Finite-difference approximation for two-dimensional generalized time-fractional Oldroyd-B fluids

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Motivation

An increasing attention has been devoted to the prediction of behaviour of viscoelastic non-Newtonian fluids in the recent years, due to their broad application in industry and biology (molten plastics, oils and greases, suspensions, emulsions, pulps, etc.).

The generalized time-fractional Oldroyd-B model is frequently used for such viscoelastic fluids. It contains two fractional time derivatives of orders $\alpha, \beta \in (0,1)$. The 2D Rayleigh-Stokes problem for a generalized Oldroyd-B fluid is considered

$$\begin{aligned}(1 + aD_t^{\alpha})u_t &= \mu(1 + bD_t^{\beta})\Delta u + f(x,y,t), \quad (x,y) \in (0,1)^2,\ t > 0,\\ u(x,y,0) &= u_t(x,y,0) = 0, \quad (x,y) \in [0,1]^2,\\ u(x,y,t) &= v(x,y,t), \quad t > 0,\ x = 0 \text{ or } x = 1 \text{ or } y = 0 \text{ or } y = 1.\end{aligned}$$

Here $u(x,y,t)$ is the unknown velocity of the unidirectional flow, $a, b, \mu > 0$ are parameters of the problem. In the numerical experiments we take them to be equal to 1. The functions $f(x,y,t)$, $v(x,y,t)$ are given. Due to the initial conditions and the properties of Caputo derivative

$$_{RL}D_t^{\alpha}u =_C D_t^{\alpha}u, \quad D_t^{\alpha}u_t = D_t^{\alpha+1}u.$$

Riemann-Liouville derivative for $\alpha > 0,\ n-1 < \alpha < n$

$$ {}_{RL}D_t^{\alpha} f(t) = \frac{1}{\Gamma(n-\alpha)} \frac{\mathrm{d}^n}{\mathrm{d}t^n} \int_0^t (t-s)^{n-\alpha-1} f(s) \mathrm{d}s. $$

Caputo derivative for $\alpha > 0,\ n-1 < \alpha < n$

$$ {}_{C}D_t^{\alpha} f(t) = \frac{1}{\Gamma(n-\alpha)} \int_0^t (t-s)^{n-\alpha-1} f^{(n)}(s) \mathrm{d}s. $$

$$ {}_{RL}D_t^{\alpha} f(t) = {}_C D_t^{\alpha} f(t) + \sum_{k=0}^{n-1} \frac{f^{(k)}(0) t^{k-\alpha}}{\Gamma(k+1-\alpha)}, $$

where $f \in C^{n-1}[0,t]$ and $f^{(n)}$ is integrable in $[0,t]$.

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1D model, Galerkin FE in space, L1 FD approximation of the fractional derivatives, $\alpha \leq \beta$, stability and convergence analysis.

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Finite Lubich derivatives of order p, $p = 2, 3, 4, 5, 6$

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First and second order ADI finite difference schemes

First and second order approximations of the fractional derivatives are implemented in the developed alternating direction implicit finite difference schemes. Let

$$t_k = k\tau,\ k = 0, 1, \ldots, K,\ T = K\tau,\ x_i = ih,\ i = 0, \ldots, N_x,\ y_j = jh, j = 0, \ldots, N_y.$$

Finite Grünwald-Letnikov derivative

$$_{GL}D_t^\alpha f(t_k) = \frac{1}{\tau^\alpha} \sum_{m=0}^{k} (-1)^m \binom{\alpha}{m} f(t_{k-m}).$$

If $f \in C^n[0, T]$, $n - 1 \leq \alpha < n$, then

$$_{RL}D_t^\alpha f(t_k) =_{GL} D_t^\alpha f(t_k) + O(\tau).$$

Let $\omega_{1,m}^\alpha = (-1)^m \binom{\alpha}{m}$, then

$$\omega_{1,0}^\alpha = 1,\ \omega_{1,m}^\alpha = \left(1 - \frac{\alpha + 1}{m}\right) \omega_{1,m-1}^\alpha,\ m = 1, 2, \ldots, K.$$

In fact $\omega^{\alpha}_{1,m},\ m=0,\dots,k$ are the first $k+1$ coefficients of Taylor series expansion of the function

$$W^{\alpha}_1(z) = (1-z)^{\alpha} = \sum_{m=0}^{\infty} \omega^{\alpha}_{1,m} z^m.$$

For $0<\alpha<1$: $\omega^{\alpha}_{1,m}<0,\ m\geq 1$, $\omega^{\alpha}_{1,m}\geq \omega^{\alpha}_{1,m-1},\ m\geq 2$, $\lim_{m\to\infty}\omega^{\alpha}_{1,m}=0$.

For $1<\alpha<2$: $\omega^{\alpha}_{1,m}>0,\ m\geq 2$, $\omega^{\alpha}_{1,m}\leq \omega^{\alpha}_{1,m-1},\ m\geq 3$, $\lim_{m\to\infty}\omega^{\alpha}_{1,m}=0$.

Finite Lubich derivatives of order p, $p=2,3,4,5,6$

If $f^{(l)}(0)=0,\ l=0,1,\dots,p-1$

$${}_{RL}D^{\alpha}_t f(t_k) = \frac{1}{\tau^{\alpha}} \sum_{m=0}^{k} \omega^{\alpha}_{p,m} f(t_{k-m}) + O(\tau^p).$$

The coefficients $\omega^{\alpha}_{p,m}$ are those of the Taylor series expansions of given generating functions $W^{\alpha}_p(z)$

$$W^{\alpha}_p(z) = \sum_{m=0}^{\infty} \omega^{\alpha}_{p,m} z^m,\ p=2,\dots 6,\quad W^{\alpha}_2 = \left(\frac{3}{2} - 2z + \frac{1}{2}z^2\right)^{\alpha}.$$

R. Wu, H. Ding, C. Li, Determination of coefficients of high-order schemes for Riemann-Liouville derivative, The Scientific World Journal, 2014 (2014) – formulas for $\omega_{p,m}^{\alpha}$ $\;p = 2, 3, \ldots, 10$

$$\omega_{2,0}^{\alpha} = \left(\frac{3}{2}\right)^{\alpha}, \quad \omega_{2,1}^{\alpha} = -\frac{4}{3}\alpha\omega_{2,0}^{\alpha},$$

$$\omega_{2,m}^{\alpha} = \frac{2}{3m}\left[-2(\alpha - m + 1)\omega_{2,m-1}^{\alpha} + \frac{1}{2}(2\alpha - m + 2)\omega_{2,m-2}^{\alpha}\right], \; m = 2, 3, \ldots$$

