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CONVERGENCE OF SEQUENCES OF CARATHÉODORY OR KOBAYASHI PSEUDOMETRICS

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In this paper are given some conditions under which, if a sequence of complex spaces $\{M_j\}$ tends to a complex space M in a suitable way, then the Carathéodory or Kobayashi pseudometric on M is a limit of the corresponding pseudometrics of M_j for $j\to\infty$.

This paper is an expanded version of our note [3]. Here we shall give complete proofs and more detailed discussions on the convergence of sequences of Carathéodory or Kobayashi pseudometrics.

1. Definitions and general remarks. On every connected complex (analytic) space one can introduce [2; 4; 5; 6] the Carathéodory and Kobayashi pseudometrics. The definitions use the well known Poincaré-Bergman metric of the unit disk $D:=\{z\in\mathbf{C}:|z|<1\}$ in the complex plane $\mathbf{C}$. The distance between two arbitrary points a and b of D in the Poincaré-Bergman metric ϱ is defined by

$$\varrho(a, b):=\frac{1}{2}\log\frac{|1-\bar{a}b|+|b-a|}{|1-\bar{a}b|-|b-a|}.$$

If M and N are complex spaces then, as usually, with $H(M, N)$ we denote the family of all holomorphic maps $f:M\to N$. Moreover, for short we shall write only "complex space" instead of "connected complex space".

Now let M be a complex space. The Carathéodory pseudometric c_M of M is defined as follows: The distance between two arbitrary points p and q of M is the number

$$c_M(p, q):=\sup_f \varrho(f(p), f(q)),\quad f\in H(M, D),$$

where the supremum is taken with respect to all holomorphic maps $f:M\to D$.

The definition of the Kobayashi pseudometric k_M of M is in a manner "dual" to that of c_M because it uses $H(D, M)$ instead of $H(M, D)$ and infimum instead of supremum. First we define a chain $\sigma:=\{a_j, b_j; f_j\}_{j=1}^k$ from p to q in M, where k is a natural number, a_j and b_j, $j=1,\dots,k$, are points of the unit disk D and f_j, $j=1,\dots,k$, are holomorphic maps of $H(D, M)$ such that $f_1(a_1)=p$; $f_j(b_j)=f_{j+1}(a_{j+1})$, $j=1,\dots,k-1$; $f_k(b_k)=q$. The length of the chain σ is given by

$$L(\sigma):=\sum_{j=1}^{k}\varrho(a_j, b_j).$$

If $V_M(p, q)$ denotes the set of all chains from p to q in M then the distance between p and q in the Kobayashi pseudometric is defined to be the number $k_M(p, q) := \inf_\sigma L(\sigma)$, $\sigma \in V_M(p, q)$, where the infimum is taken with respect to all chains $\sigma \in V_M(p, q)$.

Note that the Carathéodory and Kobayashi pseudometrics are generalisations of the Poincaré-Bergman metric since $c_D = k_D = \varrho$. Moreover, the holomorphic maps are distance-decreasing with respect to these pseudometrics, namely, for $f \in H(M, N)$ and $p, q \in M$ one has $c_M(p, q) \geq c_N(f(p), f(q))$, $k_M(p, q) \geq k_N(f(p), f(q))$.

Some of the theorems for convergence which we shall prove in this paper use the notion "kernel" of a sequence of complex spaces. The definition is the following:

Let $\{M_j\}_{j=1}^{\infty}$ be a sequence of complex spaces with the same dimension such that the interior of the intersection $\cap_{j=1}^{\infty} M_j \neq \emptyset$ is open in all M_j. Furthermore, let every two spaces of the sequence have compatible structures in their intersection. The set

$$M := \{p \in \bigcup_{j=1}^{\infty} M_j : \exists U_p \ni p, \exists j_p : \forall j > j_p : U_p \text{ is open in } M_j\}$$

we call kernel of the sequence $\{M_j\}$, i. e. the kernel M is the maximal subset of the union $\bigcup_{j=1}^{\infty} M_j$ such that every point of M (equivalently every compact subset of M) has a neighbourhood containing in all M_j with sufficiently large indices j. Obviously the kernel has in a natural manner a structure of a complex space.

Here we shall establish in some sense a "continuity" of the Carathéodory and Kobayashi pseudometrics with respect to the spaces on which they are defined. More precisely, we shall give conditions under which if a sequence $\{M_j\}$ tends in some way to M, then it follows that the sequences $\{c_{M_j}(p, q)\}$ and $\{k_{M_j}(p, q)\}$, $p, q \in M$, tend to $c_M(p, q)$ and $k_M(p, q)$ respectively.

