq \kappa_j$ for $k \neq j$, *i.e.*, the asymptotic velocities are all different. Then we have asymptotically separating, free solitons;

BSR The bound state regime takes place for $\kappa_1 = \kappa_2 = \cdots = \kappa_N = 0$, *i.e.*, all N solitons move with the same mean asymptotic velocity, and form a "bound state". The key question now will be the nature of the internal motions in such a bound state: is it quasi-equidistant or not?

MAR a variety of intermediate situations, or mixed asymptotic regimes happen when one group (or several groups) of particles move with the same mean asymptotic velocity; then they would form one (or several) bound state(s) and the rest of the particles will have free asymptotic motion.



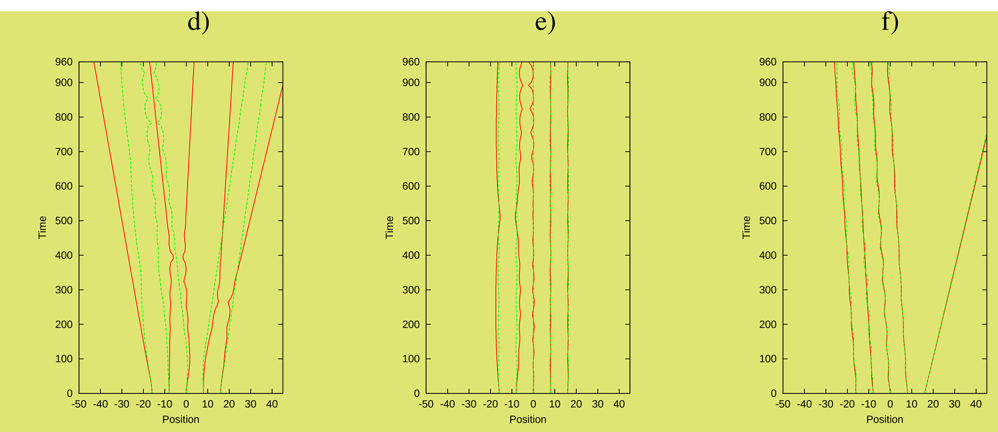


Figure 1: Various asymptotic regimes of 5-soliton trains:
a) 1+1+1+1+1; b) 1+1+2+1; c) 2+1+2; d) 3+2; e) 5; f) 4+1

3 Effects of External potentials

The results below extend our earlier ones [4, 5] on the perturbed CTC system:

$$\frac{d\lambda_k}{dt} = -4\nu_0 \left(e^{q_{k+1} - q_k}(\vec{n}_{k+1}^\dagger, \vec{n}_k) - e^{q_k - q_{k-1}}(\vec{n}_k^\dagger, \vec{n}_{k-1}) \right) + M_k + iN_k,$$
$$\frac{dq_k}{dt} = -4\nu_0\lambda_k + 2i(\mu_0 + i\nu_0)\Xi_k - iX_k, \qquad \frac{d\vec{n}_k}{dt} = \mathcal{O}(\varepsilon), \quad (12)$$

where $\lambda_k = \mu_k + i\nu_k$, $X_k = 2\mu_k\Xi_k + D_k$ and

$$N_k = -\frac{1}{2}\int_{-\infty}^{\infty} \frac{dz_k}{\cosh z_k}\Im(z_k), \qquad M_k = \frac{1}{2}\int_{-\infty}^{\infty} \frac{dz_k \sinh z_k}{\cosh^2 z_k}\Re(z_k),$$
$$\Xi_k = -\frac{1}{4\nu_k^2}\int_{-\infty}^{\infty} \frac{dz_k z_k}{\cosh z_k}\Im(z_k), \qquad D_k = \frac{1}{2\nu_k}\int_{-\infty}^{\infty} \frac{dz_k(1 - z_k\tanh z_k)}{\cosh z_k}\Re(z_k),$$
$$\Im(z_k) = \text{Im}\,\left(V(y_k)u_k e^{-i\phi_k}\right), \qquad \Re(z_k) = \text{Re}\,\left(V(y_k)u_k e^{-i\phi_k}\right) \quad (13)$$

where $y_k = z_k/(2\nu_0) + \xi_k$. Below we consider the various choices of $V(x)$.

3.1 The anharmonic and periodic potentials

These potentials for $V_4 > 0$ (no matter how small) will always restrict the N-soliton train into bound state regime. Here we have $N_k[u] = 0$, $\Xi_k[u] = 0$ and

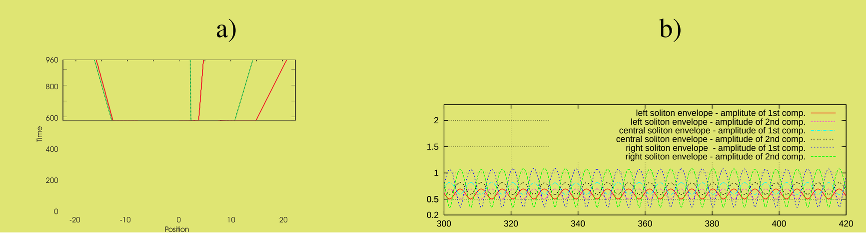
$$M_k[u] = -\frac{1}{4\nu_k}V'(\xi_k) - \frac{\pi^2}{32\nu_k^2}(V_3 + 3V_4\xi_k) + \frac{\pi A\Omega^2}{8\nu_k\sinh Z_k}\sin(\Omega\xi_k + \Omega_0),$$
$$D_k[u] = \frac{7\pi^4 V_4}{16\cdot 169\nu_k^4} - \frac{1}{2}V(\xi_k) + \frac{\pi^2}{96\nu_k^2}V''(\xi_k) - \frac{\pi^2 A\Omega^2\cosh Z_k}{16\nu_k^2\sinh^2 Z_k}\cos(\Omega\xi_k + \Omega_0).$$

where $Z_k = \Omega\pi/(4\nu_k)$.

