PLISKA
STUDIA MATHEMATICA
BULGARICA

ПЛИСКА
БЪЛГАРСКИ
МАТЕМАТИЧЕСКИ
СТУДИИ

SOME RESULTS OF CLASSICAL TYPE ABOUT GENERALIZED ANALYTIC FUNCTIONS

TOMA V. TONEV

The generalized analytic functions of Arens-Singer [1] on special groups are defined in another way, namely as the elements of uniform closure of inductive limit of polynomials on the subsets of points with unit modulus from Riemannian surfaces of functions $\sqrt[n]{z}$. We examine them from classical view point and obtain some generalizations of Schwarz's lemma, Rouché's theorem, Radó's theorem and others.

1. Introduction. Let $A(\Delta)$ be the algebra of continuous functions on the closed unit disc $\Delta=\{z\in \mathbf{C}^1 \mid |z|\leq 1\}$ which are analytic on its inner points. This is the intensively examined in the last years so-called disc algebra. In various situations in analysis one needs to deal with many-valued analytic functions on Δ instead with one-valued functions. For instance such is the theory of algebraically branch points of analytic functions, where arise Puiseux series [4], intensively applied recently in the theory of partial differential equations. Such is also the theory of Laplace transformation, where arise the so-called generalized power series [5]. The definitions of algebraic operations and of the norm in such sets of many-valued functions cause some troubles.

In this paper we propose a method avoiding these difficulties in some cases. It is useful for series, the denominators of rational powers of arguments of which are different (note that in the above examples the denominators were equal). It turns out that the obtained algebra is isomorphic to the algebra of generalized analytic functions of Arens-Singer [1], [3] on the compact group G, which character group $\widehat{G}$ is isomorphic to the group of rational numbers (point 2). Afterwards we show that with some modifications a part of classical results about disc algebra holds for the obtained algebra (points 4, 5).

2. A generalization of analytic functions. Let R be the group of rational numbers and R_+ — its nonnegative elements. By C_n we denote the subset of points with unit modulus from Riemannian surface of function $\sqrt[n]{z}$. We equip the set of natural numbers $\mathbf{N}$ with following partial ordering: $m\geq n$ iff m/n is an integer. Let $m\geq n$, i. e. let $1/m=1/n\cdot 1/k$. Then for every $z_m\in C_m$ the point $z_n=((z_m)^{1/k})^k\in C_n$ is uniquely defined. Thus arises a continuous projection π_n^m: $C_m\rightarrow C_n: z_m\rightarrow \pi_n^m(z_m)=((z_m)^{1/k})^k$. Denote by G the limit $\varprojlim_n \{C_m, \pi_n^m\}$ of obtained inverse system. Hence G is such a subset of $\prod_{n\in \mathbf{N}} C_n$, the elements $\{z_n\}_{n=1}^{\infty}$

of which satisfy the condition $\pi_n^m(z_m)=z_n$. Besause C_n are compact groups and π_n^m — continuous homomorphisms, then G is also a compact group. Let π_n be the projection $\pi_n\colon G\to C_n$, which corresponds to element $\zeta=\{z_n\}_1^\infty\in G$ its n-th co-ordinate. The continuous function $\chi_{1/n}(\zeta)=(\pi_n(\zeta))^{1/n}$ is a homomorphism from G to the unit circumference, i. e. $\chi_{1/n}$ is a character of G. Every positive and negative power of $\chi_{1/n}$ is also a character of G. Let $\chi_{m/n}$ denote the character $(\chi_{1/n})^m$. It is clear that the discrete subgroup Γ of $\widehat{G}$, generating by characters $\{\chi_{1/n}\}_{n=1}^\infty$ is isomorphic to the group of rational numbers R. Now from $G\subset\widehat{\Gamma}$ and from $\widehat{G}=\widehat{\widehat{\Gamma}}|_G\cong\Gamma|_G=\Gamma\cong\widehat{\widehat{\Gamma}}$ (see [6]) we have that $G\cong\widehat{\Gamma}$, according to the Pontrjagin duality theorem. Hence $\widehat{G}\cong R$ and $\widehat{G}_+=\{\chi_p,\ p\in R_+\}\cong R_+$.

