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CONVOLUTIONS, MULTIPLIERS AND COMMUTANTS FOR THE BACKWARD SHIFT OPERATOR

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An algebraic approach to the backward shift operator U^* is developed. The convolutions of all linear right inverse operators of U^* are found. The multiplier operators of these convolutions are determined. An explicit representation of the commutant of U^* in an invariant hyperplane is given. An application to the multiplier problem of T. A. Leontieva's expansions in a closed domain is made.

Let D be a finite domain in the complex plane. The space of local analytic functions on $\bar{D}$ will be denoted by $A(\bar{D})$. The topology in $A(\bar{D})$ is introduced, as usually, as an inductive topology. Let $\{O_n\}_{n=1}^{\infty}$ be a decreasing sequence of domains $O_n \supset \bar{O}_{n+1}$ such that every open set, containing $\bar{D}$, contains some O_n. With $B(O_n)$ we denote the space of bounded analytic functions on O_n with the norm $\|f\|_n = \sup_{z \in O_n} |f(z)|$. Then we consider $A(\bar{D})$ as the inductive limit of the spaces $B(O_n)$. It is well known that the corresponding inductive topology is such that a sequence $\tilde{f}_n$ of local analytic functions in $A(\bar{D})$ tends to $\tilde{f} \in A(\bar{D})$ iff there exists an open set $O \supset \bar{D}$ and functions f_n and f, regular in O such that f_n belongs to the class $\tilde{f}_n$, f belongs to the class $\tilde{f}$ and $f_n \to f$ uniformly on O. A linear functional Φ on $A(\bar{D})$ is continuous iff $\tilde{f}_n \to 0$ implies $\Phi(\tilde{f}_n) \to 0$.

Let the domain D contains the zero point 0. Then the operator

$$U^* f(z) = [f(z) - f(0)]/z \tag{1}$$

is said to be the backward shift operator on $A(\bar{D})$ (also, the left shift, or the adjoint shift). It had been studied by many authors (see [1]). They, usually, study directly the operator U^* itself, its invariant subspaces and cyclic vectors, mainly on disk domains. Here we intend to study the family of continuous linear right inverse operators of U^* in an arbitrary domain. Only in one case it is presupposed that D is a domain with connected complement, so that Runge's approximation theorem to be applicable.

First, we shall characterize the family of continuous right inverse linear operators of U^*.

L e m m a 1. *A continuous linear operator $T: A(\bar{D}) \to A(\bar{D})$ is a right inverse operator of the backward shift* (1), *iff it admits a representation of the form*

$$Tf(z) = zf(z) - \Phi_\zeta[\zeta f(\zeta)] \tag{2}$$

with a continuous linear functional Φ on $A(\bar{D})$, with $\Phi[1] = 1$.

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Proof. If $\Phi : A(\bar{D}) \to \mathbf{C}$ is an arbitrary continuous linear functional on $A(\bar{D})$, then it is evident that (2) is a continuous linear right inverse operator of U^*. The converse is not so evident. Let $T : A(\bar{D}) \to A(\bar{D})$ be an arbitrary continuous linear right inverse of U^*. Then it is evident, that $Tf(z) = zf(z) + \chi(f)$ with a continuous linear functional $\chi \in [A(\bar{D})]'$. A well known theorem of Sebastião e Silva [2] gives the form of these functionals. The continuous linear functionals on $A(\bar{D})$ are exactly the functionals of the form

$$\chi(f) = \frac{1}{2\pi i} \int_{C_f} L(\zeta) f(\zeta) d\zeta, \tag{3}$$

where $L(\zeta)$ is a function, analytic in $\mathbf{C} \setminus \bar{D}$ and C_f is a contour in the common domain of analiticity of $L(z)$ and $f(z)$, such that $\bar{D}$ lies inside it.

In order to represent χ in the form $\chi(f) = -\Phi_\zeta[\zeta f(\zeta)]$ with $\Phi[1] = 1$, we introduce the auxiliary functional $\Psi(f) = -(2\pi i)^{-1} \int_{C_f} \zeta^{-1} L(\zeta) f(\zeta) d\zeta$. Then it is easy to verify that the functional $\Phi(f) = (1 - \Psi[1]) f(0) + \Psi(f)$ has the property $\Phi[1] = 1$ and $\chi(f) = -\Phi_\zeta[\zeta f(\zeta)]$. Thus the lemma is proved.

