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# THE FINITE LEONTIEV TRANSFORM: OPERATIONAL PROPERTIES AND MULTIPLIERS

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*I do wish to make the point that functional analysis, or more accurately algebraic analysis, has something to say about the subject of Dirichlet series.*

*H. Helson* [1]

Let $L(\lambda)$ be an entire function of order one and of normal type, with distinct zeros $\lambda_1, \lambda_2, \dots$, and let $\gamma(z)$ be its Borel transform. If $D$ is a finite convex domain, such that all singularities of $\gamma(z)$ lie in $\bar{D}$, then under Leontiev's transform of a function $f(z) \in A(\bar{D})$ it is understood the sequence

$$T_n(f) = \frac{1}{2\pi i} \int_C \gamma(t) \left( \int_0^t e^{\lambda_n \tau} f(t-\tau) d\tau \right) dt, \qquad n = 1, 2, \dots$$

It is shown that the Leontiev transform gives a complex realization of an operational calculus for a general right inverse operator of the differentiation operator. An explicit representation of all possible convolutions of Leontiev transforms is found. The multiplier problem for the Leontiev transform is solved.

Let $L(\lambda)$ be an entire function of order 1 and of normal type with infinite sequence $\lambda_n$, $n = 1, 2, \dots$, of distinct zeros, ordered in such a way that $0 < |\lambda_1| \leq |\lambda_2| \leq \dots \leq |\lambda_n| \leq \cdots$ Without essential loss of generality we may assume $L(0) = 1$. If $L(\lambda) = \Sigma_{n=0}^{\infty} (n!)^{-1} a_n \lambda^n$, $a_0 = 1$, then its Borel transform is $\gamma(z) = \Sigma_{n=0}^{\infty} a_n / z^{n+1}$.

Let $D$ be a finite convex domain in the complex plane, such that all the singularities of $\gamma(z)$ are contained in $\bar{D}$. We may assume $0 \notin D$. Let $A(\bar{D})$ denotes the space of the functions $f(z)$, which are analytic on $\bar{D}$. As the topology of $A(\bar{D})$ we consider the inductive topology arising when we consider $A(\bar{D})$ as the inductive limit of a sequence $\{A_{\text{bd}}(O_n)\}_{n=1}^{\infty}$ of Banach spaces of the bounded analytic functions on domains $O_n \supset \bar{D}$, such that $\{O_n\}_{n=1}^{\infty}$ is a base of neighbourhoods of $D$ (see [2, p. 594]). When we speak about continuous functionals or operators on $A(\bar{D})$, we mean continuous in this inductive topology.

In 1965 A. F. Leontiev [3] introduced for each $f(z) \in A(\bar{D})$ the formal expansion

$$f(z) \sim \sum_{n=1}^{\infty} T_n(f) e^{\lambda_n z} / L'(\lambda_n) \tag{1}$$

with

(2) $$T_n(f)=\frac{1}{2\pi i}\int_C \gamma(t)dt\int_0^t e^{\lambda_n\tau}f(t-\tau)d\tau, \qquad n=1, 2, \dots,$$

where $C$ is a contour, lying in the domain of analiticity of $f(z)$, and containing $\bar{D}$ in its inside. Leontiev ([4, p. 255]) had also shown that the coefficients of (1), or the values of the functionals (2) uniquely determine the function $f(z)$, i. e. $T_n(f)=0$ for $n=1, 2, \dots$ imply $f(z)\equiv 0$.

Theorefore, it is rather natural to look on the sequence $\{T_n(f)\}_{n=1}^{\infty}$ as on a finite integral transform on $A(\bar{D})$, to a great extent like to the finite Fourier, or Sturm-Liouville transforms [5, ch. 10 and 11]. Then the correspondence (2) is said to be the finite Leontiev transform.

From this viewpoint we ask for the operational properties of the finite Leontiev transform. The main such properties are: its differential law, inversion formula, convolutions, and multipliers.

At first place, we show that each Leontiev transform (2) is connected with a right inverse operator in $A(\bar{D})$ of the differentiation operator $d/dz$.