Let $P(C_n)$ be the ring of restrictions of polynomials of argument $(z_n)^{1/n}$. It is obvious that $P(C_n)\subset P(C_m)$, if $m\geqq n$. Denote by $P(G)$ the limit $\varinjlim_n\{P(C_n),\ i_n^m\}$ of arising direct system $\{P(C_n),\ i_n^m\}$, where i_n^m is the corresponding imbedding $i_n^m\colon C(C_n)\to C(C_m)$. As a direct limit of rings, $P(G)$ can be supplied with a ring structure and every $P(C_n)$ is imbeddable in it in a natural way. Let's examine this ring more detaily. Let $z\in C_n$, $\zeta=\{z_n\}_{n=1}^\infty\in G$ and let's consider the character $\chi_{1/n}(\zeta)=(z_n)^{1/n}\in P(C_n)$. If $m\geqq n$ (i. e. if $m/n=k\in\mathbf{N}$) then $i_n^m(\chi_{1/n}(\zeta))$ $=i_n^m((z_n)^{1/n})=(\pi_n^m(z_m))^{1/n}=((z_m^{1/k})^k)^{1/n}=(z_m)^{k/k.n}=(z_m)^{k/m}=\chi_{k/m}(\zeta)\in P(C_m)$. Let now the equality $(i_n^m f)(z_m)=((z_m)^{1/m})^l=\chi_{l/m}(\zeta)$ is satisfied for some $f\in C(C_n)$, i. e. $f((z_m^{1/k})^k)=(z_m)^{\frac{1}{m}.l}=(z_m^{1/k})^{\frac{1}{n}.l}=(z_m^{1/k})^{k.\frac{1}{n}.\frac{l}{k}}=(z_n^{1/n})^{l/k}$, i. e. $f(z_n)=(\chi_{1/n}(\zeta))^{l/k}$. Now because f is one-valued, k necessarily devides l. But then $f(z_n)$ belongs to $P(C_n)$: if $l=s.k$ then $f(z_n)=(z_n^{1/n})^s=(\chi_{1/n}(\zeta))^s$. Consequently there arises a one-to-one correspondence from the subsemigroup $\widehat{G}_+=\{\chi_{m/n}\ m/n\in R_+\}$ of $\widehat{G}$ to the subset of these functions $\{\zeta\to(\pi_n\zeta)^{k/n}\}_{n,k=1}^\infty$, for which $k/n=\text{const}$. Namely the function $\{\zeta\to(\pi_n\zeta)^{k/n}\}_{n,\,k=1}^\infty$ corresponds to the character $\chi_{p/s}\in\widehat{G}$, where s is the smallest amongst all denominators of fractions k/n with $p/s=k/n$. By this $P(C_n)$ is mapped in a one-to-one way to the set of finite linear combinations of characters $\chi_{m/n}$, $m/n\in R_+$. Namely every fixed linear combination of elements from $\widehat{G}_+$ corresponds to some polynomial from $P(C_k)$, where k is the smallest common multiple of those denominators of character's indices, that take part in the combination. Let A_G be the uniform closure $\overline{P(G)}$ of algebra $P(G)$. It is clear, that A_G coincides with the algebra of uniform limits of finite linear combinations of characters from $\widehat{G}_+\cong R_+$, i. e. with the generalized analytic functions of Arens-Singer (see [1]).

Let's formulate the obtained results.