For $0 < \alpha < 1$: $\omega_{2,m}^{\alpha} < 0, \; m \geq 4$, $\omega_{2,m}^{\alpha} \geq \omega_{2,m-1}^{\alpha}, \; m \geq 5$, $\lim_{m\to\infty} \omega_{2,m}^{\alpha} = 0.$

For $1 < \alpha < 2$: $\omega_{2,m}^{\alpha} > 0, \; m \geq 4$, $\omega_{2,m}^{\alpha} \leq \omega_{2,m-1}^{\alpha}, \; m \geq 5$, $\lim_{m\to\infty} \omega_{2,m}^{\alpha} = 0.$

When $\alpha = 1$

$$\frac{\partial u}{\partial t}(t_k) = \frac{3u^k - 4u^{k-1} + u^{k-2}}{2\tau} + O(\tau^2).$$

A first order discretization of the problem is

$$\frac{U_{ij}^{k+1}-U_{ij}^{k}}{\tau}+\frac{a}{\tau^{\alpha+1}}\sum_{m=0}^{k+1}\omega_{1,m}^{\alpha+1}U_{ij}^{k+1-m}=\mu\Lambda(U_{ij}^{k+1}+\frac{b}{\tau^{\beta}}\sum_{m=0}^{k+1}\omega_{1,m}^{\beta}U_{ij}^{k+1-m})+f_{ij}^{k+1},$$

where $\Lambda = \Lambda_{xx} + \Lambda_{yy}$, $\Lambda_{xx}U_{ij} = (U_{i+1,j} - 2U_{ij} + U_{i-1,j})/h^2$, $\Lambda_{yy}U_{ij} = (U_{i,j+1} - 2U_{ij} + U_{i,j-1})/h^2$.

Multiplying by τ and dividing by $1 + a/\tau^{\alpha}$ we get

$$U_{ij}^{k+1}-\tau\mu\frac{1+b/\tau^{\beta}}{1+a/\tau^{\alpha}}\Lambda U_{ij}^{k+1}=F(U^{k},U^{k-1},\dots,U^{0},x_i,y_j,t^{k+1}).$$

Let $c=\tau\mu\frac{1+b/\tau^{\beta}}{1+a/\tau^{\alpha}}=\mu\frac{\tau^{\beta}+b}{\tau^{\alpha}+a}\tau^{1+\alpha-\beta}$. Adding the term $c^2\Lambda_{xx}\Lambda_{yy}(U_{ij}^{k+1}-U_{ij}^{k})$ we obtain

$$(I-c\Lambda_{xx})(I-c\Lambda_{yy})U^{k+1}=G(U^{k},U^{k-1},\dots,U^{0},x_i,y_j,t^{k+1}).$$

Thus, we have to solve for $j = 1, \dots, N_y - 1$ the following 1D systems

$$\begin{aligned} (I - c\Lambda_{xx})U^*_{ij} &= G, \quad i = 1, \dots, N_{x-1} \\ U^*_{0,j} &= (1 - c\Lambda_{yy}u(0, y_j, t^{k+1}) \\ U^*_{N_x,j} &= (1 - c\Lambda_{yy}u(x_{N_x}, y_j, t^{k+1}) \end{aligned}$$

and then for $i = 1, \dots, N_x - 1$

$$\begin{aligned} (I - c\Lambda_{yy})U^{k+1}_{ij} &= U^*_{ij}, \quad j = 1, \dots, N_{y-1} \\ U^{k+1}_{i,0} &= u(x_i, 0, t^{k+1}) \\ U^{k+1}_{i,N_y} &= u(x_i, y_{N_y}, t^{k+1}) \end{aligned}$$

The order of the additional term in the discretization of the equation is

$$O(c^2\tau(1 + a/\tau^\alpha)/\tau) = O(\tau^{2+2\alpha-2\beta}(\tau^\alpha + a)/\tau^\alpha) = O(\tau^{2+\alpha-2\beta}),\ a, b \neq 0.$$

We will call this discretization "method 1". In order to have first order approximation of the equation: $\alpha \geq 2\beta - 1$, i.e., when $\beta > 0.5$ we have a restriction for α.

M.R. Cui, J. Comput. Physics 231 (2012) – for the time-fractional diffusion equation instead of $\Lambda_{xx}\Lambda_{yy}(U_{ij}^{k+1} - U_{ij}^{k+1})$ the following quantity $\Lambda_{xx}\Lambda_{yy}(U_{ij}^{k+1} - 2U_{ij}^{k} + U_{ij}^{k-1})$ is used.

In our case the additional term

$$\frac{a + \tau^{\alpha}}{\tau^{\alpha+1}} c^2 \Lambda_{xx}\Lambda_{yy}(U_{ij}^{k+1} - 2U_{ij}^{k} + U_{ij}^{k-1})$$

is of order $O(\tau^{3+\alpha-2\beta})$, $a, b \neq 0$. Thus first order approximation is ensured for all $\alpha, \beta \in (0,1)$. We will call this discretization "method 2".

We also tried to use

$$\frac{a + \tau^{\alpha}}{\tau^{\alpha+1}} c^2 \Lambda_{xx}\Lambda_{yy}(U_{ij}^{k+1} - 3U_{ij}^{k} + 3U_{ij}^{k-1} - U_{ij}^{k-2}),$$

which is of order $O(\tau^{4+\alpha-2\beta})$, $a, b \neq 0$, but then in some cases the numerical solution is not stable. We will call this discretization "method 3".

A second order discretization of the problem is

$$\frac{3U_{ij}^{k+1}-4U_{ij}^{k}+U_{ij}^{k-1}}{2\tau} + \frac{a}{\tau^{\alpha+1}}\sum_{m=0}^{k+1}\omega_{2,m}^{\alpha+1}U_{ij}^{k+1-m} = \mu\Lambda(U_{ij}^{k+1}+\frac{b}{\tau^{\beta}}\sum_{m=0}^{k+1}\omega_{2,m}^{\beta}U_{ij}^{k+1-m})+f_{ij}^{k+1}.$$

Here we multiply by τ and divide by $1.5+1.5^{\alpha+1}a/\tau^{\alpha}$ and obtain similar schemes, but for

$$c=\mu\frac{\tau^{\beta}+b(1.5)^{\beta}}{1.5\tau^{\alpha}+a(1.5)^{\alpha+1}}\tau^{1+\alpha-\beta}=O(\tau^{1+\alpha-\beta}),\ a,b\neq 0.$$

Thus, the additional terms in the discretization (for method 1, 2 and 3) are of the same order, as in the previous case.