Theorem 1. *If $\Lambda: A(\bar{D})\to A(\bar{D})$ is the right inverse operator of the differentiation operator $d/dz$, defined by*

(3) $$\Lambda f(z)=\int_0^z f(t)dt-\frac{1}{2\pi i}\int_C \gamma(t)\left(\int_0^t f(\tau)d\tau\right)dt,$$

*then*

(4) $$T_n(\Lambda f)=\lambda_n^{-1}T_n(f), \quad n=1, 2,\dots$$

Proof. Let us, for brevity's sake, denote $lf(z)=\int_0^z f(t)dt$ and

$$\Phi(f)=(2\pi i)^{-1}\int_C \gamma(t)f(t)dt.$$

Then

$$T_n(\Lambda f)=\Phi\{\int_0^t e^{\lambda_n(t-\tau)}(lf)(\tau)d\tau\}-\Phi(lf)T_n\{1\}.$$

But $T_n\{1\}=-1/\lambda_n$, $n=1, 2, \dots$, and

$$\int_0^t e^{\lambda_n(t-\tau)}d\tau\int_0^\tau f(\sigma)d\sigma=\frac{1}{\lambda_n}\int_0^t e^{\lambda_n\tau}f(t-\tau)d\tau-\frac{1}{\lambda_n}\int_0^t f(\tau)d\tau.$$

Therefore

$$T_n(\Lambda f)=\frac{1}{\lambda_n}\Phi\{\int_0^t e^{\lambda_n\tau}f(t-\tau)d\tau\}=\frac{1}{\lambda_n}T_n(f), \qquad \text{q. e. d.}$$

Corollary. *If $f(z)\in A(\bar{D})$, then*

(5) $$T_n(f')=\lambda_nT_n(f)-\frac{1}{2\pi i}\int_C \gamma(t)f(t)dt.$$

Proof. Let us apply $T_n$ to the the identity

$$\Lambda f'(z)=f(z)-\frac{1}{2\pi i}\int_C \gamma(t)f(t)dt.$$

Using (4) and $T_n\{1\}=-1/\lambda_n$, we get at once (5).

The identity (5) is said to be the differential law of the Leontiev transform (2).

The right inverse operators of the differentiation operator in the real domain had been studied by the author [6] and by L. Berg [7]. Independently, they found convolutions for these operators in explicit form. The corresponding results are extendable without any difficulties in the complex domain.

Definition 1 ([6]). *A bilinear, commutative and associative operation* $f*g(z)$ *in* $A(\bar{D})$ *is said to be a convolution of a linear operator* $M:A(\bar{D})\to A(\bar{D})$, *iff the relation*

$$M(f*g)=(Mf)*g \tag{6}$$

*holds for all* $f, g\in A(\bar{D})$.

Theorem 2. *The operation*

$$f*g(z)=\frac{1}{2\pi i}\int_C \gamma(t)dt\int_z^t f(z+t-\tau)\,g(\tau)d\tau \tag{7}$$

*is a continuous convolution of the right inverse* $\Lambda$ *of the differentiation operator* $d/dz$, *defined by* (3), *and*

$$\Lambda f(z)=\{-1\}*f(z). \tag{8}$$

Proof. Under continuity of $f*g$ we mean the separate continuity, i. e that $f_n\to f$ in $A(\bar{D})$ implies $f_n*g\to f*g$ for all $g\in A(\bar{D})$. It should be noted that the contour $C$ in (7) ought be chosen in such a way, in order to lye in the common domain of analiticity of $f(z)$ and $g(z)$. The separate continuity of (7) follows easily from its explicit representation. It is not so evident the continuity in the sense that $f_n\to f$ and $g_n\to g$ imply $f_n*g_n\to f*g$ in $A(\bar{D})$. We will not use the continuity of $f*g$ in this sense, so we drop the proof.