For each of the two pseudometrics we shall prove two different theorems in the two cases when all members of the sequence $\{M_j\}$ are in M and when M is in all M_j because the methods of proof differ. The general case in which $M_j \urcorner\subset M$ and $M \urcorner\subset M_j$ for some subsequence of indices j can be reduced to these two cases by use of the inclusions $M_j \cap M \subset M_j \subset M_j \cup M$, $M_j \cap M \subset M \subset M_j \cup M$.

2. Limits of Carathéodory pseudometrics. We shall proceed with the study of the easiest case: exhausting a space M with a sequence $\{M_j\}$ whose members are in M.

Theorem 1. *Let $\{M_j\}$ be a sequence of complex spaces with kernel M and let M contains all M_j. If p and q are two arbitrary points of M then we assert that $\lim_{j\to\infty} c_{M_j}(p, q)$ exists and is equal to $c_M(p, q)$.*

Proof. Let us note first that by the definition of the kernel M, two fixed points p and q of M are also in M_j for all indices $j \geqq j_0$ (j_0 depends on p and q). Then $c_{M_j}(p, q)$ $(j \geqq j_0)$ is correctly defined and we may look for the limit of the sequence $\{c_{M_j}(p, q)\}_{j=j_0}^{\infty}$. We fix two arbitrary points p and q of M and, therefore, further we shall not write and precise the index j_0. (This shows also that the use of the notion "kernel" is in some sense necessary in such a case when we want to formulate such a theorem for convergence for arbitrary points of M.)

For every j the identity of $M_j \subset M$, $\mathrm{id}_{M_j}: M_j \to M$ is holomorphic map hence distance-decreasing with respect to the Carathéodory pseudometric, i. e. $c_{M_j}(p, q) \geq c_M(p, q)$. This shows that the sequence $\{c_{M_j}(p, q)\}$ is bounded from below by $c_M(p, q)$. If we set

$$d_M(p, q) := \limsup_{j\to\infty} c_{M_j}(p, q)$$

it is sufficient to show that $d_M(p, q) \leq c_M(p, q)$ and then the inequalities

$$c_M(p, q) \leq \liminf_{j\to\infty} c_{M_j}(p, q) \leq \limsup_{j\to\infty} c_{M_j}(p, q) = d_M(p, q) \leq c_M(p, q)$$

imply that the sequence $\{c_{M_j}(p, q)\}$ tends to $c_M(p, q)$.

We assume the contrary and then there exists a positive ε such that $d_M(p, q) - \varepsilon > c_M(p, q)$. By the definition of $d_M(p, q)$ we have for some subsequence $\{j_\nu\}_{\nu=1}^{\infty}$ of indices

$$\lim_{\nu\to\infty} c_{M_{j_\nu}}(p, q) = d_M(p, q)$$

and, therefore, for all sufficiently large $\nu(\nu > \nu_0)$ holds $c_{M_{j_\nu}}(p, q) > d_M(p, q) - \varepsilon$. Then from the definition of $c_{M_{j_\nu}}$ for such indices ν

$$c_{M_{j_\nu}}(p, q) := \sup_f \varrho(f(p), f(q)),\ f \in H(M_{j_\nu}, D)$$

it follows that there are holomorphic maps $f_\nu : M_{j_\nu} \to D$ with the property $\varrho(f_\nu(p), f_\nu(q)) > d_M(p, q) - \varepsilon$.

Here we want to use the principle of compactness for the sequence $\{f_\nu\}$ and further to take the limit (for the corresponding subsequence of indices ν) in the above inequality. But the maps f_ν are not defined on the whole M, therefore we need the following modification of the principle of compactness whose method of proof is as in [5, ch. V, theorem 3.1.].

Lemma. *Let $\{M_\nu\}$ be a sequence of complex spaces with kernel M and let M contains all M_ν. If $\{f_\nu\}$ is a sequence of holomorphic maps $f_\nu : M_\nu \to D$ with $f_\nu(r) = 0$ for some point r of M then we can choose a subsequence $\{f_{\nu_k}\}$ tending uniformly on every compact subset of M to a holomorphic map $f_0 : M \to D$, $f_0(r) = 0$.*

We shall not give here the proof of the lemma but we adduce some remarks about the needing modification when complete this proof.