Representation (2) of a right inverse of U is said to be the canonical representation.

Basic for our algebraic approach is the following convolutional operation in $A(\bar{D})$, connected with an arbitrary continuous linear right inverse operator T of U^* with the canonical representation (2).

Theorem 1. *If $\Phi \in [A(\bar{D})]'$ is a continuous linear functional with $\Phi[1] = 1$, then the operation*

$$(f * g)(z) = \Phi_\zeta \left\{ \frac{[zf(z) - \zeta f(\zeta)][zg(z) - \zeta g(\zeta)]}{z - \zeta} \right\} \tag{4}$$

is a continuous, bilinear, commutative and associative operation in $A(\bar{D})$, such that the operator T, defined by (2), *can be represented in the form*

$$Tf = \{1\} * f. \tag{5}$$

Proof. If we use a representation of the form (3) for Φ in (4), the contour $C_{f,g}$ should be chosen in such a way as to lie in the common domain of analiticity of $f(z)$ and $g(z)$. Then the separate continuity of (4) is evident. As for the bicontinuity of (4), i. e. $f_n \to f$ and $g_n \to g$ to imply $f_n * g_n \to f * g$ we can make it evident too, if we use the Cauchy formula for a representation of the divided difference $[zg(z) - \zeta g(\zeta)]/(z - \zeta)$:

$$\frac{zg(z) - \zeta g(\zeta)}{z - \zeta} = \frac{1}{2\pi i} \int_\Gamma \frac{\sigma g(\sigma) d\sigma}{(\sigma - z)(\sigma - \zeta)},$$

where Γ is a contour, lying in the domains of analyticity of $f(z)$ and $g(z)$ and containing the contour of the representation (3) for Φ inside.

The bilinearity and commutativity of (4) hold without assumption $\Phi[1] = 1$. We shall show that the same is true for the associativity too. We use the Fubini property

(6) $$\Phi_\xi \Phi_\eta\{f(\xi, \eta)\} = \Phi_\eta \Phi_\xi\{f(\xi, \eta)\}$$

valid for $f(\xi, \eta) \in A(\bar{D} \times \bar{D})$. Let f, g and h be arbitrary functions from $A(\bar{D})$. Directly, we get the identity $[(f*g)*h](z) = \Phi_\zeta \Phi_\xi\{K(f, g, h; z, \zeta, \xi)\}$ with

$$K(f, g, h; z, \zeta, \xi) = \frac{z[zf(z)-\xi f(\xi)][zg(z)-\xi g(\xi)][zh(z)-\zeta h(\zeta)]}{(z-\zeta)(z-\xi)}$$

$$-\frac{\zeta[\zeta f(\zeta)-\xi f(\xi)][\zeta g(\zeta)-\xi g(\xi)][zh(z)-\zeta h(\zeta)]}{(z-\zeta)(\zeta-\xi)}.$$

Using (6), we can make a summarization under functional's signs:

$$[(f*g)*h](z) = \frac{1}{2}\Phi_\zeta \Phi_\xi\{K(f, g, h; z, \zeta, \xi) + K(f, g, h; z, \xi, \zeta)\}.$$

Now, by a direct check, we verify the identity

$$K(f, g, h; z, \zeta, \xi) + K(f, g, h; z, \xi, \zeta) = K(h, g, f; z, \zeta, \xi) + K(h, g, f; z, \xi, \zeta),$$

thus proving the associativity relation $(f*g)*h = f*(g*h)$.

Let us note that till now we have made no use of the assumption $\Phi(1)=1$. It is needed only to ensure representation (5).

Corollary. *If Φ: $A(\bar{D}) \to \mathbf{C}$ is an arbitrary continuous linear functional, then operation* (4) *is continuous, bilinear, commutative and associative in $A(\bar{D})$.*

Lemma 2. *If Φ: $A(\bar{D}) \to \mathbf{C}$ is a linear functional with Fubini's property* (6), *then operation* (4) *has the property*

(7) $$\Phi(f*g) = 0 \quad \text{for} \quad f, g \in A(\bar{D}).$$

Proof. Let, provisionally, $k(z, \zeta)$ denotes the expression under the sign of the functional in (4). Evidently, $k(z, \zeta) = -k(\zeta, z)$. Then $\Phi(f*g) = \Phi_z \Phi_\zeta\{k(z, \zeta)\} = \Phi_\zeta \Phi_z\{k(z, \zeta)\} = -\Phi_\zeta \Phi_z\{k(\zeta, z)\} = -\Phi(f*g)$. Hence, $\Phi(f*g) = 0$.