In order to have second order approximation of the equation

- method=1: $\alpha \geq 2\beta$;
- method=2: $\alpha \geq 2\beta - 1$;
- method=3: no restriction.

Compact fourth order approximation in space

$$\Theta_x = I + \frac{h^2}{12}\Lambda_{xx}, \quad \Theta_y = I + \frac{h^2}{12}\Lambda_{yy}, \quad \Theta = \Theta_y\Lambda_{xx} + \Theta_x\Lambda_{yy},$$

$$\Theta_x U_{ij} = (U_{i+1,j} + 10U_{ij} + U_{i-1,j})/12, \quad \Theta_x \frac{\partial^2 u}{\partial x^2} = \Lambda_{xx}u + O(h^4).$$

Multiplying the equation by $\Theta_x\Theta_y$ and using Lubich formulas in time we get

$$\Theta_x\Theta_y\left(\frac{3U_{ij}^{k+1} - 4U_{ij}^{k} + U_{ij}^{k-1}}{2\tau} + \frac{a}{\tau^{\alpha+1}}\sum_{m=0}^{k+1}\omega_{2,m}^{\alpha+1}U_{ij}^{k+1-m}\right) =$$

$$= \quad \mu\Theta(U_{ij}^{k+1} + \frac{b}{\tau^{\beta}}\sum_{m=0}^{k+1}\omega_{2,m}^{\beta}U_{ij}^{k+1-m}) + \Theta_x\Theta_y f_{ij}^{k+1}.$$

Using the same additional terms as in the previous case we obtain

$$(I - \tilde{c}\Lambda_{xx})(I - \tilde{c}\Lambda_{yy})U^{k+1} = \tilde{G}(U^k, U^{k-1}, \dots, U^0, x_i, y_j, t^{k+1}),$$

where $\tilde{c} = c - h^2/12$. The coefficient matrices of the resulting linear systems of equations are strictly diagonally dominant.

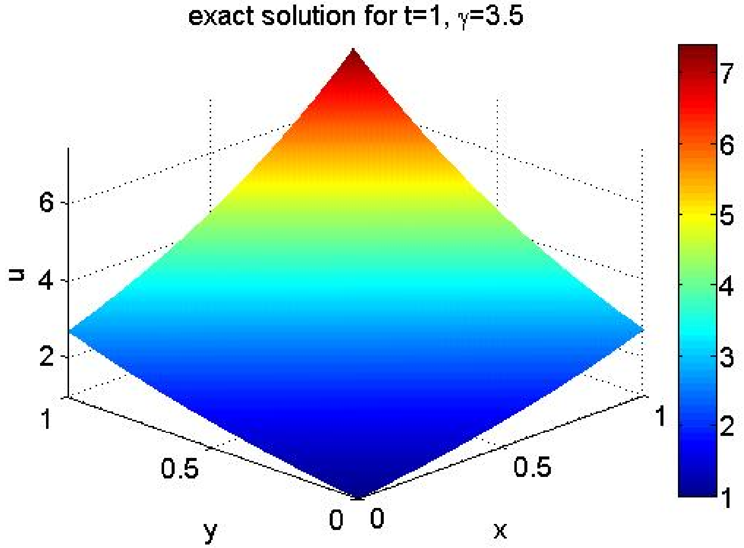
Numerical experiments

Extensive numerical experiments are performed in order to investigate the behaviour of the solutions for different values of the parameters α and β.

The initial data and the right-hand side correspond to an exact solution

$$u(x,y,t) = \mathrm{e}^{x+y} t^{\gamma+1},$$

$\gamma = 3.5$ in the numerical experiments.



The order of convergence l is computed as

$$l = \log_2 \frac{\delta(U_{s-1})}{\delta(U_s)},$$

where s is the number of the corresponding grid and

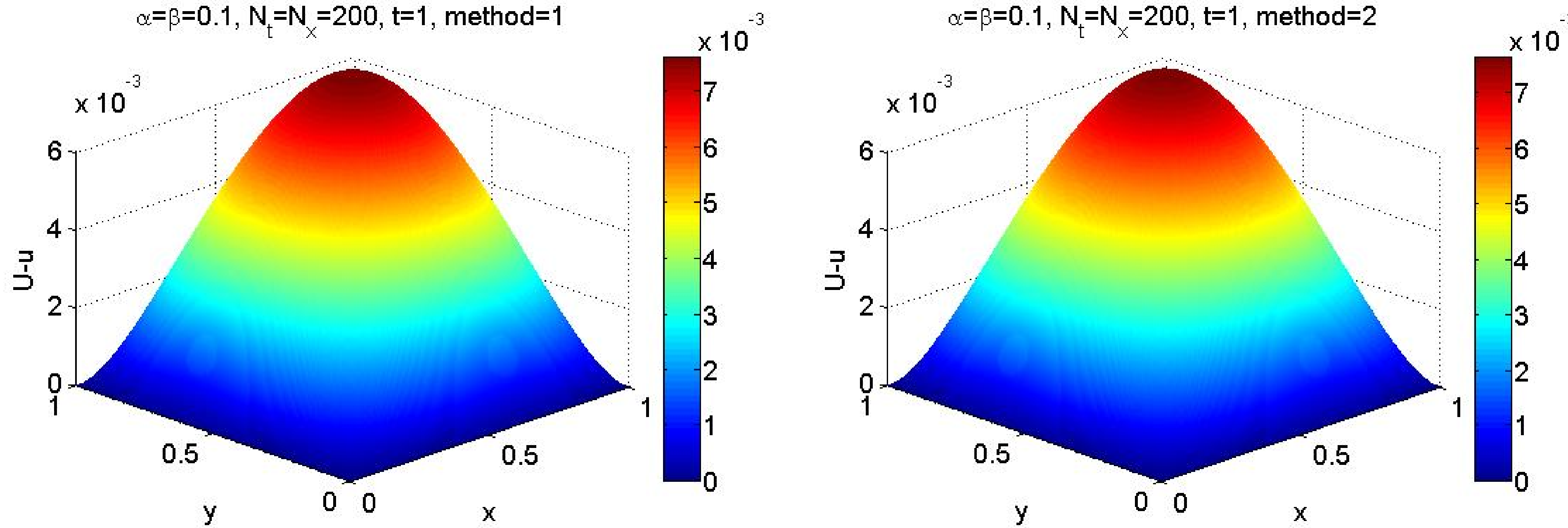
$$\delta(U) := \max\{|u(x_i,y_j,t_k) - U(x_i,y_j,t_k)|,$$
$$0 \le i \le N_x,\ 0 \le j \le N_y,\ 0 \le k \le N_t\}$$

is the maximum of the difference between the exact and the numerical solution. In all numerical experiments $N_x = N_y$.