Theorem 1. *Let C_n be the set of points with unit modulus from Riemannian surface of function $\sqrt[n]{z}$; π_n^m — the projection $z_m\to(z_m^{1/k})^k\colon C_m\to C_n$, where m, n and k are natural numbers with $m=n.k$; $P(C_n)$ — the ring of polynomials of argument $(z_m)^{1/m}$, $z_m\in C_m$; i_n^m — the natural imbedding of $P(C_n)$ into $P(C_m)$. The uniform closure of algebra $\varinjlim_n\{P(C_n),\ i_n^m\}$ coincides* *with the algebra of generalized analytic functions in Arens-Singer sence on the compact group G with character group $\widehat{G}$ algebraically isomorphic to the group of rational numbers. The group G represents as* $\lim\limits_{\overleftarrow{n}} \{C_n, \pi_n^m\}$.

Theorem 2. *There exists a uniform algebra extension of disc algebra $A(\Delta)$, in which the functions $z \to z^n$ have arbitrary natural roots.*

Namely, this is the algebra A_G, in which $A(\Delta)$ is imbeddable (via $z^n \to \chi_n$) isometrically and homomorphic. The isometric part is an immediate consequence from equalities:

$$\sup_{\zeta \in G} \left| \sum_v c_v \chi_{n_v}(\zeta) \right| = \sup_{\zeta \in G} \left| \sum_v c_v ((\pi_1(\zeta))^1)^{n_v} \right| = \sup_{z \in S^1} \left| \sum_v c_v z^{n_v} \right|.$$

Since $(\chi_{m/n})^n = \chi_m$, $\chi_{m/n} \in A$ is an n-th root of χ_m.

3. Branch points of analytic functions. Let Γ be the real line, equipped with discrete topology, and Γ_+ — its positive semiaxis. Let $G = \widehat{\Gamma}$. By Δ_G we denote the factor-space $G \times [0, 1]/G \times \{0\}$, and by $*$ — the image of $G \times \{0\} \subset G \times [0, 1]$ in Δ_G. Let's define the functions $\widetilde{\chi}_p$ on Δ_G in the following way: $\widetilde{\chi}_p(g, \lambda) = \lambda^p \chi_p(g)$, $p \neq 0$, $(g, \lambda) \neq *$; $\widetilde{\chi}_p(*) = 0$, $p \neq 0$ and $\widetilde{\chi}_0 \equiv 1$ $(\chi_0 \equiv 1)$. The algebra A_G of generalized analytic functions (in Arens-Singer sense) on G is the uniform closure of finite linear combinations of characters χ_a, $a \in \Gamma_+$ from $\widehat{G}$. The spectrum (=maximal ideal space) sp A_G of A_G coincides with Δ_G and the Gelfand image $\widehat{\chi}_a$ of character χ_a, $a \in \Gamma_+$ — with the function $\widetilde{\chi}_a$ (see e. g. [3]). The set $\Delta_G(\varepsilon) = \{(g, \lambda) \in \Delta_G \mid \lambda \leqq \varepsilon\}$ appears naturally in the examining of branch points of analytic functions. Let f be an analytic function on the punctured unit disc $\Delta^* = \Delta \setminus \{0\}$. We consider that f has not any other singularities in Δ. Then f is representable as a Laurent series with respect to the powers of z. Let's correspond the function $F(g, \lambda) = f(\widetilde{\chi}_1(g, \lambda))$ to f. Obviously $F \in C(\Delta_G^*)$. F is representable as a "Laurent series" with respect to the integer powers of $\widetilde{\chi}_1$ and the correspondence $f \to F$ is an isometrical homomorphism. Let f does not be a one-sheeted function on Δ^*, i. e. let $f \notin C(\Delta^*)$, or equivalently, $F \notin C(\Delta_G^*) = C(\Delta_G \setminus \{*\})$. One from both possibilities below may occur: a) There exists such $n > 0$, that the function $f(\chi_{1/n}(g))$ is constant over all co-sets of Ker $\chi_{1/n}$ in G; b) There is not such a number $n > 0$. In the first case we shall correspond to f the function Φ: $(g, \lambda) \to f(\widetilde{\chi}_{1/n}(g, \lambda))$, which is one-valued on $\Delta_G^* = \Delta_G \setminus \{*\}$, because of certain standard analytical considerations (e. g. [4]), and hence is representable as an absotutely convergent "Laurent series": $\Phi(g, \lambda) = \Sigma_{-\infty}^{\infty} c_m (\widetilde{\chi}_{1/n})^m (g, \lambda) = \Sigma_{-\infty}^{\infty} c_m \widetilde{\chi}_{m/n}(g, \lambda)$. Hence $\Phi \in A_G(\Delta_G^*)$. Note that since $(\widetilde{\chi}_{m/n})^n = (\widetilde{\chi}_1)^m$ we may formally write that $\widetilde{\chi}_{m/n} = \widetilde{\chi}_1^{m/n}$, from where $\Phi(g, \lambda) = \Sigma_{-\infty}^{\infty} c_m \widetilde{\chi}_1^{m/n}(g)$. This is exactly the case when f has a branch point in 0 from $(n-1)$-th order. In the case b) f has a logarithmic branch point, independently from the existence or nonexistence of such an $a \in \Gamma_+$, with $f(\chi_a(g))$ constant over all co-sets of Ker χ_a in G. Note that if the first possibility occurs, then the one-valued function $\Phi(g, \lambda) = f(\chi_a(g, \lambda))$ is representable as a Laurent series with respect to the powers of $\widetilde{\chi}_a$: $\Phi(g, \lambda) = \Sigma_{-\infty}^{\infty} c_m \widetilde{\chi}_a^m(g, \lambda) = \Sigma_{-\infty}^{\infty} c_m \widetilde{\chi}_{ma}(g, \lambda)$, hence $\Phi \in A_G(\Delta_G^*)$. If $m. a$ is not a rational number we cannot write $\widetilde{\chi}_{ma} = \widetilde{\chi}_1^{ma}$ because Γ