Lemma 3. *If f, $g \in A(\bar{D})$, then*

(8) $$U^*(f*g) = (U^*f)*g + \Phi(f)g.$$

Proof. By an easy check. Let us note that (8) is a special case of a general identity for a right inverse operator with convolution [3, th. 1].

Lemma 4. *If Φ: $A(\bar{D}) \to \mathbf{C}$ is a non-zero continuous linear functional, then $A(\bar{D})$ with the multiplication* (4) *is an algebra without annihilators.*

Proof. The term "algebra without annihilators" is understood in a sense that if $f*g=0$ for all $f \in A(\bar{D})$, then $g=0$, or, informally, no non-zero element plays the role of 0 in the multiplication. In our case we shall show something more: there are non-divisors of zero in $A(\bar{D})$. Let us choose a $\lambda_0 \in \mathbf{C} \setminus \bar{D}$, such that $\Phi_\zeta[1/(\zeta-\lambda_0)] \neq 0$. Such λ_0 does exist. Indeed, if it were $\Phi_\zeta[1/(\zeta-\lambda)] \overset{\text{def}}{=} L(\lambda) \equiv 0$, then, according to the representation formula of the linear continuous functionals in $A(\bar{D})$ (see [2]) we would have $\Phi=0$, contrary to the hypothesis. The function $\varphi_0(z) = 1/(z-\lambda_0)$ is a non-divisor of zero of (4). Indeed, the operator

(9) $$T_0 f(z) = \left\{\frac{1}{z-\lambda_0}\right\} * f(z)$$

is a right inverse of $U_0^* = (U^* - 1/\lambda_0)/L(\lambda_0)$ in $A(\bar{D})$. Therefore, the equation $\varphi_0(z) * f(z) = 0$ is satisfied only by $f(z) \equiv 0$. Hence $A(\bar{D})$ with multiplication (4) is an algebra without annihilators.

Definition 1 [4, p. 13]. *An operator $M: A(\bar{D}) \to A(\bar{D})$ is said to be a multiplier of the convolution* (4), *iff the relation*

(10) $$(Mf) * g = f * (Mg)$$

holds for all $f, g \in A(\bar{D})$.

Lemma 5. *The multipliers of the convolution* (4) *are linear operators in $A(\bar{D})$ and they form a commutative algebra.*

Proof. Since $A(\bar{D})$ with multiplication (4) is an algebra without annihilators, the statement follows immediately from a general theorem [4, p. 13].

Now we shall characterize the multiplier algebra of (4) for an arbitrary non-zero continuous linear functional Φ.

Theorem 2. *An operator $M: A(\bar{D}) \to A(\bar{D})$ is a multiplier of a convolution* (4) *with non-zero continuous linear functional Φ, iff it can be represented in the form*

(11) $$Mf(z) = \mu f(z) + m(z) * f(z)$$

with $\mu = \text{const}$ and with $m(z) \in A(\bar{D})$. This representation is unique.

Proof. Let $\lambda_0 \in \mathbf{C} \setminus \bar{D}$ be chosen so that $\Phi_\zeta[1/(\zeta - \lambda_0)] = L(\lambda_0) \neq 0$. The possibility of such a choice is shown in the proof of lemma 4. Let $M: A(\bar{D}) \to A(\bar{D})$ be a multiplier of (4), and T_0 is the linear operator (9). Applying M to both sides of (9), we get

(12) $$T_0 M f = (M\varphi_0) * f,$$

where lemma 5 and relation (10) are used. But T_0 is a right inverse operator of $U_0 = (U^* - 1/\lambda_0)/L(\lambda_0)$. Applying this operator to both sides of (12) we get $Mf = [(U^* - 1/\lambda_0)/L(\lambda_0)][(M\varphi_0) * f]$. Using lemma 3, from the last equality we get a representation of the form (11).

Let us show that representation (11) is unique. Suppose that

(13) $$\mu_1 f(z) + m_1(z) * f(z) = \mu_2 f(z) + m_2(z) * f(z)$$

for all $f(z) \in A(D)$. First, let f be such that $\Phi(f) \neq 0$. Applying Φ to (13), we get $\mu_1 \Phi(f) = \mu_2 \Phi(f)$, using (7). Hence $\mu_1 = \mu_2$. Then (13) can be written in the form $[m_1(z) - m_2(z)] * f(z) = 0$. According to lemma 4, $m_1(z) - m_2(z) \equiv 0$ and the uniqueness is proved.