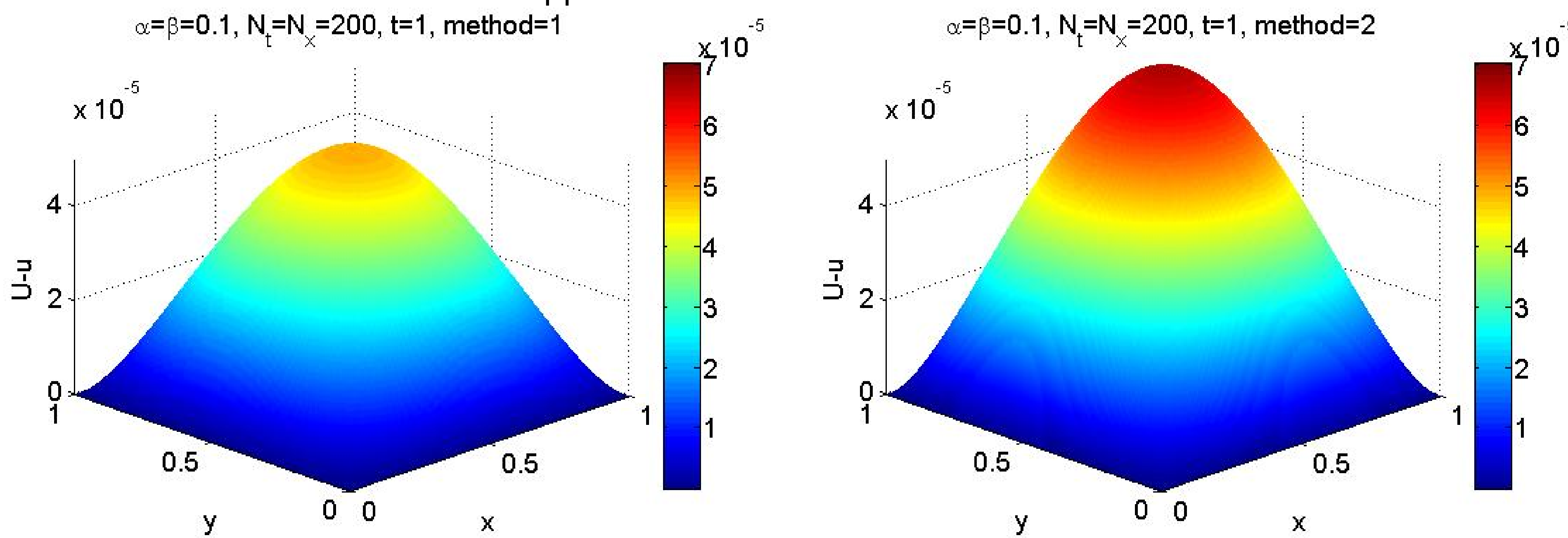
The error $\delta(U)$ and the order l for $\alpha = 0.1,\ \beta = 0.1,\ 2 + \alpha - 2\beta = 1.9$

N_t	N_x	method 1		method 2		method 3	
First order Grünwald-Letnikov approximation							
100	100	1.5066e-2		1.5180e-2		1.5241e-2	
200	100	7.5939e-3	0.9884	7.6216e-3	0.9940	2.7827e+0	unstable
400	100	3.8127e-3	0.9940	3.8196e-3	0.9967		
800	100	1.9113e-3	0.9963	1.9131e-3	0.9975		
100	100	1.5066e-2		1.5180e-2		1.5241e-2	
200	200	7.5929e-3	0.9886	7.6206e-3	0.9942	2.9450e+0	unstable
400	400	3.8107e-3	0.9946	3.8176e-3	0.9972		
800	800	1.9088e-3	0.9974	1.9106e-3	0.9986		
Second order Lubich approximation							
100	100	1.9894e-4		2.6217e-4		2.8736e-4	
200	200	4.9245e-5	2.0143	6.6214e-5	1.9853	0.3733e+0	unstable
400	400	1.2141e-5	2.0201	1.6640e-5	1.9925		
800	800	2.9853e-6	2.0239	4.1709e-6	1.9962		
Compact approximation in space, second order Lubich approximation in time							
100	25	1.9567e-4		2.5861e-4		2.7605e-4	
200	25	4.8423e-5	2.0147	6.5314e-5	1.9853	1.8655e-2	unstable
400	25	1.1933e-5	2.0207	1.6411e-5	1.9927		
800	25	2.9307e-6	2.0256	4.1110e-6	1.9971		

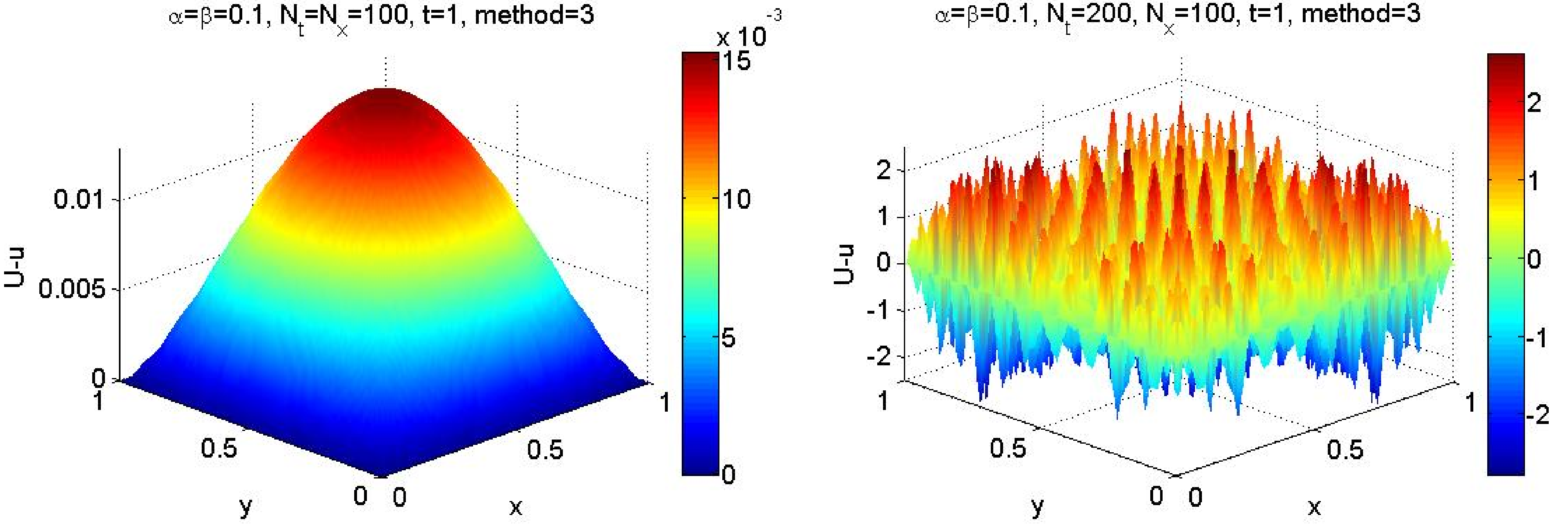
First order approximation of the fractional derivatives