In our case the maps of the sequence $\{f_\nu\}$ have not the property $f_\nu(r) = 0$ for some $r \in M$ supposing in the lemma. But it is sufficient to compose f_ν with automorphisms g_ν of the unit disk D such that $g_\nu(f_\nu(q)) = 0$, i. e.

$$g_\nu(z) := [z - f_\nu(q)]/[1 - \overline{f_\nu(q)} z]$$

and the sequence $\{\tilde{f}_\nu := g_\nu \circ f_\nu\}$ has the property $\tilde{f}_\nu(q) = 0$, i. e. we obtain that our fixed point q plays the role of the point r from the lemma. Furthermore, since the Poincaré-Bergman metric is invariant with respect to the automorphisms of D we have $\varrho(\tilde{f}_\nu(p), \tilde{f}_\nu(q))=\varrho(f_\nu(p), f_\nu(q))$. Therefore, we may suppose that the initial maps f_ν satisfy the assumptions of the lemma. Then there is a subsequence $\{f_{\nu_k}\}$ which tends to a holomorphic map $f_0: M \to D$ and

$$\varrho(f_0(p), f_0(q))= \lim_{k\to\infty} \varrho(f_{\nu_k}(p), \quad f_{\nu_k}(q)) \geqq d_M(p, q)-\varepsilon > c_M(p, q).$$

But $f_0 \in H(M, D)$ must participate when we take the supremum in the definition of $c_M(p, q)$, hence

$$\varrho(f_0(p), f_0(q)) > c_M(p, q) := \sup_f \varrho(f(p), f(q)) \geqq \varrho(f_0(p), f_0(q)),$$

which is a contradiction and the theorem is proved.

Remarks to the lemma. The difficulties in the proof of this lemma come from the fact that the maps f_ν are defined in different subspaces of M. Here we construct a new sequence $\{\tilde{M}_\nu\}$ of complex spaces with the same kernel M which has also the following two properties essentially used in the proof:

1) $\{\tilde{M}_\nu\}$ is increasing, i. e. $\tilde{M}_\nu \subset \tilde{M}_{\nu+1}$ for every ν;

2) $M_\nu \supset \tilde{M}_\nu$ for every ν.

Such a sequence we obtain by setting $\tilde{M}_\nu := \cap_{\mu=\nu}^{\infty} M_\mu$. Following the method of the Kobayashi's proof one must carefully do each step because almost everywhere one has to use the Carathéodory pseudometrics of the new spaces $\tilde{M}_\nu$ and the initial spaces M_ν simultaneously.

We can prove the lemma by more general assumptions, for example, without the condition $M_\nu \subset M$, furthermore the image $f_\nu(r)$ of the point r can be arbitrary point of some fixed compact subset of M instead of to be the origin. Finally, the notion "kernel" can be defined for sequences of topological spaces in the same manner and we can prove such a proposition as the lemma for distance-decreasing maps between topological spaces with pseudometrics. Then one obtains a principle of compactness which is a generalization of [5, ch. V, theorem 3.1.] for maps defined in different spaces.

As an important particular case of theorem 1 we have

Corollary. *Let $\{M_j\}$ be increasing sequence of complex spaces with compatible structures and let M be the union $\cup_{j=1}^{\infty} M_j$. Then for every two points p and q of M the limit $\lim_{j\to\infty} c_{M_j}(p, q)$ exists and is equal to $c_M(p, q)$.*

Proof. Obviously M has a structure of complex space and coincides with the kernel of the sequence $\{M_j\}$. Thus, the assumptions of the theorem are fulfilled.

The following theorem is a sufficient condition for coincidence of c_M and $\lim_{j\to\infty} c_{M_j}$ in the case when M contains in all M_j. It seems that this case is quite more difficult to study than the previous one.

Theorem 2. *Let $\{M_j\}$ be a sequence of complex spaces containing the complex space M and let the spaces be with compatible structures in their intersections. Moreover, let every holomorphic map $f: M \to D$ can be represented as a limit of some sequence $\{f_j\}$ of holomorphic maps $f_j: M_j \to D$. Then if $\lim_{j\to\infty} c_{M_j}(p, q)$ exists for every two points p and q of M we assert that it is equal to $c_M(p, q)$.*

Proof. If we set

$$d_M(p,q):=\lim_{j\to\infty} c_{M_j}(p,q), \qquad p,\ q\in M$$

then it is sufficient to show that $d_M(p,q)=c_M(p,q)$.