Conversely, if an operator M in $A(\bar{D})$ is of the form (11), it is evidently a linear continuous operator and a multiplier of (4).

Corollary 1. *Each multiplier of the convolution* (4) *is a continuous linear operator in $A(\bar{D})$.*

C o r o l l a r y 2. *If the continuous linear functional $\Phi(f)$ satisfies the condition $\Phi\{1\}=1$, then each multiplier of convolution* (4) *can be represented in the form*

$$Mf(z)=U^*(r(z)*f(z)) \tag{14}$$

with $r(z)=M\{1\}$.

P r o o f. In the same way, as of theorem 2, but with T instead of T_0. This time we should use relation (5) instead of (9).

In the next two theorems we suppose that $\bar{D}$ has a connected complement.

T h e o r e m 3. *Let Φ: $A(\bar{D})\to\mathbf{C}$ be a continuous non-zero linear functional. A continuous linear operator M: $A(\bar{D})\to A(\bar{D})$ has as an invariant subspace the hyperplane $A_\Phi=\{f\in A(\bar{D}),\ \Phi(f)=0\}$ and commutes with the backward shift operator U^* in A_Φ iff it is a multiplier of convolution* (4).

P r o o f. Let M: $A(\bar{D})\to A(D)$ be a continuous linear operator, and the hyperplane $\Phi(f)=0$ is its invariant subspace, i. e. if $\Phi(f)=0$, then $\Phi(Mf)=0$. Let us again choose $\lambda_0\in\mathbf{C}\setminus\bar{D}$ such that $L(\lambda_0)=\Phi_\zeta[1/(\zeta-\lambda_0)]\neq 0$. We shall show that the commutation relation $MU^*=U^*M$ in A_Φ implies the commutation relation $MT_0f=T_0Mf$ for $f\in A(\bar{D})$, where T_0 is defined by (9). Indeed, according to lemma 2, $\Phi(T_0f)=0$ for each $f\in A(\bar{D})$. Hence

$$M\{(U^*-1/\lambda_0)/L(\lambda_0)\}T_0f=\{(U^*-1/\lambda_0)/L(\lambda_0)\}MT_0f$$

or $Mf=\{(U^*-1/\lambda_0)/L(\lambda_0)\}MT_0f$. Let us apply T_0 to both sides of the last identity. Using the fact that T_0 and $U_0=(U^*-1/\lambda_0)/L(\lambda_0)$ are mutually inverse on A_Φ, we get $T_0Mf=(T_0U_0)MT_0f=MT_0f$, since $\Phi(MT_0f)=0$ for $f\in A(\bar{D})$.

In order to prove the multiplier relation (10) we begin with the evident identity $(M\varphi_0)*\varphi_0=\varphi_0*(M\varphi_0)$. Using the associativity of $f*g$ and the commutation relation $MT_0=T_0M$, we obtain the following series of identities:

$$M(T_0^m\varphi_0)*(T_0^n\varphi_0)=(T_0^m\varphi_0)*M(T_0^n\varphi_0), \qquad m, n=0, 1, 2, \ldots$$

From the bilinearity of $f*g$ it follows that multiplier relation (10) holds for functions of the form $f_m=\Sigma_{k=0}^m a_{m,k}T_0^k\varphi_0$ and $g_n=\Sigma_{l=1}^n b_{n,l}T_0^l\varphi_0$, where $a_{m,k}$ and $b_{n,l}$ are constants, i. e. $(Mf_m)*g_n=f_m*(Mg_n)$. It remains to show that the linear span of $\{T_0^n\varphi_0\}_{n=1}^\infty$ is dense in $A(\bar{D})$ in its inductive topology. Each function $T_0^n\varphi_0$ is a polynomial of $1/(z-\lambda_0)$, i. e. $T_0^n\varphi_0(z)=\Sigma_{k=0}^n c_{n,k}[1/(z-\lambda_0)]^k$ with $c_{n,n}=-\lambda_0^nL(\lambda_0)\neq 0$. Let Ψ: $z\mapsto 1/(z-\lambda_0)$. If $k(z)$ is an arbitrary function from $A(\bar{D})$, then the function $h(\zeta)=k(\lambda_0+1/\zeta)$ is from $A(\overline{\Psi(D)})$. According to Runge's approximation theorem, there exists a sequence of polynomials $p_n(\zeta)=\Sigma_{k=0}^n a_{n,k}\zeta^k$ converging to $h(\zeta)$ in the inductive topology of $A(\overline{\Psi(D)})$. Then the sequence $q_n(z)=p_n[1/(z-\lambda_0)]$ converges to $k(z)$ in $A(\bar{D})$. Hence the linear span of $\{T_0^n\varphi_0\}_{n=0}^\infty$ is dense in $A(\bar{D})$. Now, using the continuity of $f*g$ in $A(\bar{D})$, we can assert that the multiplier relation (14) holds in $A(\bar{D})$.