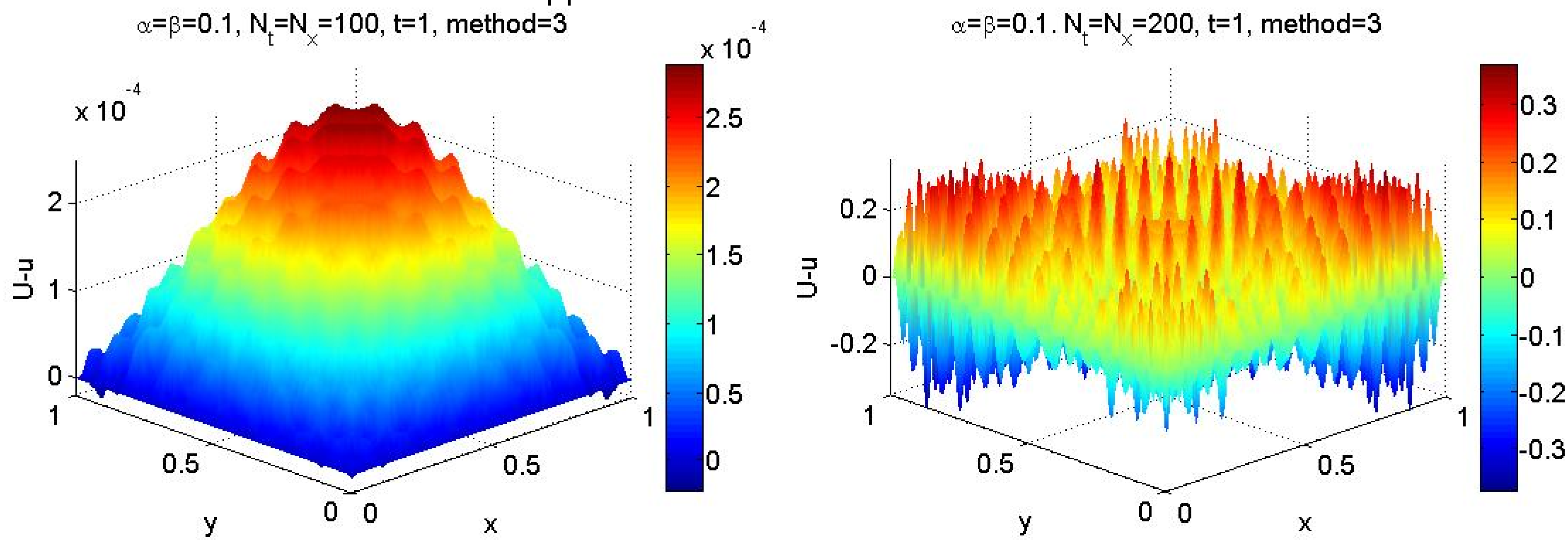
Second order approximation of the fractional derivatives