is discrete. In the second possibility such representation is impossible and it is not clear whether we can correspond any generalized analytic function to f or not.

4. Some results of classical type. Let Γ now be an additive and dense subgroup of real line $\mathbf{R}^1$. We equip Γ with discrete topology and assume that $G=\widehat{\Gamma}$ is the character group of Γ. Let $\Gamma_+=\Gamma\cap\mathbf{R}^1_+$. Suppose that $f\in A_G$ and that $P_f=\{\{P_n^\alpha\}_{\alpha\in\Sigma_\beta}\}_{\beta\in I}$ is the set of all sequences of finite linear combinations P_α of characters χ_a from Γ_+, that uniformly converge to f. Let a_α denote the smallest amongst all indices of χ_a, taking part in $\{P_n^\alpha\}_\alpha$ and let $a_f=\lim\sup\{a_\alpha \mid \alpha\in\Sigma_\beta,\ \beta\in I\}$. The following theorem is a generalization of classical Schwarz's lemma.

Theorem 3. *Let f be an element from A_G, for which $\max_G|f|\leq 1$ and $\widehat{f}(*)=0$. If $a_f>0$, then for each $(g,\lambda)\in\Delta_G$ we have $|\widehat{f}(g,\lambda)|\leq\lambda^{a_f}$.*

Proof. Since $f(*)=0$ we may assume that the character $\chi_0\equiv 1$ does not take part in P_α, $\alpha\in\Sigma_\beta$, $\beta\in I$. Let $0<a<a_f$, $a\in\Gamma_+$. Then we may assume that the indices a_ν of characters taking part in P_α are $\geq a$. But then the function $f(g)/\chi_a(g)$ will belong to A_G, and hence for $(g,\lambda)\in\Delta_G$ we'll have: $|\widehat{f}(g,\lambda)/\widehat{\chi_a}(g,\lambda)|\leq\max_G|f(g)/\chi_a(g)|=\max_G|f(g)|\leq 1$, and $|\widehat{f}(g,\lambda)|\leq|\widehat{\chi_a}(g,\lambda)|=|\lambda^a\chi_a(g)|=\lambda^a$ (here we applied the fact that $\widehat{\chi_a}(g,\lambda)=\lambda^a\chi_a(g)$). Now $|\widehat{f}(g,\lambda)|\leq\lambda^{a_f}$ because the last inequality holds for every $a<a_f$. The theorem is proved.

