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# AN IMPROVEMENT OF SENDOV'S ESTIMATION FOR PARAMETRIC APPROXIMATION OF PARTIALLY ANALYTIC FUNCTIONS

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The following estimate for the best parametric approximation, introduced by Bl. Sendov, is given:

$$\varepsilon_{n,n}(f) \leqq \exp[-cn^{2/3}(\ln n)^{1/3}],$$

where $f$ is a function continuous in $[-1, 1]$ and identically equal to two analytic functions respectively in $[-1, 1]$ and $[0, 1]$. It is proved that this estimate is exact in one particular case of parametric approximations.

**1.** The parametric approximation of continuous functions was introduced by Bl. Sendov [1]. Denote by $H_n$ the set of all algebraic polynomials of degree not greater than $n$, and by $\hat{H}_n$ denote the subset of $H_n$, as follows: $P \in \hat{H}_n$ if $P(-1)=-1$, $P(1)=1$, $P'(x) \geqq 0$ for $|x| \leqq 1$.

If $f \in C[-1, 1]$, define $\varepsilon_{m,n}(f) = \inf_{P \in \hat{H}_m} \inf_{Q \in H_n} \max_{|x| \leqq 1} |f(P(x)) - Q(x)|$.

It can be easily seen that for every $f \in C[-1, 1]$ and a pair of integers $m$, $n$, there exists a pair of polynomials $P^* \in \hat{H}_m$ and $Q^* \in H_n$ for which the best parametric approximation $\varepsilon_{n,m}(f)$ of the function $f$ can be obtained. Provided one knows the pair of polynomials of the best parametric approximation $P^*$ and $Q^*$, it is easy to find the approximate value of $f$ at any point $y \in [-1, 1]$. It is sufficient to obtain the unique root of the equation $y = P^*(x)$, and, if it is $x_0$, then $f(y) \approx Q^*(x_0)$.

It is well-known that for functions having particular features at a finite number of points, the parametric approximation is better than the usual polynomial approximation.

In [2] Bl. Sendov has proved the following

Theorem 1. *If $f$ is a partially analytic function of the form*

$$f(x) = \begin{cases} \varphi_1(x), & -1 \leqq x \leqq 0, \\ \varphi_2(x), & 0 \leqq x \leqq 1, \end{cases}$$

*where $\varphi_1$, $\varphi_2$ are analytic functions in a disc having its centre at the origin and radius $r > 1$, and $\varphi_1(0) = \varphi_2(0)$, then for any positive integer $n$*

$$\varepsilon_{n,n}(f) \leqq e^{-c_1\sqrt{n}}, \quad c_1 > 0$$

*holds.*

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In [3] J. Szabados proves

Theorem 2. *Let $f$ be continuous in $[-1, 1]$ and $-1=\xi_0<\xi_1<\ldots<\xi_s=1$, $(s\geqq 2)$, $\tau=\min_{1\leq j\leq s}(\xi_j-\xi_{j-1})$, $I_j=[\xi_{j-1}, \xi_j]$, $j=1,\ldots, s$. If on any of the intervals $I_j$, $f$ coincides with a function analytic in the disc $c_{j,r}=\{z: |z-\eta_j|\leqq(\xi_j-\xi_{j-1})r/2\}$ $(j=1, \ldots, s; r>1)$, $\eta_j=(\xi_j-\xi_{j-1})/2$ then*

$$\varepsilon_{n,n}(f)\leqq e^{-c_2\sqrt{n}}, \quad c_2=c_2(s, \tau)>0.$$

In the present paper the estimate from Theorem 1 is improved, i. e. if $f$ satisfies the conditions of Theorem 1, then

$$\varepsilon_{n,n}(f)\leqq e^{-c_3\sqrt[3]{n^2\ln n}}. \tag{1}$$

It can be proved that if $f$ satisfies the conditions of Theorem 2 for $s=2$, then the estimate (1) also holds. Note that for $s\geqq 3$ the results of J. Szabados has been improved in [4] by proving that under the conditions of Theorem 2

$$\varepsilon_{n,n}(f)\leqq e^{-c_4\sqrt{n\ln n}}, \quad c_4=c_4(s, \tau).$$

When finding (1), it is actually proved that

$$\varepsilon^*_{n,n}(f)=\inf_{0\leqq k\leqq[n/2]}\ \inf_{Q\in H_n}\ \max_{|x|\leqq 1}|f(x^{2k+1})-Q(x)|\leqq e^{-c_3\sqrt[3]{n^2\ln n}} \tag{2}$$

hence follows (1), since for each $k$, $0\leqq k\leqq[n/2]$, $x^{2k+1}\in\hat{H}_n$.