Since $M\subset M_j$ for every j, the identity of M, $\mathrm{id}_M: M\to M_j$ is holomorphic map hence distance-decreasing with respect to the Carathéodory pseudometric, i. e. $c_M(p,q)\geqq c_{M_j}(p,q)$, $\forall j$. If we fix two arbitrary points p and q of M and take the limit for $j\to\infty$ in this inequality then we obtain $d_M(p,q)\leqq c_M(p,q)$. To show the inequality $d_M(p,q)\geqq c_M(p,q)$ and, thus, to complete the proof we use the fact that the Carathéodory pseudometric is characterized by the following property [6]:

If d is a pseudometric on M with

$$(*) \qquad d(p,q)\geqq \varrho(f(p),f(q)), \qquad \forall p,\ q\in M, \quad \forall f\in H(M,D)$$

then $d(p,q)\geqq c_M(p,q)$, i. e. c_M is the smallest pseudometric on M which satisfies $(*)$.

Therefore, it is sufficient to show that our d_M is a pseudometric on M and that d_M satisfies $(*)$.

Since c_{M_j} are pseudometrics, for arbitrary points p, q and r of $M\subset M$ and every j we have

$$c_{M_j}(p,q)\geqq 0, \quad c_{M_j}(p,q)=c_{M_j}(q,p), \quad c_{M_j}(p,r)\leqq c_{M_j}(p,q)+c_{M_j}(q,r).$$

If we fix the points p, q and r then for $j\to\infty$ we obtain

$$d_M(p,q)\geqq 0, \quad d_M(p,q)=d_M(q,p), \quad d_M(p,r)\leqq c_{M_j}(p,q)+c_{M_j}(q,r),$$

i. e. d_M is a pseudometric on M.

It remains to show that d_M satisfies $(*)$. Let $f: M\to D$ be holomorphic map. By the assumption f can be represented as a limit of some sequence $\{f_j\}$ of holomorphic maps $f_j: M_j\to D$, i. e.

$$f(r)=\lim_{j\to\infty} f_j(r), \qquad \forall r\in M.$$

Moreover, the maps $f_j\in H(M_j,D)$ are distance-decreasing with respect to the Carathéodory pseudometric and since $c_D=\varrho$ we have $c_{M_j}(p,q)\geqq\varrho(f_j(p),f_j(q))$. Then for $j\to\infty$ using also the continuity of the Poincaré-Bergman metric we obtain

$$d_M(p,q):=\lim_{j\to\infty} c_{M_j}(p,q)\geqq \lim_{j\to\infty}\varrho(f_j(p),f_j(q))$$

$$=\varrho(\lim_{j\to\infty} f_j(p),\ \lim_{j\to\infty} f_j(q))=\varrho(f(p),f(q)),$$

i. e. d_M satisfies $(*)$ and the proof of the theorem is complete.

Remark. In this theorem we suppose that $\lim_{j\to\infty} c_{M_j}(p,q)$ exists for every two points p and q of M. It is possible to obtain a variant of the theorem in which we suppose only that $\lim_{j\to\infty} c_{M_j}(p,q)$ exists for two fixed points p and q of M and then we assert that this limit is equal to $c_M(p,q)$ again. The proof has the same steps without the verification that d_M is a pseudometric on M, but one can omit this verification also in the above proof. The following arguments show this.

In the same way we obtain for the fixed number $d_M(p, q)$ the inequalities $d_M(p, q) \leqq c_M(p, q)$, $d_M(p, q) \geqq \varrho(f(p), f(q))$, $\forall f \in H(M, D)$. If we take the supremum with respect to all holomorphic maps $f: M \to D$ in the right side of the second inequality then by the definition of the Carathéodory pseudometric we obtain the desired inequality $d_M(p, q) \geqq c_M(p, q)$ and the result follows.

Moreover, in this case we can make the more weak assumption that for every $f \in H(M, D)$ there exists some sequence $\{f_j\}$, $f_j \in H(M_j, D)$, which tends to f only at the fixed points p and q of M, i. e.

$$f(p) = \lim_{j \to \infty} f_j(p), \qquad f(q) = \lim_{j \to \infty} f_j(q)$$

and the conclusion is the same.

Now we shall give some consequences from theorem 2. First in an important particular case the existence of $\lim_{j \to \infty} c_{M_j}(p, q)$ is not in the assumptions but in the conclusions of the proposition.