In [7] I. Glicksberg proved the following general Schwarz's lemma for uniform algebras A: If f and g are two elements from A with f/g bounded on $\operatorname{sp}A\setminus g^{-1}(0)$, then $\sup|f/g(\operatorname{sp}A\setminus g^{-1}(0))|=\sup|f/g(\partial A\setminus g^{-1}(0))|$. We could use that result to obtain the inequality $|\widehat{f}/\widehat{\chi_a}|_{\Delta_G}\leq\|f/\chi_a\|_G$ in the proof of theorem 3, if this inequality was not an immediate consequence of the obvious fact that G is a boundary of A_G.

Let f and g be two elements of uniform algebra A. There is a generalization of classical Rouché's theorem for uniform algebras. Namely if the inequality $|f+g|<|f|+|g|$ holds on the Šilov boundary ∂A of A, then f and g are simultaneously invertible or not [8]. In the situation of algebra A_G the non-invertible case of this result can be precised in a way similar to the classical one. We call a function $f\in A_G$ differentiable in the point $g_0\in G$ with respect to the character $\chi_a\in\Gamma_+$, if the differential ratio $(f(g)-f(g_0))/(\chi_a(g)-\chi_a(g_0))$ is convergent when $\chi_a(g)\to\chi_a(g_0)$. As usual the derivative $f'_{\chi a}$ of f with respect to χ_a we call the existed limit.

Theorem 4. *Let $f=\chi_a^m\varphi$ and $h=\chi_a^n\psi$ are two elements of A_G, where φ and ψ are invertible and differentiable with respect to χ_a elements of A_G, which derivatives are also in A_G. Then $m=n$ if $|f+h|<|f|+|h|$ on G.*

Proof. Following Glicksberg [8], we construct the integral

$$J(t)=\int\limits_G\frac{(fh^{-1}+t)'}{fh^{-1}+t}\chi_a(g)\,d\sigma(g),$$

where $d\sigma$ is the Haar measure on G. $J(t)$ tends to zero when $t\to\infty$ and

$$J(0)=\int\limits_G\left(\frac{f'}{f}-\frac{h'}{h}\right)\chi_a(g)\,d\sigma=\int\limits_G\left(m+\frac{\varphi'}{\varphi}\chi_a-n-\frac{\psi'}{\psi}\chi_a\right)d\sigma=m-n.$$

The theorem is proved.

Theorem 5. *Let* $f \neq 0$ *be differentiable with respect to* $\chi_a \in \Gamma_+$ *element of* A_G, *which derivative* f'_{χ_a} *is also in* A_G. *If* f *takes the form* $f = (\chi_a - \lambda_1^a \chi_a (g_1))^{m_1} (\chi_a - \lambda_2^a \chi_a(g_2))^{m_2} \dots (\chi_a - \lambda_k^a \chi_a(g_k))^{m_k} . \varphi(g)$, *where* $m_i \geqq 1$, $(g_i, \lambda_i) \in \Delta_G$ *and* $\varphi \in A_G^{-1}$, *then for each* $1 \leqq i_0 \leqq k$

$$m_{i0} \prod_{\substack{s=1 \\ s \neq i_0}}^{k} (\lambda_{i_0}^a \chi_a(g_{i_0}) - \lambda_s^a \chi_a(g_s)) = \int_G f'(g)/f(g) . (\chi_a - \lambda_1^a \chi_a(g_1)^{m_1} \dots (\chi_a - \lambda_k^a \chi_a(g_k))^{m_k} d\mu_{i_0},$$

where μ_{i_0} *is the representing measure on* G *of the point* $(\lambda_{i_0}, g_{i_0}) \in \Delta_G$.