In the present paper it has been proved that there exists a function $f^*$ satisfying the conditions of Theorem 1, for which $\varepsilon^*_{n,n}(f^*)\geqq\exp(-c_5\sqrt[3]{n^2\ln n})$, i. e. the estimate (2) is exact.

2. By $E_n(f; [a, b])$ denote, as usual, the uniform best approximation of the continuous function $f$ by polynomials from $H_n$ in $[a, b]$.

Lemma 1. *Let $f(u)$ be an analytic function in a disc with radius $a$, $a>1$ and centre the origin, and let $f$ assume real values for real values of $u$. Then there exists a constant $c_6$, such that for any positive integer $k$*

$$E_n(\varphi_k; [-1, 1])\leqq e^{-c_6 n/\sqrt{k}},$$

*where $\varphi_k(z)=f(z^k)$, $z=x+iy$.*

Proof. The following generalization of Bernstein, given in [5], will be used. Let $f(z)$ be an analytic function which is regular in the interior of an ellipse with focuses at the points $\pm 1$ and half-sum of the axes $1/q$, $(z=x+iy)$. Then

$$E_{n-1}(f(x); [-1, 1])\leqq\frac{8}{\pi}\sum_{m=0}^{\infty}(-1)^m\frac{1}{2m+1}\cdot\frac{q^{(2m+1)n}}{1+q^{2(2m+1)n}}. \tag{3}$$

For every $k$ substitute $u=z^k$. We obtain $\varphi_k(z)=f(z^k)$. For each $k$ the function $\varphi_k(z)$ is analytic in a disc having its centre at the origin and a radius $b=a^{1/k}$. But then it is regular in the interior of an ellipse with focuses at $\pm 1$ and half-sum of the axes $b+\sqrt{b^2-1}$ as well. So, from (3), for $1/q=b+\sqrt{b^2-1}$

$$E_{n-1}(\varphi_k(x);[-1,1])\leq\frac{8}{\pi}\sum_{m=0}^{\infty}(-1)^m\frac{1}{2m+1}\cdot\frac{q^{(2m+1)n}}{1+q^{2(2m+1)n}}$$

$$\leq\frac{8}{\pi}q^n\sum_{m=0}^{\infty}(-1)^m\frac{1}{2m+1}\cdot\frac{q^{2mn}}{1+q^{2(2m+1)n}}\leq c_7q^n$$

$$=c_7(a^{1/k}+\sqrt{a^{2/k}-1})^{-n}\leq c_7[(1+\frac{\sqrt{a^{2/k}-1}}{a})^{\sqrt{k}}]^{-n/\sqrt{k}},$$

where $c_7$ is an absolute constant. But $a^{2/k}-1\geq c_8/k$. We obtain

$$E_n(\varphi_k(x);[-1,1])\leq c_7'[(1+c_9/\sqrt{k})^{\sqrt{k}}]^{-(n+1)/\sqrt{k}}\leq e^{-c_9n/\sqrt{k}}.$$

Thus the Lemma is proved.

Theorem 3. *Let $f\in C[-1,1]$, $f(0)=0$ and*

$$f(x)=\begin{cases}f_1(x) & \text{for } -1\leq x\leq 0,\\ f_2(x) & \text{for } 0\leq x\leq 1,\end{cases}$$

*where $f_1$ and $f_2$ are analytic functions in a disc having its centre at the origin and a radius $a$, $a>1$, assuming real values for real values of their arguments. Then*

$$\varepsilon_{n,n}(f)\leq\exp(-c_{10}\sqrt[3]{n^2\ln n}).$$

Proof. Consider the functions $f_1(z^k)$ and $f_2(z^k)$. From Lemma 1 it follows that there exist polynomials $Q_1\in H_{n/2}$ and $Q_2\in H_{n/2}$ and an absolute constant $c_{11}$, such that

(4) $$\|f_1(x^k)-Q_1(x)\|_{[-1,1]}\leq e^{-c_{11}n/\sqrt{k}},$$

(5) $$\|f_2(x^k)-Q_2(x)\|_{[-1,1]}\leq e^{c_{11}n/\sqrt{k}}.$$

Consider the function

$$\sigma(x)=\begin{cases}0, & -1\leq x\leq 0,\\ 1, & 0<x\leq 1.\end{cases}$$

From [4] it is known that for every $\delta>0$ there exists a polynomial $\sigma_n(x)\in H_{n/2}$, such that

(6) $$|\sigma_n(x)|\leq 1 \quad \text{for } x\in[-\delta,\delta],$$

(7) $$|\sigma_n(x)|\leq e^{-c_{12}n\delta} \quad \text{for } x\in[-1,\delta],$$

(8) $$|\sigma_n(x)-1|\leq e^{-c_{12}n\delta} \quad \text{for } x\in[\delta,1].$$

Form the polynomial $Q(x)=Q_1(x)+(Q_2(x)-Q_1(x))\sigma_n(x)$, $Q\in H_n$.