Corollary 1. *Let $\{M_j\}$ be decreasing sequence of complex spaces containing the complex space M and let the spaces have compatible structures in their intersections. If every holomorphic map $f: M \to D$ can be represented as a limit of some sequence $\{f_j\}$ of holomorphic maps $f_j: M_j \to D$ then for every two points p and q of M, $\lim_{j \to \infty} c_{M_j}(p, q)$ exists and is equal to $c_M(p, q)$.*

Proof. It is sufficient to show that $\lim_{j \to \infty} c_{M_j}(p, q)$ exists for every two points p and q of M and the assumptions of theorem 2 will be satisfied. But from the inclusions $M_j \supset M_{j+1}$ and $M_j \supset M$ for every j it follows that $(c_{M_j}\ p, q) \leqq c_{M_{j+1}}(p, q)$, $c_{M_j}(p, q) \leqq c_M(p, q)$, $\forall j$, i. e. the sequence $\{c_{M_j}(p, q)\}$ is increasing and bounded from above by $c_M(p, q)$ hence converges.

Now we shall give another consequence with more strong assumption for approximation which we shall use also later in the theorem for the Kobayashi pseudometric in the case when $M_j \supset M$.

Corollary 2. *Let $\{M_j\}$ be a sequence of complex spaces containing the complex space M and let the spaces be with compatible structures in their intersections. Moreover, let there exists some sequence $\{\varphi_j\}$ of holomorphic maps $\varphi_j: M_j \to M$ which tends to the identity of M. Then if $\lim_{j \to \infty} c_{M_j}(p, q)$ exists for every two points p and q of M we assert that it is equal to $c_M(p, q)$.*

Proof. We shall show that every holomorphic map $f: M \to D$ can be represented as a limit of some sequence $\{f_j\}$ of holomorphic maps $f_j: M_j \to D$. It is sufficient to set $f_j := f \circ \varphi_j$ and then

$$\lim_{j \to \infty} f_j(r) = \lim_{j \to \infty} f(\varphi_j(r)) = f(\lim_{j \to \infty} \varphi_j(r)) = f(r),$$

i. e. the assumptions of the theorem are satisfied.

Of course, here we can make the same remark as that after theorem 2, i. e. we can assume the existence of $\lim_{j \to \infty} c_{M_j}(p, q)$ and the convergence of $\{\varphi_j\}$ to the identity of M only for two fixed points p and q of M. Moreover, if here the sequence is decreasing as in corollary 1 we can prove the existence of $\lim_{j \to \infty} c_{M_j}(p, q)$ without assuming this.

Finally, in the general case when $M_j \urcorner\subset M$ and $M \urcorner\subset M_j$ for some subsequence of indices j it is possible to formulate and prove new theorems for coinciding of $\lim_{j\to\infty} c_{M_j}(p, q)$ and $c_M(p, q)$ taking the assumptions of theorem 1 and theorem 2, its variant or corollary 2, but we shall not do this here.

3. Limits of Kobayashi pseudometrics. First we prove a theorem for the Kobayashi pseudometric which is formulated as theorem 1 for the Carathéodory pseudometric but the method of proof is similar to that of theorem 2. Here one can see a display of the "duality" in the definitions of the two pseudometrics.

Theorem 3. *Let $\{M_j\}$ be a sequence of complex spaces with kernel M and let M contains all M_j. If p and q are two arbitrary points of M then we assert that* $\lim_{j\to\infty} k_{M_j}(p, q)$ *exists and is equal to* $k_M(p, q)$.

Proof. Since M is the kernel of the sequence $\{M_j\}$, two arbitrary points p and q of M are in M_j for all sufficiently large indices j. We have (for such indices) $k_{M_j}(p, q) \geqq k_M(p, q)$ because the identity of $M_j \subset M$ is holomorphic map and hence distance-decreasing with respect to the Kobayashi pseudometric. This shows that the sequence $\{k_{M_j}(p, q)\}$ is bounded from below by $k_M(p, q)$ and we have

$$\liminf_{j\to\infty} k_{M_j}(p, q) \geqq k_M(p, q).$$

Therefore, it is sufficient to show that

$$\limsup_{j\to\infty} k_{M_j}(p, q) \leqq k_M(p, q)$$

and then the inequalities

$$k_M(p, q) \leqq \liminf_{j\to\infty} k_{M_j}(p, q) \leqq \limsup_{j\to\infty} k_{M_j}(p, q) \leqq k_M(p, q)$$

imply that the limit $\lim_{j\to\infty} k_{M_j}(p, q)$ exists and is equal to $k_M(p, q)$.