Proof. The function under the sign of integral is representable as: $f'/f . (\chi_a - \lambda_1 \chi_a(g_1)) \dots (\chi_a - \lambda_k \chi_a(g_k)) = m_1 (\chi_a - \lambda_2^a \chi_a(g_2)) \dots (\chi_a - \lambda_k^a \chi_a(g_k)) + m_2 (\chi_a - \lambda_1^a \chi_a(g_1)) (\chi_a - \lambda_3^a \chi_a(g_3)) \dots (\chi_a - \lambda_k^a \chi_a(g_k)) + \dots + m_x (\chi_a - \lambda_1^a \chi_a(g_1)) \dots (\chi_a - \lambda_{k-1}^a \chi_a(g_{k-1})) + \frac{\varphi'}{\varphi} (\chi_a - \lambda_1^a \chi_a(g_1) \dots (\chi_a - \lambda_k^a \chi_a(g_k)) \in A_G$. The desired result can be obtained by integrating with respect to the measure μ_{i_0}.

5. Functional analytic properties of generalized analytic functions. Let Γ be a dense additive subgroup of $\mathbf{R}^1$, equipped with discrete topology as in point 4. We mentioned above, that sp $A = G \times [0, 1]/G \times \{0\}$. There are known many other facts about algebra A_G. For instance: a) Šilov boundary of A_G coincides with group G (see [12]); b) A_G is a Dirichlet algebra (i. e. Re A_G is dense in $C_R(G)$) [12]; c) A_G is an antisymmetric algebra (i. e. every real function of A_G is constant on G); d) A_G is analytic on Δ_G (i. e. every f in A_G, with $\hat{f}$ vanishing on a nonempty open subset of Δ_G vanishes identically), see [10]; e). The locally maximum modulus principle holds for A_G (i. e. for every $\varphi \in \Delta_G \setminus G$ there exists a compact neighbourhood $U \subset \Delta_G \setminus G$, for which $|f| \leqq \sup |f(bU)|$ on U for every $f \in A$). As a consequence of that fact in A_G hold the Glicksberg's results about generalization of the classical Phragmén-Lindelöf theorem for uniform algebras; f) A_G is a maximal algebra of $C(G)$ (i. e. every uniform algebra B on G with $A_G \subset B \subset C(G)$ coincides either with A_G or with $C(G)$). The proofs of that fact are not very short [2, 3], and therefore we give a shorter one, modelling the proof of Wermer's maximality theorem for disc algebra $A(\Delta)$.

Theorem 6 (Hoffman, Singer, [2]). *If the group* Γ *is algebraically isomorphic to some subgroup of real numbers, then algebra* A_G *of generalized analytic functions on the group* $G = \hat{\Gamma}_d$ *is a maximal subalgebra of* $C(G)$.

Proof. Let B be a uniform algebra on G with $A_G \subset B \subset C(G)$. Let $\varphi(\chi_a) \neq 0$ holds for every linear multiplicative functional $\varphi \in$ sp B and for every character $\chi_a \in \Gamma_+$. This means that all characters of G are invertible in B, i. e. that B contains the whole Γ, since $1/\chi_a = \overline{\chi}_a = \chi_{-a}$. According to the Weierstrass-Stone's theorem, the generated by Γ algebra is uniformly dense in $C(G)$. In this case $B = C(G)$. In the other case there exists a linear multiplicative functional $\varphi \in$ sp B and a character $\chi_a \in \Gamma_+$, such that $\varphi(\chi_a) = 0$. Let $\chi_b \in \Gamma_+$, $b \neq a$, $b \neq 0$, is another character of Γ. Let for the integer m holds $bm - a \geqq 0$. Then $\chi_b^m = \chi_{bm} = \chi_a . \chi_{bm-a}$ and $\varphi(\chi_b^m) = \varphi(\chi_a) . \varphi(\chi_{bm-a}) = 0$, i. e. $\varphi(\chi_b)^m = 0$ simultaneously with $\varphi(\chi_a)$. Thus, if $\varphi(\chi_a) = 0$ for some $\chi_a \in \Gamma_+$, then φ is identically zero on $\Gamma_+ \setminus \{0\}$. Since $\varphi(\chi_0) = \varphi(1) = 1$ the restriction of φ on the algebra A_G