Conversely, if M: $A(\bar{D})\to A(\bar{D})$ is a multiplier of the convolution (4), then, according to theorem 2, it has a representation of the form (11). The operators (11) evidently, have A_Φ as an invariant subspace. If $\Phi(f)=0$, then, by (8), it follows

$$U^*Mf=\mu U^*f+m*(U^*f)+\Phi(f)m=MU^*f,$$

i. e. $U^*M=MU^*$ in A_Φ. The theorem is proved.

Corollary. *A linear continuous operator* $M\colon A(\bar{D})\to A(\bar{D})$ *with an invariant hyperplane* $\Phi(f)=0$ *with continuous linear functional* Φ, *commutes with* U^* *in this hyperplane, iff* M *is an operator of the form* (11).

Now we shall characterize all continuous convolutions of the right inverse operator T of U^*.

Definition 2. *A bilinear, commutative and associative operation* $f\tilde{*}g$ *in* $A(\bar{D})$ (*i. e.* $\tilde{*}\colon A(\bar{D})\times A(\bar{D})\to A(\bar{D})$) *is said to be a convolution of the operator* T, *defined by* (2), *iff the relation* $T(f\tilde{*}g)=(Tf)\tilde{*}g$ *holds for all* $f, g\in A(\bar{D})$.

Such definition of convolution of linear operator, mapping a linear space into itself is given by one of the authors in [5].

Now we shall find all separately continuous convolutions of T in $A(\bar{D})$. Let us remind, that a convolution $f\tilde{*}g$ in $A(\bar{D})$ is said to be separately continuous, iff $f_n\to f$ in $A(\bar{D})$ implies $f_n\tilde{*}g\to f\tilde{*}g$ for all $g\in A(\bar{D})$.

Theorem 4. *A continuous bilinear, commutative and associative operation* $f\tilde{*}g$ *in* $A(\bar{D})$ *is a convolution of the right inverse operator* T *of* U^*, *defined by* (2), *iff it has the representation*

$$f\tilde{*}g=U^{*2}(r*f*g)\ \text{with}\ r=1\tilde{*}1\in A(\bar{D}), \tag{15}$$

where by $*$ *the operation* (4) *is denoted.*

Proof. Let $f\tilde{*}g$ be a continuous convolution of T in $A(\bar{D})$. If we fix f, we may look on $f\tilde{*}g$ as on an operator M_fg, defined on $A(\bar{D})$, i. e. $M_fg=f\tilde{*}g$. From the convolution relation $T(f\tilde{*}g)=(Tf)\tilde{*}g$ it follows that $M_fT=TM_f$, i. e. M_f and T commute on $A(\bar{D})$. As in the proof of theorem 3, it follows that M_f is a multiplier of the convolution $f*g$. Hence, it has the representation (14), i. e.

$$M_fg=f\tilde{*}g=U^*[(M_f\{1\})*g].$$

In particular, $M_f\{1\}=M_{\{1\}}f=U^*\{M_{\{1\}}\{1\}*f\}=U^*[(1\tilde{*}1)*f]$. Therefore,

$$f\tilde{*}g=U^*[U^*((1\tilde{*}1)*f)*g].$$

Now, using formula (8) and lemma 2 we get (15).

Conversely, if $r\in A(\bar{D})$ is arbitrary, the operation in $A(\bar{D})$ defined by (15) is a (continuous) convolution of T.

Especially interesting are the following two convolutions of T in $A(\bar{D})$.