Estimate $|f(x^k)-Q(x)|$, for $x\in[-1,1]$. For $x\in[-1,\delta]$, from (4) and (7) we obtain

(9) $$|f(x^k)-Q(x)|=|f_1(x^k)-Q_1(x)-(Q_2(x)-Q_1(x))\sigma_n(x)|$$

$$\leq|f_1(x^k)-Q_1(x)|+|Q_2(x)-Q_1(x)|\,|\sigma_n(x)|\leq e^{-c_{11}n/\sqrt{k}}+Me^{-c_{12}n\delta},$$

since $Q_1$ and $Q_2$ are obviously bounded in $[-1,1]$.

For $x \in [\delta, 1]$, from (5) and (8), we get

(10) $$|f(x^k)-Q(x)|\leqq|f_2(x^k)-Q_2(x)|+|Q_2(x)-Q_1(x)||1-\sigma_n(x)|$$
$$\leqq e^{-c_{11}n\sqrt{k}}+Me^{-c_{12}n\delta}.$$

For $x \in [-\delta, \delta]$

(11) $$|f(x^k)-Q(x)|=|f(x^k)-Q_1(x)(1-\sigma_n(x))-Q_2(x)\sigma_n(x)|$$
$$\leqq|f(x^k)|+|f_1(x^k)-Q_1(x)||1-\sigma_n(x)|+|f_2(x^k)||\sigma_n(x)|$$
$$+|f_2(x^k)-Q_2(x)||\sigma_n(x)|+|f_1(x^k)||1-\sigma_n(x)|.$$

But $f_1$, $f_2$ and $f \in \mathrm{Lip}_{c_{11}} 1$ and $f_1(0)=f_2(0)=f(0)$. Therefore

(12) $$|f_i(x^k)|=|f_i(x^k)-f(0)|\leqq c_{13}|x|^k\leqq c_{13}\delta^k, \quad i=1,\ 2,$$
$$|f(x^k)|\leqq c_{13}\delta^k.$$

Besides, from (6) for $x \in [-\delta, \delta]$

(13) $$|\sigma_n(x)|\leqq 1 \text{ and } |1-\sigma_n(x)|\leqq 1+|\sigma_n(x)|\leqq 2.$$

From (11), (12) and (13), for $x \in [-\delta, \delta]$ we obtain

(14) $$|f(x^k)-Q(x)|\leqq c_{14}\delta^k+\exp(-c_{15}n/\sqrt{k}).$$

Or, from (9), (10) and (14), we get for every $\delta>0$, $1\leqq k\leqq n$, $k$ — odd

(15) $$\|f(x^k)-Q(x)\|_{[-1,1]}\leqq e^{-c_{16}n/\sqrt{k}}+e^{-c_{17}n\delta}+c_{14}\delta^k.$$

If $k=n^{2/3}(\ln n)^{-2/3}$, $\delta=n^{-1/3}(\ln n)^{1/3}$,

(16) $$\exp(-c_{16}n/\sqrt{k})=\exp[-c_{16}n^{2/3}(\ln n)^{1/3}],$$
$$\exp(-c_{17}n\delta)=\exp[-c_{17}n^{2/3}(\ln n)^{1/3}],$$
$$\delta^k=\left(\frac{\ln n}{n}\right)^{n^{2/3}\cdot(\ln n)^{-2/3}/3}=\exp\{1/3[\ln\ln n-\ln n]\,n^{2/3}(\ln n)^{-2/3}\}$$
$$\leqq\exp[-c_{18}n^{2/3}(\ln n)^{1/3}].$$

From (15) and (16), since for $k$ — odd, $1\leqq k\leqq n$, $x_k \in \hat{H}_n$ it follows that $\varepsilon_{n,n}(f)\leqq\exp(-c_{10}\sqrt[3]{n^2\ln n})$.