corresponds to every function $\widehat{f}$ its meaning in the point $*$. Every representing measure of φ on G will present the homomorphism "meaning in the point $*$" of algebra A_G. Because A_G is a Dirichlet algebra, the Haar measure $d\sigma$ on G is the uniquely representing measure of $\varphi|_{A_G}$ on G. Consequently every representing measure of φ on G will coincide with $d\sigma$. Hence the Haar measure will be multiplicative on the algebra B. Let $f \in B$ and let $a>0$. Then $\int_G \chi_a(g) f(g) d\sigma(g) = (\int_G \chi_a(g) d\sigma(g)) \cdot (\int_G f(g) d\sigma(g)) = 0$. As a continuous function f is uniformly approximable by finite linear combinations P_α of characters of G. Then if $a>0$, $0 = \int_G \chi_a f\, d\sigma = \lim_\alpha \int_G \chi_a P_\alpha d\sigma$. If the character $\chi_{-a} = \overline{\chi_a}$ takes part in polynomials P_α with coefficient c_α, then $c_\alpha \to 0$ since $\int_G \chi_a P_\alpha d\sigma = c_\alpha \int_G \chi_a \overline{\chi_a} d\sigma = c_\alpha$, i. e. we may drop all negative indexed characters from P_α without breaking the convergence to f. Censequently $f \in A_G$ and in this case $B = A_G$. The theorem is proved.

Remember that a uniform algebra is called analytically closed in $C(\mathrm{sp}\, A)$ iff every continuous function on $\mathrm{sp}\, A$ is in A, if it satisfies some equality $F(f_1, \ldots, f_n, f) = 0$ on $\mathrm{sp}\, A$, where $f_i \in A$ and F is an analytic function in some neighbourhood of the range of the map $\varrho : x \to (\widehat{f_1}(x), \ldots, \widehat{f_n}(x), f(x)) \subset \mathbf{C}^{n+1}$ with $[(\partial/\partial z_{n+1})^k F]\, \varrho(x) \neq 0$ for some $k \geqq 1$ [7]. As shown in [7], if A is a maximal subalgebra of $C(\mathrm{sp}\, A)$ having its Šilov boundary proper in $\mathrm{sp}\, A$, and if A is analytic on $\mathrm{sp}\, A$, then A is analytically closed in $C(\mathrm{sp}\, A)$. Applying this to the maximal analytic algebra A_G with $\mathrm{sp}\, A_G = \Delta_G \neq G = \partial A_G$, we obtain the following:

Corollary. *The algebra A_G is analytically closed.*

It is known [3], that there exists an imbedding j of $\mathbf{C}' = \{z \,|\, \mathrm{Im} z \geqq 0\}$ as a dense subset in Δ_G, such that for every $f \in A_G$, $f \circ j^{-1}$ is an analytic function on $\mathbf{C}'$. If $f(\lambda, g) = \Sigma\, c_k \widetilde{\chi}_{p_k}(\lambda, g)$, then $f \circ j^{-1}(x+it) = \Sigma\, c_k e^{ip_k(x+it)}$, where $p_k \geqq 0$, $j(x) = g$, and $t = -\ln \lambda$. Also if $\varepsilon > 0$ and λ_0 is such that $|f(\lambda, g) - f(*)| < \varepsilon$ for $0 \leqq \lambda < \lambda_0$, then $|f \circ j^{-1} - f(*)| < \varepsilon$ for $t > t_0 = -\ln \lambda_0$.