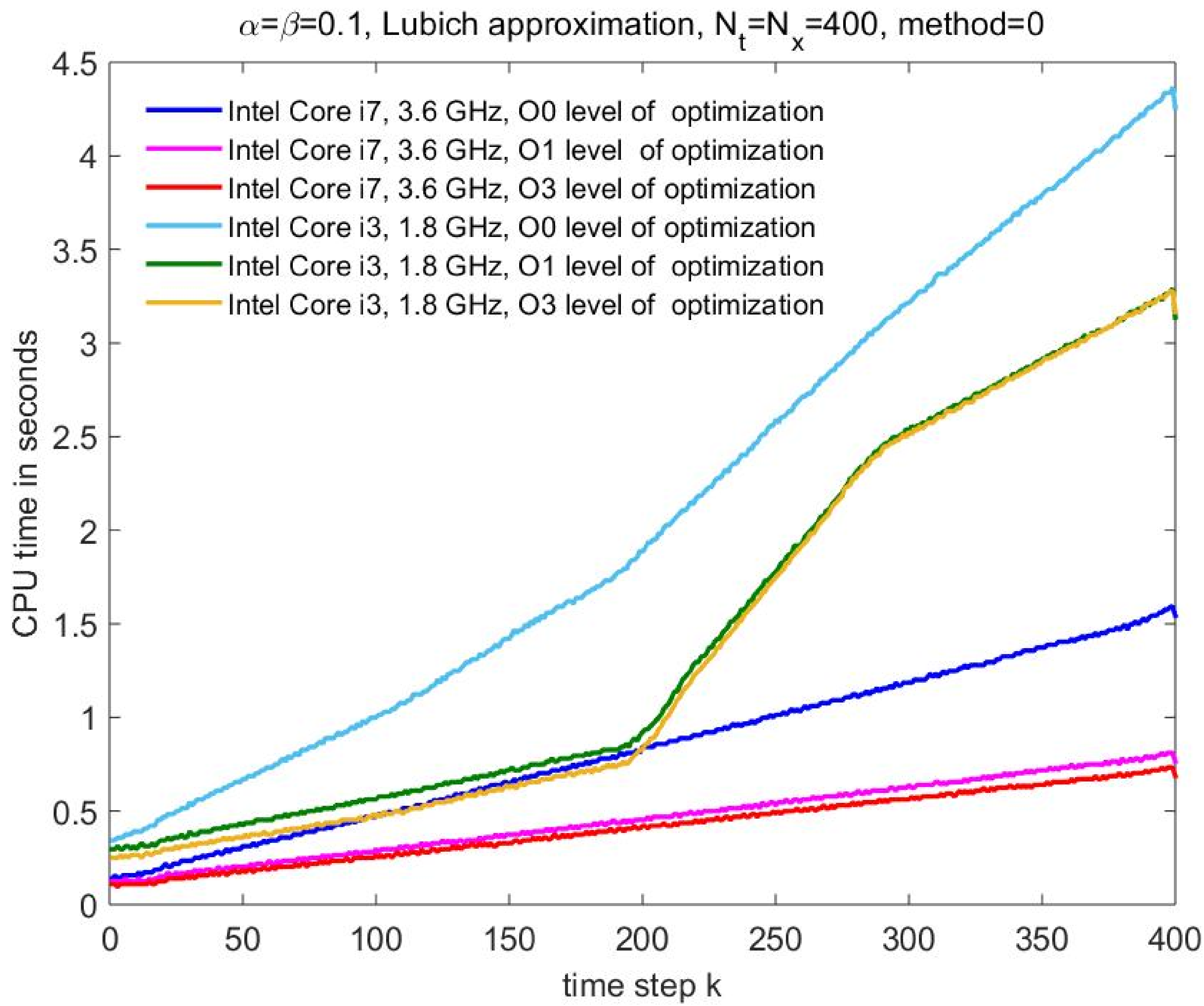


First order approximation of the fractional derivatives



Second order approximation of the fractional derivatives





α=β=0.1, Lubich approximation, N_t=N_x=400, method=0
Intel Core i7, 3.6 GHz, O0 level of optimization
Intel Core i7, 3.6 GHz, O1 level of optimization
Intel Core i7, 3.6 GHz, O3 level of optimization
Intel Core i3, 1.8 GHz, O0 level of optimization
Intel Core i3, 1.8 GHz, O1 level of optimization
Intel Core i3, 1.8 GHz, O3 level of optimization
CPU time in seconds
time step k
0
0.5
1
1.5
2
2.5
3
3.5
4
4.5
0
50
100
150
200
250
300
350
400

The error $\delta(U)$ and the order l for $\alpha = 0.1,\ \beta = 0.1,\ 2 + \alpha - 2\beta = 1.9$

N_t	N_x	explicit method	
First order Grünwald-Letnikov approximation			
3200	25	4.0845e-4	
12800	50	1.0242e-4	1.9957
51200	100	stable	
6400	25	1.8117e-4	
25600	50	4.5442e-5	1.9952
12800	25	6.7573e-5	
51200	50	1.6956e-5	1.9947

Note, here we refine $\tau = 1/N_t$ 4 times and $h = 1/N_x$ 2 times from case to case, thus the second order convergence is natural.

The error $\delta(U)$ and the order l for $\alpha = 0.5,\ \beta = 0.5,\ 2 + \alpha - 2\beta = 1.5$

N_t	N_x	method 1		method 2		method 3	
First order Grünwald-Letnikov approximation							
100	100	1.6514e-2		1.6871e-2		1.6884e-2	
200	100	8.3589e-3	0.9823	8.4709e-3	0.9940	8.4728e-3	0.9947
400	100	4.2082e-3	0.9901	4.2450e-3	0.9968	7.3226e-3	unstable
800	100	2.1134e-3	0.9936	2.1259e-3	0.9977		
100	100	1.6514e-2		1.6871e-2		1.6884e-2	
200	200	8.3577e-3	0.9825	8.4698e-3	0.9941	8.4740e-3	0.9945
400	400	4.2061e-3	0.9906	4.2430e-3	0.9972	8.4691e-3	unstable
800	800	2.1110e-3	0.9946	2.1234e-3	0.9987		
Second order Lubich approximation							
100	100	1.1377e-4		2.6673e-4		2.7561e-4	
200	200	2.4027e-5	2.2434	6.7712e-5	1.9780	6.9273e-5	1.9923
400	400	1.0677e-5	1.1702	1.7091e-5	1.9852	4.0513e-5	unstable
800	800	5.4765e-6	0.9632	4.2989e-6	1.9912		
Compact approximation in space, second order Lubich approximation in time							
100	25	1.1301e-4		2.6322e-4		2.7206e-4	
200	25	2.3901e-5	2.2413	6.6823e-5	1.9779	6.8398e-5	1.9919
400	25	1.0859e-5	1.1382	1.6865e-5	1.9863	1.7143e-5	1.9963
800	25	5.5245e-6	0.9750	4.2398e-6	1.9920	4.2887e-6	1.9990
1600	25	2.3754e-6	1.2177	1.0615e-6	1.9979	1.0701e-6	2.0028
3200	25	9.4697e-7	1.3268	2.6336e-7	2.0110	2.6487e-7	2.0144
6400	25	3.6323e-7	1.3824	6.3305e-8	2.0566	6.3572e-8	2.0588
12800	25	1.3647e-7	1.4122	1.3968e-8	2.1802	1.4015e-8	2.1814

The error $\delta(U)$ and the order l for $\alpha = 0.5,\ \beta = 0.5,\ 2 + \alpha - 2\beta = 1.5$

N_t	N_x	explicit method	
First order Grünwald-Letnikov approximation			
3200	25	3.5547e-4	
12800	50	8.9109e-5	1.9961
51200	100	stable	
6400	25	1.5467e-4	
25600	50	3.8789e-5	1.9955
12800	25	5.4313e-5	
51200	50	1.3630e-5	1.9945

Note, here we refine $\tau = 1/N_t$ 4 times and $h = 1/N_x$ 2 times from case to case, thus the second order convergence is natural.

The error $\delta(U)$ and the order l for $\alpha = 0.9,\ \beta = 0.9,\ 2+\alpha-2\beta = 1.1$