The bilinearity and continuity of (7) are evident. We shall give an elaborate proof of the associativity only. Let $\alpha$, $\beta$, $\gamma$ be three mutually different numbers. It is not difficult to find that (see [6])

$$(e^{\alpha z}*e^{\beta z})*e^{\gamma z}=e^{\alpha z}*(e^{\beta z}*e^{\gamma z}). \tag{9}$$

Now, let $f$, $g$, $h\in A(\bar{D})$ be arbitrary. In order to prove the associativity relation $(f*g)*h=f*(g*h)$ we shall take the contour $C$ to lie in the common domain of analiticity of $f$, $g$ and $h$ and to contain $\bar{D}$ in its inside. Let us differentiate (9) $m$ times in $\alpha$, $n$ times in $\beta$ and $p$ times in $\gamma$. Thus, we get $(z^m e^{\alpha z}*z^n e^{\beta z})*z^p e^{\gamma z}=z^m e^{\alpha z}*(z^n e^{\beta z}*z^p e^{\gamma z})$. If we let first $\alpha\to 0$, then $\beta\to 0$, and at last $\gamma\to 0$, we get $(z^m*z^n)*z^p=z^m*(z^n*z^p)$ for $m, n, p=0, 1, 2, \ldots$. From the bilinearity of (7) it follows that the associativity relation is true for polynomials. It remains to use Runge's approximation theorem for the closed domain, bounded by the contour $C$.

The proof of (8) is a matter of a simple verification.

The following lemma exhibits an essential connection between convolution (7) and the Leontiev transform (2).

Lemma 1. *If* $f(z)\in A(\bar{D})$, *then*

$$f(z)*e^{\lambda_n z}=T_n(f)e^{\lambda_n z},\qquad n=1, 2, \ldots \tag{10}$$

Proof. Let us transform the left-hand side of (10):

$$f(z)*e^{\lambda_n z}=\Phi_t\{\int_z^t e^{\lambda_n(z+t-\tau)}f(\tau)d\tau\}=\Phi_t\{\int_0^t-\int_0^z\}=e^{\lambda_n z}\Phi_t\{\int_0^t e^{\lambda_n(t-\tau)}f(\tau)d\tau\}$$

$$-\Phi_t\{e^{\lambda_n t}\}.\int_0^z e^{\lambda_n(z-\tau)}f(\tau)d\tau=T_n(f)e^{\lambda_n z},$$

since $\Phi_t\{e^{\lambda_n t}\}=L(\lambda_n)=0$.

Theorem 3. *The linear functionals* $T_n(f)$, $n=1, 2, \ldots$, *are multiplicative in the convolution algebra* $A(\bar{D})$ *with multiplication* (7), *i. e.*

(11) $$T_n(f*g)=T_n(f)T_n(g), \qquad n=1, 2, \ldots$$

Proof. "Multiplying" term-by-term the identity $f*e^{\lambda_n z}=T_n(f)e^{\lambda_n z}$ by $g$ we get $(f*g)*e^{\lambda_n z}=T_n(f)T_n(g)e^{\lambda_n z}$, using the associativity of (7). By (10), we have $T_n(f*g)e^{\lambda_n z}=T_n(f)T_n(g)e^{\lambda_n z}$, thus proving (11).

Relations (11) expresses the convolution property of the Leontiev transform. They say that $f*g$ is the convolution of the Leontiev transform. Sometimes, the notion of convolution of a finite integral transform is understood in broader sense ([5, p. 320]).

Definition 2. *A bilinear, commutative and associative operation* $f\tilde{*}g$ *in* $A(\bar{D})$ *is said to be a convolution of the Leontiev transform* (2), *iff*

(12) $$T_n(f\tilde{*}g)=\mu_n T_n(f)T_n(g), \qquad n=1, 2, \ldots,$$

*where the numerical sequence* $\mu_n$ *is one and the same for all* $f, g\in A(\bar{D})$.

We aim to find an explicit representation of all separately continuous convolutions of the Leontiev transform in $A(\bar{D})$. But this cannot be done earlier than we have a solution for the multiplier problem of the Leontiev transform.