3. Let for $x \in [-1, 1]$, $f^*(x)=|x|/(x-a)$, $a>1$. This function possesses the conditions of Theorem 1. We will prove that

$$\varepsilon^*_{n,n}(f^*)=\inf_{1\leqq k\leqq[n/2]}\ \inf_{Q\in H_n}\ \max_{|x|\leqq 1}|f(x^{2k+1})-Q(x)|\geqq\exp(-c_{19}\sqrt[3]{n^2\ln n}).$$

L e m m a **2.** *There exists an absolute constant $c_{20}$, such that for every odd $k$, $1\leqq k\leqq n$,*

$$\varepsilon^*_n(k)=\inf_{Q\in\tilde{H}n}\|x^k/(x^k-a)-Q(x)\|_{[0,1]}\geqq\exp[-c_{20}n/\sqrt{k}],$$

*where $\tilde{H}_n$ is the set of all even polynomials of degree not greater than $n$.*

Proof. Since $x^k/(x^k-a)=a/(x^k-a)+1$, then

$$\varepsilon_n^*(k)=a\inf_{Q\in\widetilde{H}_n}\left\|\frac{1}{x^k-a}-Q(x)\right\|_{[0,1]}$$

$$=a\inf_{Q\in\widetilde{H}_n}\left\|\frac{1}{(x-b)(x^{k-1}+x^{k-2}b+\ldots+b^{k-1})}-Q(x)\right\|_{[0,1]},$$

where $b=a^{1/k}$. But

$$\|x^{k-1}+x^{k-2}b+\ldots+b^{k-1}\|_{[0,1]}=1+b+\ldots+b^{k-1}=(b^k-1)/(b-1),$$

or

$$\varepsilon_n^*(k)\geqq a\frac{b-1}{b^k-1}\inf_{Q\in H_{n+k-1}}\left\|\frac{1}{x-b}-Q(x)\right\|_{[0,1]} \tag{17}$$

$$\geqq a\frac{b-1}{b^k-1}\inf_{Q\in H_{2n}}\left\|\frac{1}{x-b}-Q(x)\right\|_{[0,1]}.$$

But

$$\inf_{Q\in H_{2n}}\left\|\frac{1}{x-b}-Q(x)\right\|_{[0,1]}=\inf_{Q\in H_{2n}}\left\|\frac{2}{y+1-2b}-Q(y)\right\|_{[-1,1]}$$

$$=2\inf_{Q\in H_{2n}}\left\|\frac{1}{y-c}-Q(y)\right\|_{[-1,1]},$$

where $c=2b-1$. In [6] it is proved that

$$\inf_{Q\in H_{2n}}\left\|\frac{1}{y-c}-Q(y)\right\|_{[-1,1]}=\frac{1}{(c^2-1)(c+\sqrt{c^2-1}\,)^n}. \tag{18}$$

From (17) and (18) we obtain

$$\varepsilon_n^*(k)\geqq 2a\frac{b-1}{b^k-1}\cdot\frac{1}{[(2b-1)^2-1][2b-1+\sqrt{(2b-1)^2-1}]^n} \tag{19}$$

$$=\frac{a}{2(a-1)\,b}\cdot\frac{1}{[2b-1+\sqrt{4b^2-4b}]^n},$$

where $b=a^{1/k}$. Let $a=1+c_{21}$, $c_{21}>0$. Then $b=(1+c_{21})^{1/k}$.

But

$$(1+c_{21})^{1/k}-1=\frac{1}{k}c_{21}+\frac{(1/k-1)}{k.2!}c_{21}^2+\ldots+\frac{(1/k-1)\ldots(1/k-i+1)}{k.i!}c_{21}^i \tag{20}$$

$$=\frac{1}{k}c_{21}\left(1+\frac{1/k-1}{1!}c_{21}+\ldots+\frac{(1/k-1)\ldots(1/k-i+1)}{i!}c_{21}^{i-1}+\ldots\right)$$

$$\leq \frac{1}{k} c_{21}\Big(1+\frac{1/k-1}{1!}c_{21}+\ldots+\frac{(1/k-1)\ldots(1/k-i+1)}{(i-1)!}c_{21}^{i-1}+\ldots\Big)$$

$$=\frac{1}{k} c_{21}(1+c_{21})^{1/k-1}\leq \frac{c_{22}}{k},$$

where $c_{22}$ does not depend on $k$.