N_t	N_x	method 1		method 2		method 3	
First order Grünwald-Letnikov approximation							
100	100	1.7633e-2		1.9048e-2		1.9101e-2	
200	100	8.9931e-3	0.9714	9.5726e-3	0.9927	9.5843e-3	0.9949
400	100	4.5484e-3	0.9835	4.7989e-3	0.9962	4.8016e-3	0.9972
800	100	2.2915e-3	0.9891	2.4036e-3	0.9975	2.4042e-3	0.9980
100	100	1.7633e-2		1.9048e-2		1.9101e-2	
200	200	8.9916e-3	0.9716	9.5712e-3	0.9929	9.5829e-3	0.9951
400	400	4.5462e-3	0.9839	4.7969e-3	0.9966	4.7996e-3	0.9975
800	800	2.2889e-3	0.9900	2.4011e-3	0.9984	2.4017e-3	0.9989
Second order Lubich approximation							
100	100	6.9947e-4		2.4425e-4		2.8676e-4	
200	200	3.9464e-4	0.8257	6.1964e-5	1.9789	7.2168e-5	1.9904
400	400	2.0133e-4	0.9710	1.5686e-5	1.9819	1.8100e-5	1.9954
800	800	9.8224e-5	1.0354	3.9630e-6	1.9848	4.5301e-6	1.9984
Compact approximation in space, second order Lubich approximation in time							
100	25	7.0215e-4		2.4094e-4		2.8326e-4	
200	25	3.9517e-4	0.8293	6.1123e-5	1.9789	7.1285e-5	1.9905
400	25	2.0140e-4	0.9724	1.5472e-5	1.9821	1.7876e-5	1.9957
800	25	9.8211e-5	1.0361	3.9086e-6	1.9849	4.4733e-6	1.9986
1600	25	4.6874e-5	1.0671	9.8432e-7	1.9895	1.1165e-6	2.0024
3200	25	2.2130e-5	1.0828	2.4617e-7	1.9995	2.7705e-7	2.0108
6400	25	1.0386e-5	1.0914	6.5094e-8	1.9191	7.2309e-8	1.9379

The error $\delta(U)$ and the order l for $\alpha = 0.9,\ \beta = 0.9,\ 2 + \alpha - 2\beta = 1.1$

N_t	N_x	explicit method	
First order Grünwald-Letnikov approximation			
3200	25	2.8623e-4	
12800	50	7.1717e-5	1.9968
51200	100	stable	
6400	25	1.2005e-4	
25600	50	3.0093e-5	1.9961
12800	25	3.6987e-5	
51200	50	9.2816e-6	1.9946

Note, here we refine $\tau = 1/N_t$ 4 times and $h = 1/N_x$ 2 times from case to case, thus the second order convergence is natural.

The error $\delta(U)$ and the order l for $\alpha = 0.9,\ \beta = 0.1,\ 2 + \alpha - 2\beta = 2.7$

N_t	N_x	method 1		method 2		method 3	
First order Grünwald-Letnikov approximation							
100	100	3.9699e-2		3.9704e-2		3.9704e-2	
200	100	1.9948e-2	0.9929	1.9949e-2	0.9930	1.9949e-2	0.9930
400	100	9.9992e-3	0.9964	9.9993e-3	0.9964	9.9993e-3	0.9964
800	100	5.0067e-3	0.9980	5.0067e-3	0.9980	5.0067e-3	0.9980
100	100	3.9699e-2		3.9704e-2		3.9704e-2	
200	200	1.9948e-2	0.9929	1.9949e-2	0.9930	0.4513e+0	unstable
400	400	9.9981e-3	0.9965	9.9981e-3	0.9966		
800	800	5.0050e-3	0.9983	5.0050e-3	0.9983		
Second order Lubich approximation							
100	100	4.8860e-4		4.9018e-4		4.9025e-4	
200	200	1.2267e-4	1.9938	1.2291e-4	1.9957	1.9030e-3	unstable
400	400	3.0738e-5	1.9968	3.0773e-5	1.9979		
800	800	7.6938e-6	1.9983	7.6991e-6	1.9989		
Compact approximation in space, second order Lubich approximation in time							
100	25	4.8529e-4		4.8687e-4		4.8693e-4	
200	25	1.2183e-4	1.9940	1.2207e-4	1.9958	1.2207e-4	1.9960
400	25	3.0523e-5	1.9969	3.0559e-5	1.9980	3.0559e-5	1.9980
800	25	7.6379e-6	1.9986	7.6432e-6	1.9993	7.6433e-6	1.9993

The error $\delta(U)$ and the order l for $\alpha = 0.9,\ \beta = 0.1,\ 2 + \alpha - 2\beta = 2.7$

N_t	N_x	explicit method	
First order Grünwald-Letnikov approximation			
400	100	2.2392e-3	
800	100	1.1167e-3	1.0037
1600	100	5.5678e-4	1.0040
3200	100	2.7715e-4	1.0064
800	200	1.1186e-3	
1600	200	5.5858e-4	1.0019
3200	200	2.7890e-4	1.0020

The error $\delta(U)$ and the order l for $\alpha = 0.1,\ \beta = 0.9,\ 2 + \alpha - 2\beta = 0.3$

N_t	N_x	method 1		method 2		method 3	
First order Grünwald-Letnikov approximation							
100	200	1.6180e-2		1.4388e-3		2.6537e-3	
200	200	1.4920e-2	0.1170	8.3133e-4	0.7914	1.3414e-3	0.9843
400	200	1.3455e-2	0.1491	4.6130e-4	0.8497	6.7413e-4	0.9926
800	200	1.1942e-2	0.1721	2.4956e-4	0.8863	3.3812e-4	0.9955
1600	200	1.0468e-2	0.1901	1.3281e-4	0.9100	1.6961e-4	0.9953
3200	200	9.0875e-3	0.2040	6.9969e-5	0.9246	8.5240e-5	0.9926
6400	200	7.8248e-3	0.2158	3.6700e-5	0.9309	4.3130e-5	0.9828
100	100	1.6178e-2		1.4411e-3		2.6559e-3	
200	200	1.4920e-2	0.1168	8.3133e-4	0.7937	1.3414e-3	0.9855
400	400	1.3456e-2	0.1490	4.6070e-4	0.8516	6.7354e-4	0.9939
800	800	1.1942e-2	0.1722	2.4882e-4	0.8887	3.3733e-4	0.9976

The explicit method is unstable even for $N_t = 2000000$, $N_x = 25$.

The error $\delta(U)$ and the order l for $\alpha = 0.1,\ \beta = 0.9,\ 2+\alpha-2\beta = 0.3$