Definition 3. *A numerical sequence* $\mu_1, \mu_2, \ldots, \mu_n, \ldots$ *is said to be a multiplier sequence of the Leontiev transform* (2), *iff for each* $f\in A(\bar{D})$ *there exists a function* $g\in A(\bar{D})$, *such that* $T_n(g)=\mu_n T_n(f)$, $n=1, 2, \ldots$

Each multiplier sequence $\mu_n$, $n=1, 2, \ldots$, defines a linear operator $M: A(\bar{D})\to A(\bar{D})$ with $Mf=g$. For it $T_n\{Mf\}=\mu_n T_n(f)$, $n=1, 2, \ldots$ Now we shall show that each multiplier of the Leontiev transform (2) is a multiplier of convolution (7) in the sense of the following

Definition 4 ([8, p. 13]). *A linear operator* $M: A(\bar{D})\to A(\bar{D})$ *is said to be a multiplier of the convolution algebra* $A(\bar{D})$ *with the multiplication* (7), *iff* $f*g$ *is a convolution of* $M$.

In other words relation (6) is characteristic for the multipliers of the corresponding convolution algebra.

Theorem 4. *Each multiplier of the Leontiev transform* (2) *is a multiplier of the convolution* (7) *too.*

Proof. Let $M: A(\bar{D})\to A(\bar{D})$ be a multiplier operator of the Leontiev transform (2), determined by a multiplier sequence $\mu_n$, $n=1, 2, \ldots$ Then

$$T_n[M(f*g)-(Mf)*g]=\mu_n T_n(f*g)-T_n(Mf)T_n(g)=0,$$

due to (11) and the multiplier property $T_n(Mf)=\mu_n T_n(f)$, $n=1, 2, \ldots$ From the uniqueness theorem of L e o n t i e v (see [3, p. 255]) it follows $M(f*g)-(Mf)*g=0$, i. e. relation (6). Hence $M$ is a multiplier of the convolution $f*g$.

Thus, the multiplier problem for the Leontiev transform (2) is reduced to the multiplier problem for the convolution (7).

T h e o r e m 5. *A linear operator* $M: A(\bar{D})\to A(\bar{D})$ *is a multiplier of convolution* (7) *iff it has a representation of the form*

$$Mf(z)=\mu f(z)+m(z)*f(z) \tag{13}$$

*with* $\mu=const.$, *and* $m(z)\in A(\bar{D})$. *Representation* (13) *is unique.*

P r o o f. Since the convolution (7) is an operation without non-zero annihilators (or, in the terminology of Larsen ([8, p. 13]), the convolution algebra $A(\bar{D})$ is without order), then the multipliers of (7) form a commutative ring. Let $M: A(\bar{D})\to A(\bar{D})$ be an arbitrary multiplier of $f*g$. From (8) it follows that $\Lambda$ is a multiplier of (7) too. Hence $M$ and $\Lambda$ commute, i. e. $M\Lambda=\Lambda M$. From (8) we get $M\Lambda f=M(\{-1\}*f)$, or

$$\Lambda(Mf)=(M\{-1\})*f, \tag{14}$$

where multiplier relation (6) is used. If we denote $n(z)=M\{-1\}$ and differentiate (14), we get

$$Mf(z)=\frac{d}{dz}[n(z)*f(z)]. \tag{15}$$

This representation is equivalent to (13) with $\mu=\Phi(n)$ and $m(z)=n'(z)$.

Conversely, if $M$ if an operator of the form (13), then evidently, it is a multiplier of (7).

C o r o l l a r y 1. *If* $M: A(\bar{D})\to A(\bar{D})$ *is a multiplier operator of the Leontiev's transform* (2), *then it is a continuous operator on* $A(\bar{D})$ *and has representation* (13).

The continuity follows from representation (13) and from the separate continuity of (7).

C o r o l l a r y 2. *A sequence* $\{\mu_n\}_{n=1}^{\infty}$ *is a multiplier sequence of Leontiev transform* (2), *iff there is a function* $m(z)\in A(\bar{D})$ *such that*

$$\mu_n=\mu+T_n(m), \qquad n=1,2,\dots \tag{16}$$

*with an arbitrary constant* $\mu$.