Corollary 1. *The operation*

$$f\tilde{*}g(z)=z\Phi_\zeta\left\{\frac{[f(z)-f(\zeta)]\,[g(z)-g(\zeta)]}{z-\zeta}\right\}+\Phi(f)g(z)+\Phi(g)f(z)-\Phi\{f(\zeta)g(\zeta)\} \tag{16}$$

is a convolution of T *in* $A(\bar{D})$ *with function* 1 *as unit element, i. e.* $1\tilde{*}f=f$ *for all* $f\in A(\bar{D})$.

Proof. Let us take $r=1$ in representation formula (15). We get $f\tilde{*}g=U^*(f*g)$. By formula (8) and some elementary algebra, we arrive to (16).

Corollary 2. *The operation*

$$f\,\widetilde{\widetilde{*}}\,g(z)=\Phi_\zeta\left\{\frac{[f(z)-f(\zeta)]\,[g(z)-g(\zeta)]}{z-\zeta}\right\}+\Phi(f)U^*g+\Phi(g)U^*f$$

is a convolution of T *in* $A(\bar{D})$ *with unit element* $z-\Phi(\zeta)$, *i. e.* $[z-\Phi(\zeta)]\,\widetilde{\widetilde{*}}\,f=f$ *for each* $f\in A(\bar{D})$.

This convolution is received from (15) by the choice

$$r=T\{1\}=z-\Phi(\zeta).$$

At last, we shall consider an application of the above results to the problem of expanding of functions from $A(\bar{D})$ in a system of fractions $\{1/(z-\lambda_n)\}_{n=1}^{\infty}$ with prescribed poles. In the recent years this problem has been studied by T. A. Leontieva [6].

Let $\Phi\colon A(\bar{D})\to\mathbf{C}$ be a continuous linear functional in $A(\bar{D})$ with Sebastião e Silva's representation

$$\Phi(f)=\frac{1}{2\pi i}\int_{C_f}L(\lambda)f(\lambda)d\lambda, \tag{17}$$

where $L(\lambda)$ is a function, analytic in $\mathbf{C}\setminus\bar{D}$ with $L(\infty)=0$. Without loss of generality, we may assume that $\lim_{z\to\infty}zL(z)=1$, i. e. $\Phi(1)=1$. Let $L(\lambda)$ has an infinite sequence of zeros $\lambda_1, \lambda_2, \ldots, \lambda_n, \ldots$ which certainly cluster on $\partial\bar{D}$.

The functional (17) determines the right inverse operator of U^* given by $Tf(z)=zf(z)-\Phi[\zeta f(\zeta)]$.

Let us denote

$$\varphi_n(z)=1/(z-\lambda_n),\qquad n=1, 2, \ldots \tag{18}$$

Lemma 6. *If* $f(z)\in A(\bar{D})$, *then*

$$f*\varphi_n=\chi_n(f)\varphi_n,\qquad n=1, 2, \ldots, \tag{19}$$

where $\chi_n(f)$ *are continuous linear functionals, given by*

$$\chi_n(f)=\frac{\lambda_n}{2\pi i}\int_{C_f}\frac{\lambda L(\lambda)f(\lambda)}{\lambda-\lambda_n}d\lambda\cdot \tag{20}$$

Proof. By a direct check, using (4) and the condition $\Phi(1)=1$.

Theorem 5. *Linear functionals* (20) *are multiplicative with respect to convolution* (4), *i. e.* $\chi_n(f*g)=\chi_n(f)\chi_n(g)$, $n=1, 2, \ldots$ *for all* $f, g\in A(\bar{D})$.

Proof. Let us multiply (19) convolutionally by g. Using the associativity of (4), we get $(f*g)*\varphi_n=\chi_n(f)\,(g*\varphi_n)$ or $\chi_n(f*g)\varphi_n=\chi_n(f)\chi_n(g)\varphi_n$, where again (19) is used, this time for $f*g$, and g.

Corollary. *The system* (18) *is convolutionally orthogonal with respect to the corresponding convolution* (4), *i. e.*

$$\varphi_m*\varphi_n=\begin{cases}0, & m\neq n;\\ \lambda_n^2L'(\lambda_n), & m=n.\end{cases}$$

T. A. Leontieva [6] considered formal expansions of the functions $(z)\in A(\bar{D})$ of the form

$$f(z)\sim\sum_{n=1}^{\infty}\frac{\chi_n(f)}{\lambda_n^2L'(\lambda_n)}\frac{1}{z-\lambda_n} \tag{21}$$

and gave sufficent conditions in order the series in (21) to represent $f(z)$ in D. Later on, we will not enter into details about the convergence problem for the series (21). In order to ensure an uniqueness theorem, we shall suppose the convergence of T. A. Leontieva's expansion (21) only in a neighbourhood of a point in D. These neighbourhoods may be different for different functions, but they should contain a common point.