3.2 The interchannel interaction and the wide wells

Below we assume that the constants c_1 characterizing the interchannel interaction is real. The corresponding integrals take the form $N_k = 0$, $\Xi_k = 0$ and:

$$M_k = 2c_s\nu_k(P_0(z_k - y_f) - P_0(z_k - y_i)), \quad P_0(w) = \frac{\sinh(w) - w\cosh(w)}{\sinh^2(w)},$$
$$D_k = \frac{2c_s}{\nu_k}(R_0(z_k - y_f) - R_0(z_k - y_i) - \frac{c_1}{4}\sin(2\theta_k)\cos(2\beta_k),$$
$$R_0(w) = \frac{e^{-w}\sinh^2(w) + w^2\cosh(w) - 2w\sinh(w)}{2\sinh^2(w)}.$$



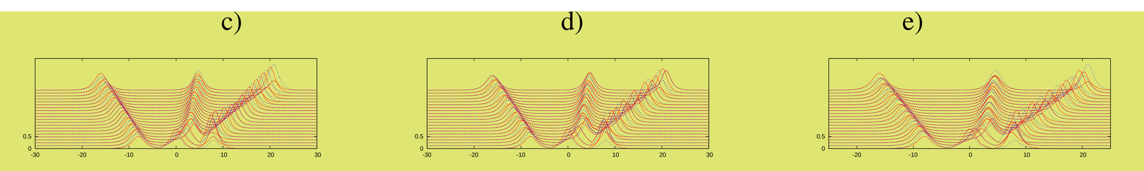


Figure 2: Inter-channel interaction of three solitons in AFR regime

4 The criterium for adiabatically small potentials

The hypothesis is formulated as $|H_V| \ll |H_0|$, see eq. (3). Both integrals can be evaluated through the parameters of the soliton train. For example, in the case of anharmonic potential one gets:

$$H_0 \simeq 8\sum_{k=1}^{N}\left(\mu_k^2\nu_k^2 - \frac{\nu_k^3}{3}\right),$$
$$H_{V_{\rm anh}} = \sum_{k=1}^{N}\left(4\nu_k V_{\rm anh}(\xi_k) + \frac{\pi^2}{24\nu_k}V''_{\rm anh}(\xi_k) + \frac{7\pi^4}{960\nu_k^3}V_4\right). \quad (14)$$

The first remark is, that both $|H_0|$ and $|H_V|$ depend on the soliton parameters. But while the leading term of $|H_0|$ depends only on ν_k and μ_k, $|H_V|$ depends substantially also on the positions ξ_k of the solitons.

We evaluated the ratio $|H_V|/|H_0|$ and found that if it is of the order of ε then the perturbed CTC matches very well the soliton trajectories of the perturbed Manakov model. If this ratio is of the order of 1 then the potential strongly prevails the soliton interactions and determines the soliton dynamics. Further analysis is needed to confirm or disproof this hypothesis.

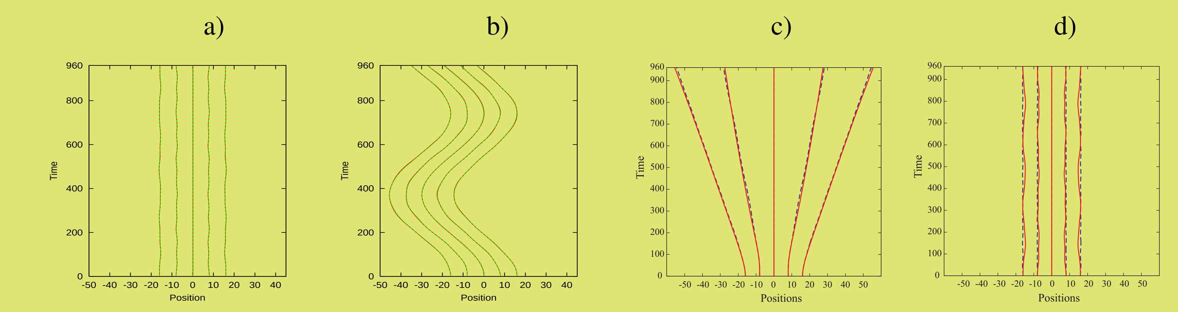


Figure 3: a) $V_{H_0} = 0.000036x^2$, $\frac{|H_0|}{|H_0|} \approx 2.78\%$; b) $V_{H_0} = 0.000036(x+15)^2$, $\frac{|H_0|}{|H_0|} \approx 7.64\%$; $V_{\rm per} = A\cos\frac{\pi}{8}x$: c) $A = -0.0001$, $\frac{|H_{V\rm per}|}{|H_0|} \approx 0.047\%$; d) $A = -0.0075$, $\frac{|H_{V\rm per}|}{|H_0|} \approx 3.53\%$

5 Conclusions

- The CTC describes adequately the Manakov soliton trains consisting of at least 5 solitons up to distances about $10\varepsilon^{-1}$;
- The PCTC describes adequately the effects of interchannel interactions and adiabatically small potentials up to distances about $10\varepsilon^{-1}$.

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