P r o o f. To each multiplier sequence $\{\mu_k\}_{k=1}^{\infty}$ of (2) there corresponds a multiplier $M$ of (7) with representation (13). If we apply (2) to (13), we get

$$T_n(Mf)\overset{\text{def}}{=}\mu_n T_n(f)=\mu T_n(f)+T_n(m)T_n(f),$$

i. e. (16). The converse is evident.

Representation formulas (13) or (15) can be written at once, using a general representation formula for the multipliers of a convolution of right inverse operator [9, theor. 2], but we prefer to proceed directly.

T h e o r e m 6. *If a continuous linear operator* $M: A(\bar{D})\to A(\bar{D})$, *commutes with the right inverse operator* $\Lambda$ *of* $d/dz$ *in* $A(\bar{D})$, *then M is a multiplier of* (7).

Proof. We begin with the evident identity $(M\{1\})*\{1\}=\{1\}*M\{1\}$. Using the commutating relation $M\Lambda=\Lambda M$ and the fact that $f*g$ is a convolution of $\Lambda$, we can write $(M\Lambda^p\{1\})*(\Lambda^q\{1\})=(\Lambda^p\{1\})*(M\Lambda^q\{1\})$ for $p, q=0, 1, 2, \dots$ The functions $\Lambda^n\{1\}$, $n=0, 1, 2, \dots$ are polynomials exactly of $n$-th degree. Therefore, the relation $(Mf_p)*g_q=f_p*(Mg_q)$ holds for polynomials $f_p$ and $g_q$ of arbitrary degrees $p$ and $q$. Let $f, g\in A(\bar{D})$ be arbitrary. We take a contour $C$ in the common domain of analiticity of $f$ and $g$ and containing $\bar{D}$ inside. According to Runge's theorem we can choose polynomial sequences $\{f_p(z)\}_{p=1}^{\infty}$ and $\{g_q(z)\}_{q=1}^{\infty}$, converging uniformly on the closed domain, bounded by $C$, to $f(z)$ and $g(z)$ correspondingly. Now, from the separate continuity of $f*g$ it follows that $(Mf)*g=f*(Mg)$. The last relation implies the complete convolution relation (6) (see [8, p. 15]).

Corollary. *A continuous linear operator $M: A(\bar{D})\to A(\bar{D})$ for which the hyperplane $\Phi(f)=0$ is an invariant subspace, commutes with the differentiation operator $d/dz$ in this hyperplane iff $M$ is a multiplier of convolution* (7).

The proof in the real case can be seen in [10]. In the complex case it proceeds in the same way. That's why we omit it.

Now we are ready to characterize all continuous convolutions of the Leontiev transform (2).

Theorem 7. *If $f\tilde{*}g$ is a continuous convolution of the Leontiev transform* (2) *in $A(\bar{D})$, then it has a representation of the form*

$$f\tilde{*}g=\frac{d^2}{dz^2}[\varrho*(f*g)] \tag{17}$$

*with $\varrho\in A(\bar{D})$, where by $*$ operation* (7) *is denoted. Conversely, each operation $f\tilde{*}g$ of the form* (17) *in $A(\bar{D})$ is a convolution of* (2).

Proof. Let $f\tilde{*}g$ be an arbitrary convolution of (2) in $A(\bar{D})$. If $f\in A(\bar{D})$ is arbitrary, but fixed, let us consider the linear operator $M_f g=f\tilde{*}g$. From (12) it follows that $M_f g$ is a multiplier operator of the Leontiev transform (2). According to theorem 5, $M_f$ can be represented in the form

$$M_f g=\frac{d}{dz}[(M_f\{-1\})*g].$$

But $M_f\{-1\}=M_{\{-1\}}f=\frac{d}{dz}\{M_{\{-1\}}\{-1\}*f\}$, and hence

$$f\tilde{*}g=M_f g=\frac{d}{dz}\{(\frac{d}{dz}(\varrho*f))*g\}=\frac{d^2}{dz^2}[\varrho*(f*g)]$$

with $\varrho=M_{\{-1\}}\{-1\}=\{1\}\tilde{*}\{1\}$.