N_t	N_x	method 1		method 2		method 3	
Second order Lubich approximation							
100	100	2.0857e-2		1.5429e-3		3.6779e-6	
200	200	1.8862e-2	0.1450	6.6520e-4	1.2138	2.2757e-6	0.6926
400	400	1.6888e-2	0.1595	2.8149e-4	1.2407	1.0317e-6	1.1413
800	800	1.4984e-2	0.1726	1.1800e-4	1.2543	3.5302e-7	1.5472
200	25	1.8817e-2		6.1245e-4		5.3154e-5	
400	50	1.6876e-2	0.1571	2.6870e-4	1.1886	1.3806e-5	1.9449
800	100	1.4980e-2	0.1719	1.1481e-4	1.2267	3.5509e-6	1.9590
1600	200	1.3181e-2	0.1846	4.8419e-5	1.2456	9.0821e-7	1.9671
Compact approximation in space, second order Lubich approximation in time							
100	25	2.0859e-2		1.5421e-3		3.3490e-6	
200	25	1.8863e-2	0.1451	6.6429e-4	1.2150	1.7695e-6	0.9204
400	25	1.6888e-2	0.1596	2.8099e-4	1.2413	8.2782e-7	1.0960
800	25	1.4983e-2	0.1727	1.1777e-4	1.2545	3.0164e-7	1.4565
1600	25	1.3181e-2	0.1849	4.9114e-5	1.2618	9.2731e-8	1.7017
3200	25	1.1502e-2	0.1966	2.0423e-5	1.2659	2.4340e-8	1.9297
6400	25	9.9635e-3	0.2072	8.4768e-6	1.2686	4.3421e-9	2.4869
12800	25	8.5713e-3	0.2171	3.5146e-6	1.2702	3.8719e-8	unstable
25600	25	7.3276e-3	0.2262	1.4568e-6	1.2706		

The error $\delta(U)$ and the order l for $\alpha = 0.1,\ \beta = 0.5,\ 2 + \alpha - 2\beta = 1.1$

N_t	N_x	method 1		method 2		method 3	
First order Grünwald-Letnikov approximation							
100	100	7.9286e-3		9.3329e-3		9.3855e-3	
200	100	4.0811e-3	0.9581	4.6970e-3	0.9906	4.7099e-3	0.9947
400	100	2.0789e-3	0.9731	2.3568e-3	0.9949	1.3588e-2	unstable
800	100	1.0537e-3	0.9804	1.1815e-3	0.9962		
100	100	7.9286e-3		9.3329e-3		9.3855e-3	
200	200	4.0791e-3	0.9588	4.6951e-3	0.9912	4.7100e-3	0.9947
400	400	2.0763e-3	0.9742	2.3543e-3	0.9959	1.6863e-2	unstable
800	800	1.0509e-3	0.9824	1.1787e-3	0.9981		
Second order Lubich approximation							
100	100	1.0077e-3		1.2271e-4		1.6838e-4	
200	200	5.0801e-4	0.9881	3.1768e-5	1.9496	4.2509e-5	1.9859
400	400	2.4698e-4	1.0405	8.1576e-6	1.9614	1.2408e-4	unstable
800	800	1.1820e-4	1.0632	2.0836e-6	1.9691		
Compact approximation in space, second order Lubich approximation in time							
100	25	1.0090e-3		1.1936e-4		1.6487e-4	
200	25	5.0794e-4	0.9902	3.0920e-5	1.9487	4.1755e-5	1.9813
400	25	2.4678e-4	1.0414	7.9417e-6	1.9610	4.9238e-5	unstable
800	25	1.1807e-4	1.0636	2.0266e-6	1.9704		

The explicit method is unstable even for $N_t = 2000000$, $N_x = 25$.

The error $\delta(U)$ and the order l for $\alpha = 0.5,\ \beta = 0.9,\ 2 + \alpha - 2\beta = 0.7$

N_t	N_x	method 1		method 2		method 3	
First order Grünwald-Letnikov approximation							
100	100	2.1472e-3		8.3602e-3		8.6590e-3	
200	100	8.0142e-4	1.4218	4.2534e-3	0.9749	4.3451e-3	0.9948
400	100	3.8416e-4	1.0609	2.1490e-3	0.9850	2.1773e-3	0.9968
800	100	5.0113e-4	stable	1.0822e-3	0.9897	1.0910e-3	0.9969
1600	100	4.4304e-4	0.1777	5.4450e-4	0.9910	5.4721e-4	0.9955
3200	100	3.3983e-4	0.3826	2.7436e-4	0.9889	2.7520e-4	0.9916
6400	100	2.4209e-4	0.4893	1.3891e-4	0.9819	1.3916e-4	0.9837
100	100	2.1472e-3		8.3602e-3		8.6590e-3	
200	200	8.0097e-4	1.4226	4.2514e-3	0.9756	4.3431e-3	0.9955
400	400	3.8691e-4	1.0498	2.1464e-3	0.9860	2.1747e-3	0.9979
800	800	5.0414e-4	stable	1.0793e-3	0.9918	1.0880e-3	0.9991

The explicit method is unstable even for $N_t = 2000000$, $N_x = 25$.

The error $\delta(U)$ and the order l for $\alpha = 0.5,\ \beta = 0.9,\ 2 + \alpha - 2\beta = 0.7$

N_t	N_x	method 1		method 2		method 3	
Second order Lubich approximation							
100	100	6.2613e-3		1.6712e-4		1.3899e-4	
200	200	4.2141e-3	0.5712	6.3000e-5	1.4075	3.5898e-5	1.9531
400	400	2.7456e-3	0.6181	2.2155e-5	1.5077	9.1539e-6	1.9714
800	800	1.7531e-3	0.6472	7.4867e-6	1.5652	2.3225e-6	1.9787
Compact approximation in space, second order Lubich approximation in time							
100	25	6.2559e-3		1.7000e-4		1.3554e-4	
200	25	4.2056e-3	0.5729	6.3684e-5	1.4165	3.5025e-5	1.9523
400	25	2.7384e-3	0.6190	2.2317e-5	1.5128	8.9306e-6	1.9716
800	25	1.7492e-3	0.6466	7.5270e-6	1.5680	2.2570e-6	1.9844
1600	25	1.1029e-3	0.6654	2.4788e-6	1.6024	5.6545e-7	1.9969
3200	25	6.8965e-4	0.6774	8.0463e-7	1.6232	1.3905e-7	2.0238
6400	25	4.2890e-4	0.6852	2.5988e-7	1.6305	3.1960e-8	2.1213

Conclusions

- The most time-consuming part of the numerical method is the computation of the finite fractional derivatives;
- The order of convergence of the proposed numerical schemes seems to be in accordance with their order of approximation;
- The first and the second method for the choice of the additional term in the ADI schemes seem to be unconditionally stable;
- The third method is not unconditionally stable, but it seems to work well in some cases, where the first and second method for the choice of the additional term destroy the second order (Lubich) approximation of the equation in time.
- The future work includes
  - Stability and convergence analysis;
  - Development of iterative solvers (AMG) for the 2D implicit schemes instead of using ADI methods;
  - Development of techniques for truncation of the "tail" in the finite fractional derivatives formulas;
  - Numerical experiments for practical problems with application in industry.

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Some papers and presentations, supported by the grant, may be found at

http://www.math.bas.bg/~nummeth/nonlinear/

Thank you for your attention!