For every two points p and q of M we set

$$d_M(p, q) := \limsup_{j\to\infty} k_{M_j}(p, q)$$

and we have to show that $d_M(p, q) \leqq k_M(p, q)$. We shall use the fact that the Kobayashi pseudometric is characterized by the following extremal property [6]:

If d is a pseudometric on M with

$$(*) \qquad \varrho(a, b) \geqq d(f(a), f(b)), \quad \forall a, b \in D, \quad \forall f \in H(D, M),$$

then $d_M(p, q) \leqq k_M(p, q)$, $\forall p, q \in M$, i. e. k_M is the largest pseudometric on M which satisfies $(*)$.

Therefore, it remains to show that our d_M is a pseudometric on M and that d_M satisfies $(*)$.

Since k_{M_j} are pseudometrics, for arbitrary points p, q and r of M and every sufficiently large j we have

$$k_{M_j}(p, q) \geqq 0, \quad k_{M_j}(p, q) = k_{M_j}(q, p), \quad k_{M_j}(p, r) \leqq k_{M_j}(p, q) + k_{M_j}(q, r).$$

Taking $\limsup_{j\to\infty}$, for fixed points p, q and r of M we obtain

$$d_M(p, q) \geqq 0, \quad d_M(p, q) = d_M(q, p),$$
$$d_M(p, r) := \limsup_{j\to\infty} k_{M_j}(p, r) \leqq \limsup_{j\to\infty} [k_{M_j}(p, q) + k_{M_j}(q, r)]$$
$$\leqq \limsup_{j\to\infty} k_{M_j}(p, q) + \limsup_{j\to\infty} k_{M_j}(q, r) = d_M(p, q) + d_M(q, r),$$

i. e. d_M is a pseudometric on M.

Now it remains to show that d_M satisfies (*). Let $f: D \to M$ be holomorphic map. The image of the origin $f(0)$ is a point of the kernel M of the sequence $\{M_j\}$. Then there is an open neighbourhood V of $f(0)$ contained in all M_j with sufficiently large indices j. The map f is in particular continuous and, therefore, $f^{-1}(V)$ is open in D. With U we denote the connected component of $f^{-1}(V)$ which contains the origin and with U_j the connected component of $f^{-1}(M_j)$ which contains U. Then it is clear that every U_j contains circles with centre at the origin and let $r_j \leqq 1$ be the maximal radius of such a circle, i. e. $D_{r_j} := \{z \in \mathbf{C} : |z| < r_j\} \subset U_j \subset D$. We shall prove that the sequence $\{r_j\}$ tends to 1 for $j \to \infty$.

Let $0 < r < 1$. The closed disk $\bar{D}_r := \{z \in \mathbf{C} : |z| \leqq r\}$ is compact in D and since f is continuous map, its image $f(\bar{D}_r)$ is compact in M. This compact has a finite covering of neighbourhoods participating when one defines the kernel M of the sequence $\{M_j\}$ and there is such an index j_0 that $f(\bar{D}_r)$ is in M_j for $j > j_0$. Then $\bar{D}_r$ itself is a circle which contains in U_j for $j > j_0$. But D_{r_j} is the maximal circle in U_j and we have $\bar{D}_r \subset D_{r_j} \subset U_j \subset D$, $\forall j > j_0$. This shows that $r < r_j \leqq 1$ for $j > j_0$ and since $r < 1$ is arbitrary we have the desired equality $\lim_{j\to\infty} r_j = 1$.

Now we can show for arbitrary points a and b of D that

$$\lim_{j\to\infty} k_{D_{r_j}}(a, b) = \varrho(a, b).$$

The homoteties $\sigma_j : D_{r_j} \to D$, $\sigma_j(z) := z/r_j$ are biholomorphic maps hence isometries of the Kobayashi pseudometric and then

$$k_{D_{r_j}}(a, b) = k_D(\sigma_j(a), \sigma_j(b)) = \varrho(a/r_j, b/r_j).$$

The Poincaré-Bergman metric is continuous and it follows from the above equalities and $\lim_{j\to\infty} r_j = 1$ that

$$\lim_{j\to\infty} k_{D_{r_j}}(a, b) = \lim_{j\to\infty} \varrho\left(a/r_j, b/r_j\right) = \varrho(a, b).$$

It remains to use the inclusions $f(D_{r_j}) \subset M_j$ and the fact that the restrictions of f to D_{r_j}, $f|_{D_{r_j}} : D_{r_j} \to M_j$ are holomorphic, hence distance-decreasing with respect to the Kobayashi pseudometric, i. e. $k_{D_{r_j}}(a, b) \geqq k_{M_j}(f(a), f(b))$. From here and the definition of d_M we obtain the desired inequality

$$\varrho(a, b) = \lim_{j\to\infty} k_{D_{r_j}}(a, b) \geqq \limsup_{j\to\infty} k_{M_j}(f(a), f(b)) = d_M(f(a), f(b))$$

and the proof of the theorem is complete.