Let us mention at last about inversion formulas for the Leontiev transform (2). This problem is treated in chapter 4 of Leontiev's book [4]. We here say only that if the formal expansion (1) represents the function $f(z)$ in $\bar{D}$, i. e. if

$$f(z)=\sum_{n=1}^{\infty} T_n(f)e^{\lambda_n z}/L'(\lambda_n), \tag{18}$$

then we can consider (18) as an inversion formula.

From the viewpoint of analysis it is desireable to extend above results for the space $A(\bar{D})$ to the space $\tilde{A}(D)$ of the functions, which are analytic in the open domain $D$ and continuous on $\bar{D}$. Then, instead of (2), we take

$$(19) \qquad T_n(f)=\frac{1}{2\pi i}\int_{\partial\bar{D}}\gamma(t)\,(\int_0^t e^{\lambda_n \tau}f(t-\tau)d\tau)dt.$$

In order to ensure an unicity theorem, additional restrictions on the function $L(\lambda)$ are needed. Leontiev [4, p. 260] had proved that if $L(\lambda)$ is such that $|L(re^{i\varphi})|<Ae^{h(\varphi)r}/r^{\mu}$, $\mu>1$, $r>0$, where $h(\varphi)$ is the indicator function of $L(\lambda)$, and $A$ is a constant, then if $T_n(f)=0$, $n=1,2,\dots$, for $f\in\tilde{A}(D)$, then $f(z)\equiv 0$.

Operation (7) is a convolution of (19) too. As for the representation formula (13), some mild restrictions on the boundary behaviour of $f(z)$ are needed. A sufficient condition is $f'(z)$ and $f''(z)$ to be continuous on $\bar{D}$. Here an illustrative example of an application of representation formula (13) will be given.

Theorem 8 (see [4, p. 320]). *In order that Leontiev's expansion* (1) *to be uniformly convergent on $\bar{D}$ for each function $f(z)$, analytic in $D$, and continuous in $\bar{D}$ together with its two first derivatives $f'(z)$ and $f''(z)$, provided*

$$(20) \qquad \Phi(f)=\frac{1}{2\pi i}\int_{\partial\bar{D}}\gamma(t)f(t)dt=0,$$

*it is necessary and sufficient this to hold only for Leontiev's expansion of the linear function $z-\Phi_\zeta(\zeta)$. If Leontiev's expansion of $z-\Phi_\zeta(\zeta)$ is known to be uniformly convergent on $\bar{D}$ to $z-\Phi(\zeta)$, then Leontiev's expansion of $f(z)$ is uniformly convergent to $f(z)$ for each function, which satisfies the conditions of the theorem.*

Proof. It is easy to find that

$$(21) \qquad z-\Phi(\zeta)\sim\sum_{n=1}^{\infty}e^{\lambda_n z}/\lambda_n^2 L'(\lambda_n).$$

Let us assume that the series is uniformly convergent on $\bar{D}$. If $f(z)\in\tilde{A}(D)$ is a function with continuous $f'(z)$ and $f''(z)$ in $\bar{D}$, then it can be represented in the form

$$(22) \qquad f(z)=[z-\Phi(\zeta)]\Phi(f')+[z-\Phi(\zeta)]*f''(z),$$

where the condition (20) is taken into account. It is easy to see that

$$(23) \qquad [z-\Phi(\zeta)]*f(z)\sim\sum_{n=1}^{\infty}\frac{e^{\lambda_n z}*f''(z)}{\lambda_n^2 L'(\lambda_n)}$$

and hence the Leontiev expansion (23) is uniformly converegent on $\bar{D}$. If the expansion of $z-\Phi(\zeta)$ converges uniformly to $z-\Phi(\zeta)$, then from (22) it is clear that the Leontiev's expansion of $f(z)$ is uniformly convergent to $f(z)$ on $\bar{D}$.