Lemma 7. *If T. A. Leontieva's expansion* (21) *for each* $f\in A(\bar{D})$ *is convergent to* f *in a neighbourhood of one and the same point of* D, *then expansion* (21) *is unique, i. e.* $\chi_n(f)=0$ *for* $n=1, 2, \ldots$ *imply* $f\equiv 0$.

Proof. Evident, from the principle of analytic continuation.

Later on, we suppose the uniqueness of (21).

Theorem 6. *If* $M\colon A(\bar{D})\to A(\bar{D})$ *is a multiplier of the convolution* (4) *then there exists a numerical sequence* $\{\mu_n\}_{n=1}^{\infty}$, *such that* $\chi_n(Mf)=\mu_n\chi_n(f)$, $n=1, 2, \ldots$ *for each* $f\in A(\bar{D})$.

Proof. From representation (11) and relation (21), we have

$$\chi_n(Mf)=\mu\chi_n(f)+\chi_n(m*f)=\mu\chi_n(f)+\chi_n(m)\chi_n(f).$$

Hence $\mu_n=\mu+\chi_n(m)$, where $\mu=$const, and $m\in A(\bar{D})$.

Definition 3. *An operato r*M: $A(\bar{D})\to A(\bar{D})$ *is said to be a coefficient multiplier of T. A. Leontieva's expansion* (21), *iff the expansion of* Mf *is of the form*

$$Mf(z)\sim\sum_{n=1}^{\infty}\frac{\mu_n\chi_n(f)}{\lambda_n^2L'(\lambda_n)}\frac{1}{z-\lambda_n},$$

where $\{\mu_n\}_{n=1}^{\infty}$ *is a numerical sequence, not depending on* f.

Theorem 6 asserts that each multiplier of convolution (4) is a coefficient multiplier of T. A. Leontieva's expansion too.

Theorem 7. *If the expansion* (21) *is unique in* $A(\bar{D})$, *then each coefficient multiplier of it is a multiplier of* (4) *too.*

Proof. Let $f, g\in A(\bar{D})$ be arbitrary, and $M\colon A(\bar{D})\to A(\bar{D})$ is a coefficient multiplier of T. A. Leontieva's expansion (21). Then

$$\chi_n[M(f*g)-(Mf)*g]=\mu_n\chi_n(f*g)-\chi_n(Mf)\chi_n(g)=\mu_n[\chi_n(f)\chi_n(g)-[\mu_n\chi_n(f)]\chi_n(g)=0$$

for $n=1, 2, \ldots$. From the hypothesis, that there it is ensured the uniqueness of (21), it follows $M(f*g)-(Mf)*g=0$. Hence M is a multiplier of the convolution $f*g$.

Corollary. *An operator* $M\colon A(\bar{D})\to A(\bar{D})$ *is a coefficient multiplier of T. A. Leontieva's expansion* (21), *iff it has a representation of the form* $Mf(z)=\mu f(z)+m(z)*f(z)$ *with* $\mu=$const, *and* $m(z)\in A(\bar{D})$.

From the last representation it follows that the coefficient multipliers of T. A. Leontieva's expansion are continuous linear operators.

It seems that the algebraic approach, proposed here, could be used for study of other problems, related to T. A. Leontieva's expansion too.

REFERENCES

1. R. G. Douglas, H. S. Shapiro, A. L. Shields. Cyclic vectors and invariant subspaces for the backward shift operator. *Ann. Inst. Fourier*, **20**, 1970, 37—76.
2. J. Sebastião e Silva. As funções analiticas e a análise functional. *Port. Math.*, 9, 1950, 1—130.
3. I. H. Dimovski. Convolutions of right inverse operators and representation of their multipliers. *C. R. Acad. bulg. Sci.*, **31**, 1978, 1377—1380.
4. R. Larsen. An introduction to the theory of multipliers. Berlin—Heidelberg—New York, 1971.
5. I. H. Dimovski. On an operational calculus for a differential operator. *C. R. Acad. bulg. Sci.*, **21**, 1968, 513—516.
6. Т. А. Леонтьева. Представление функций, аналитических в замкнутой области, рядами рациональных функции. *Мат. заметки*, 4, 1968, 191—200.

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