From (19) and (20) we get

$$\varepsilon_n^*(k)\geq \frac{a}{2(a-1)b}\Big[a^{1/k}+\frac{c_{22}}{k}+2\sqrt{\frac{a^{1/k}\cdot c_{22}}{k}}\Big]^{-n}$$

$$=\frac{a}{2(a-1)b}\Big[a^{1/k}\Big(1+\frac{c_{22}}{a^{1/k}}\cdot\frac{1}{k}+2\sqrt{\frac{c_{22}}{a^{1/k}}}\cdot\frac{1}{\sqrt{k}}\Big)\Big]^{-n}$$

$$\geq \frac{a}{2(a-1)b}\Big[\Big(1+\frac{c_{23}}{\sqrt{k}}\Big)^{\sqrt{k}}\Big]^{-n/\sqrt{k}} a^{-n/k}$$

$$\geq e^{-c_{24}n/\sqrt{k}}\cdot a^{-n/k}\geq e^{-c_{26}n/\sqrt{k}}.$$

Thus the lemma is proved.

Lemma 3. *There exists an absolute constant $c_{25}$, such that for every odd $k$, $1\leq k\leq n$*

$$\varepsilon_n^*(k)\geq (k/ne)^{c_{25}k}.$$

Proof. Since $x\leq 1$, $a>1$, then

$$\varepsilon_n^*(k)=\inf_{Q\in \tilde{H}_n}\Big\|\frac{x^k}{x^k-a}-Q(x)\Big\|_{[0,1]} \tag{21}$$

$$\geq \frac{1}{a-1}\inf_{Q\in\tilde{H}_n}\|x^k-Q(x)x^k+aQ(x)\|_{[0,1]}.$$

Let inf in (21) be reached for $Q^*\in\tilde{H}_n$.

It can be easily seen that

$$Q^*(0)\leq 1/2. \tag{22}$$

But then, since $Q\in\tilde{H}_n$, $k$ is odd, the polynomial $x^k-Q(x)x^k+aQ(x)$ is a polynomial of degree not greater than $2n$ and with a coefficient $1+\delta$ in front of $x^k$, where $-1/2\leq\delta\leq 1/2$.

But then, from (21) and (22)

$$\varepsilon_n^*(k)\geq \frac{1}{2(a-1)}\inf_{P\in H_{2n}^k}\|x^k-P(x)\|_{[0,1]}, \tag{23}$$

where $H_{2n}^k$ is the set of all polynomials from $H_{2n}$ with coefficient equal to 0 in front of $x^k$. Besides, from [7] it is known that $\inf_{P\in H_{2n}^k}\|x^k-P(x)\|_{[0,1]}$ $=1/A_k$, where

(24) $A_k = 2^{2k}\, 2n(2n+k-1)(2n+k-2)\ldots(2k+1)/(2n-k)! \leqq 1/(k/ne)^{c_{26}k}$.

From (23) and (24), we obtain $\varepsilon_n^*(k) \geqq (k/ne)^{c_{25}k}$.

Theorem 4. *If* $f^*(x) = |x|/(x-a)$, $a>1$, *then*

$$\varepsilon_{n,n}^*(f^*) \geqq \exp(-c_{27}\sqrt[3]{n^2 \ln n}).$$

Proof. We will consider $k$ to be odd.

(25)
$$\varepsilon_{n,n}^*(f^*) = \inf_{1\leqq k\leqq n}\ \inf_{Q\in H_n}\ \max_{|x|\leqq 1} |f(x^k) - Q(x)|$$
$$= \inf_{1\leqq k\leqq n}\ \inf_{Q\in \tilde{H}_n}\ \max_{0\leqq x\leqq 1} |f(x^k) - Q(x)|.$$

From Theorem 3 it follows that

(26)
$$\varepsilon_{n,n}^*(f^*) \leqq \exp[-c_{28}\sqrt[3]{n^2 \ln n}].$$

Then, by Lemma 2 and (26), it follows that the $k$ for which inf is reached in (25) must be such that $n/\sqrt{k} \geqq c_{29}\sqrt[3]{n^2 \ln n}$. Or,

(27)
$$k \leqq c_{30}(n/\ln n)^{2/3}.$$

But from Lemma 2 it follows that if $k$ satisfies (27), then

$$\varepsilon_{n,n}^*(f^*) \geqq [(\frac{n}{\ln n})^{2/3} \cdot \frac{1}{n}]^{c_{31}(n/\ln n)^{2/3}} \geqq (n^{-1/3})^{c_{32}(n/\ln n)^{2/3}}$$

$$= \exp[-c_{32}\ln n(n/\ln n)^{2/3}/3] = \exp(-c_{27}\sqrt[3]{n^2 \ln n}).$$

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