Formula (21) is a special case of a general Taylor formula for the operator $\Lambda$. To write it explicitly, we introduce a class of Bernoulli-type polynomials. The polynomial $B_n(z)=n!\,\Lambda^n\{1\}$, $n=0,1,2,\dots$, is said to be $n$-th generalized Bernoulli polynomial for the operator $\Lambda$, defined by (3). For example, $B_1(z)=z-\Phi(\zeta)$.

Theorem 9. *If $f(z) \in A(\bar{D})$, then for each n*

$$f(z)=\sum_{k=0}^{n-1}\Phi[f^{(k)}]B_k(z)/k!+R_n(z),$$

*where*

$$R_n(z)=\Lambda^n f(z)=-\Phi_t\{\int_z^t \frac{B_{n-1}(t+z-\tau)}{(n-1)!}f^{(n)}(\tau)d\tau\} \tag{24}$$

*with* $\Phi(f)=(2\pi i)^{-1}\int_C \gamma(t)f(t)dt$.

Proof. We may use the general Taylor formula [11] for a right inverse operator, but it is simpler to proceed directly. If $I$ is the identity operator of $A(\bar{D})$, we use the evident operator identity

$$I=(I-\Lambda\frac{d}{dz})+(\Lambda\frac{d}{dz}-\Lambda^2(\frac{d}{dz})^2)+\cdots+[\Lambda^{n-1}(\frac{d}{dz})^n-\Lambda^n(\frac{d}{dz})^n]+\Lambda^n(\frac{d}{dz})^n,$$

which easily can be written in the form

$$f(z)=\sum_{k=0}^{n-1}\Lambda^k(I-\Lambda\frac{d}{dz})f^{(k)}(z)+\Lambda^n f^{(n)}(z).$$

But $(I-\Lambda\frac{d}{dz})f(z)=\Phi(f)$ and hence $f(z)=\sum_{k=0}^{n-1}\Phi(f^{(k)})\Lambda^k\{1\}+\Lambda^n f^{(n)}(z)$. Then the remainder term can be written in the form $R_n(z)=\Lambda^n f^{(n)}(z)=-(\Lambda^{(n-1)}\{1\})*f^{(n)}(z)$, thus proving formula (24).

It seems that the idea to consider the Dirichlet expansions as eigenfunction expansions is due to A. P. Hromov [12]. Eventually, there remain other possibilities for applications of the algebraic approach to the subject of Dirichlet series and relative Leontiev transform.

## REFERENCES


1. H. Helson. Foundations of the theory of Dirichlet series. *Acta Math.*, **118**, 1967, 61—67.
2. Р. Эдвардс. Функциональный анализ. Москва, 1969.
3. А. Ф. Леонтьев. О представлении произвольных функций рядами Дирихле. *Доклады АН СССР*, **164**, 1965, 40—42.
4. А. Ф. Леонтьев. Ряды экспонент. Москва, 1976.
5. R. V. Churchill. Operational mathematics. New York, 1972.
6. I. H. Dimovski. On an operational calculus for vector-valued functions. *Math. Balkanica*, **4**, 1974, 129—135.
7. L. Berg. Generalized convolutions. *Math. Nachr.*, **72**, 1976, 239—245.
8. R. Larsen. An introduction to the theory of multipliers. Berlin, Heidelberg, New York, 1971.
9. I. H. Dimovski. Convolutions of right inverse operators and representation of their multipliers. *C. R. Acad. bulg. Sci.*, **31**, 1978. 1377—1380.
10. I. H. Dimovski. Representation of operators which commute with differentiation in an invariant hyperplane. *C. R. Acad. bulg. Sci.*, **31**, 1978, 1246—1248.
11. D. Przeworska-Rolewicz. Algebraic theory of right invertible operators. *Studia Math.*, **48**, 1973, 129—143.
12. А. П. Хромов. Оператор дифференцирования и ряды типа Дирихле. *Мат. заметки*, **6**, 1969, 759—766.


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