Now we shall give a consequence from this theorem which is analogical to the corollary of theorem 1 and has the same proof.

Corollary. *Let $\{M_j\}$ be increasing sequence of complex spaces with compatible structures and let M be the union $\cup_{j=1}^{\infty} M_j$. Then for every two points p and q of M the limit $\lim_{j\to\infty} k_{M_j}(p, q)$ exists and is equal to $k_M(p, q)$.*

In the case when M contains in all M_j we shall prove a theorem which is analogical to corollary 2 of theorem 2 but under more strong assumptions than those of theorem 2.

Theorem 4. *Let $\{M_j\}$ be a sequence of complex spaces containing the complex space M and let the spaces be with compatible structures in their intersections. Moreover, let there exists some sequence $\{\varphi_j\}$ of holomorphic maps $\varphi_j: M_j \to M$, which tends to the identity of M. Then if the limit $\lim_{j\to\infty} k_{M_j}(p, q)$ exists for every two points p and q of M we assert that it is equal to $k_M(p, q)$.*

Proof. If we set

$$d_M(p, q) := \lim_{j\to\infty} k_{M_j}(p, q), \qquad \forall p, q \in M,$$

then it is sufficient to show that $d_M(p, q) = k_M(p, q)$. Since $M \subset M_j$, for every j the identity of M, $\mathrm{id}_M: M \to M_j$ is holomorphic map hence distance-decreasing with respect to the Kobayashi pseudometric, i. e. $k_M(p, q) \geqq k_{M_j}(p, q)$, $\forall j$. If we fix two arbitrary points p and q of M and take limit for $j \to \infty$ in this inequality then we obtain $d_M(p, q) \leqq k_M(p, q)$.

To show that this inequality is in fact an equality we suppose that there is a positive ε such that $d_M(p, q) + \varepsilon < k_M(p, q)$. By the definition of $d_M(p, q)$ for all sufficiently large indices j we have $k_{M_j}(p, q) < d_M(p, q) + \varepsilon$. Let j be arbitrary fixed such an index. Since

$$k_{M_j}(p, q) := \inf_{\sigma} L(\sigma), \qquad \sigma \in V_{M_j}(p, q)$$

we can find a chain $\sigma_j := \{a_\mu, b_\mu; f_\mu\}_{\mu=1}^{m}$ from p to q in M_j with length

$$L(\sigma_j) := \sum_{\mu=1}^{m} \varrho(a_\mu, b_\mu) < d_M(p, q) + \varepsilon.$$

The maps f_μ belong to $H(D, M_j)$ and composing them with the map $\varphi_j \in H(M_j, M)$ (from the assumptions of the theorem) we obtain maps $g_\mu := \varphi_j \circ f_\mu$ belonging to $H(D, M)$. Then $\tilde{\sigma}_j := \{a_\mu, b_\mu; g_\mu\}_{\mu=1}^{m}$ is a chain from $\varphi_j(p)$ to $\varphi_j(q)$ in M ($\tilde{\sigma}_j \in V_M(\varphi_j(p), \varphi_j(q))$). Its length is the same as that of σ_j because

$$L(\tilde{\sigma}_j) := \sum_{\mu=1}^{m} \varrho(a_\mu, b_\mu) =: L(\sigma_j).$$

Now using the definition of $k_M(\varphi_j(p), \varphi_j(q))$ we obtain

$$k_M(\varphi_j(p), \varphi_j(q)) \leqq L(\tilde{\sigma}_j) = L(\sigma_j) < d_M(p, q) + \varepsilon,$$

i. e. for all sufficiently large indices j we have $k_M(\varphi_j(p), \varphi_j(q)) < d_M(p, q) + \varepsilon$. Since the Kobayashi pseudometric is continuous [1] it follows for $j \to \infty$

$$k_M(p, q) = k_M(\lim_{j\to\infty}\varphi_j(p), \lim_{j\to\infty}\varphi_j(q)) = \lim_{j\to\infty} k_M(\varphi_j(p), \varphi_j(q)) \leq d_M(p, q) + \varepsilon < k_M(p, q).$$

This is a contradiction and the theorem is proved.