Theorem 7. *The algebra A_G is isometrically isomorphic to these continuous functions f on Δ_G, for which $f \circ j^{-1}$ is an analytic on $\mathbf{C}'$ function.*

Proof. Let B be the uniform algebra on Δ_G, described in the theorem. Because B contains all the functions $\widetilde{\chi}_p$, $p \geqq 0$, we have that $C(G) \supset B \supset A_G$. Obviously $B \neq C(G)$. From the maximality of algebra A_G, we obtain that $B = A_G$, Q. E. D.

Let K be a compact subset of Δ_G. We define the algebra $R_G(K)$ as the uniform closure in $C(K)$ of ratios of finite linear combinations of restrictions on K of extensions $\widetilde{\chi}_a$, $a \in \Gamma_+$. A well known theorem of Radó says (see e. g. [3]) that if a continuous function f on the closed unit disc is analytic on the subset of inner points of the set, where it does not vanish, then f is analytic on the whole open unit disc. The following is an extension for algebra $A_G = A_G(\Delta_G)$ of Radó's theorem.

Theorem 8. *If the continuous function f belongs to the algebra $R_G(\Delta_G \setminus \mathrm{int}\, f^{-1}(0))$, then f is a generalized analytic function on Δ_G.*

Proof. Let F be the function $F(x+it) = f(\lambda, g)$, where $j(x) = g$, and $t = -\ln \lambda$. If $j(\mathbf{C}') \cap Z(f) = \emptyset$, then F is analytic on $\mathbf{C}'$, i. e. $f \in A_G$ (theorem 7).

Let $K'=j(\mathbf{C}')\cap Z(f)\neq\emptyset$. Then F is analytic on $\mathbf{C}'\setminus K'$, and $F\underset{K'}{\equiv}0$. Now according to the classical theorem of Radó, F is analytic on $\mathbf{C}'$. We obtained that $f\in C(\Delta_G)$ and that $f\circ j^{-1}$ is analytic on $\mathbf{C}'$, i. e. that $f\in A_G$. The theorem is proved.

REFERENCES

1. R. Arens, I. M. Singer. Generalized analytic functions. *Trans. Amer. Math. Soc.*, **81**, 1956, 379-393.
2. K. H. Hoffman, I. M. Singer. Maximal algebras of continuous functions. *Acta Math.*, **103**, 1960, 217-241.
3. T. W. Gamelin. Uniform algebras. Engl. Cliffs, 1969.
4. А. И. Маркушевич. Теория аналитических функций. Москва, 1968.
5. В. И. Крылов, Н. С. Скобля. Методы приближенного преобразования Фурье и обращения преобразования Лапласа. Москва, 1974.
6. W. Rudin. Fourier analysis on groups. New York, 1962.
7. I. Glicksberg. Maximal algebras and a theorem of Radó. *Pacif. J. Math.*, 14, 1964, 919-941.
8. I. Glicksberg. A remark on Rouché's theorem. *Amer. Math. Monthly*, **83**, 1976, 186-187.
9. I. Glicksberg. A Phragmén-Lindelöf theorem for function algebras. *Pacif. J. Math.*, **22**, 1967, 401-406.
10. J. Wermer. Banach algebras and analytic functions. *Advances in Mathematics*, 1, 1961.
11. T. V. Tonev. Algebras of generalized analytic functions. *Banach Center Publ.* ,8. (to appear).
12. Д. Л. Ламбов, Т. В. Тонев. Некоторые функционально-алгебраические свойства алгебры обобщенных аналитических функций. *Доклады БАН*, **31**, 1978, 803—806.

Centre for Mathematics and Mechanics
1090 Sofia P. O. Box 373

Received 24. 10. 1977