Remark. One can see that in this proof the points p and q can be fixed still from the beginning. Therefore, this theorem also admits as theorem 2 a variant in which we suppose that $\lim_{j\to\infty} k_{M_j}(p, q)$ exists for some two fixed points p and q of M and then we assert that this limit is equal to $k_M(p, q)$. Furthermore, we can suppose in such a case that there is a sequence $\{\varphi_j\}$ of holomorphic maps $\varphi_j: M_j \to M$ which tends to the identity of M only at the points p and q, i. e.

$$\lim_{j\to\infty}\varphi_j(p) = p, \quad \lim_{j\to\infty}\varphi_j(q) = q$$

and the conclusion is the same.

As after theorem 2 we shall give in an important particular case a consequence in which the existence of $\lim_{j\to\infty} k_{M_j}(p, q)$ is not in the assumptions, but in the conclusions.

Corollary. *Let $\{M_j\}$ be decreasing sequence of complex spaces and let the spaces be with compatible structures in their intersections. If there is a sequence $\{\varphi_j\}$ of holomorphic maps $\varphi_j: M_j \to M$ which tends to the identity of M then for every two points p and q of M the limit $\lim_{j\to\infty} k_{M_j}(p, q)$ exists and is equal to $k_M(p, q)$.*

The proof is the same as that of corollary 1 of theorem 2.

Finally, it is possible to formulate and prove new theorems for coinciding of $\lim_{j\to\infty} k_{M_j}(p, q)$ and $k_M(p, q)$ in the case when $M_j \urcorner\subset M$ and $M \urcorner\subset M_j$ for some subsequence of indices j taking the assumptions of theorem 3 and theorem 4 or its variant, but we shall not do this here.

4. Final remarks. A difference between the previous note [3] and this paper is that we prove here the theorems for the case when M contains in all members of the sequence $\{M_j\}$ without the assumption that M is the kernel of $\{M_j\}$. Only assumptions for approximation play the role of a condition for "tending" of the sequence $\{M_j\}$ to M.

A supplement to the definition of a kernel is the following:

We say that the sequence $\{M_j\}_{j=1}^{\infty}$ tends to its kernel M if every subsequence $\{M_{j_\nu}\}_{\nu=1}^{\infty}$ has the same kernel M in $\cup_{j=1}^{\infty} M_j$ (not in $\cup_{\nu=1}^{\infty} M_{j_\nu}$!).

This supplement is not essential in the case when M contains all M_j (as in theorem 1 and theorem 3) because it is easy to verify that if M is the kernel of the sequence $\{M_j\}$, then every subsequence has the same kernel M, i. e. $\{M_j\}$ tends to its kernel M.

In the general case the kernels of all subsequences $\{M_{j_\nu}\}$ contain the kernel M of the initial sequence $\{M_j\}$, and the condition that $\{M_j\}$ tends to its kernel M is an essential additional assumption for the sequence $\{M_j\}$.

It remains as open question whether there exists some relation between the comparatively strong topological condition for tending of $\{M_j\}$ to its kernel M and our assumptions for approximation in theorem 2 and theorem 4.

More generally we can ask whether it is possible to replace the approximation in these two theorems which have analytic character with other assumptions which have topological or geometrical one.

REFERENCES

1. T. Barth. The Kobayashi distance induces the standard topology. *Proc. Amer. Math. Soc.*, **35**, 1972, 439—441.
2. C. Carathéodory. Über das Schwarzsche Lemma bei analytischen Funktionen von zwei komplexen Veränderlichen, *Math. Ann.*, **97**, 1926, 76—98.
3. V. Z. Hristov. Limits of Carathéodory and Kobayashi pseudometrics. *C. R. Acad. bulg. Sci*, **29**, 1976, 951—954.
4. S. Kobayashi. Invariant distances on complex manifolds and holomorphic mappings. *J. Math. Soc Japan*, **19**, 1967, 460—480.
5. S. Kobayashi. Hyperbolic manifolds and holomorphic mappings. New York, 1970.
6. S. Kobayashi. Intrinsic distances, measures and geometric function theory. *Bull. Amer. Math. Soc.*, **82**, 1976, 357—416.

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