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CONVERGENCE RATE FOR SPLINE COLLOCATION TO FREDHOLM INTEGRAL EQUATION OF SECOND KIND

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An error estimation is obtained in L_p $(1 \leqq p < \infty)$ norm for the approximate solution of Fredholm integral equation by means of linear, quadratic and cubic splines, determined by the collocation method. In the case, when the solution of the integral equation has derivatives, the estimations between them and the derivatives of the splines are given.

We shall suppose that the equation

$$(1) \qquad y(x)-\lambda\int_0^1 K(x, t)\, y(t)dt=f(x), \quad 0\leqq x\leqq 1,$$

has an unique solution, i. e. λ is not an eigen-value and let us denote $x_i=ih$, $i=0, 1, \dots, n$, $\bar{x}_i=x_i-h/2$, $i=0, 1, \dots, n+1$, $\bar{x}_0=0$, $\bar{x}_{n+1}=1$, $h=1/n$. By S_k, $k=1, 2, 3$, we shall denote respectively linear, quadratic and cubic spline, if

a. $S_k \in C^{k-1}_{[0,\, 1]}$;

(2) b. S_k is polynomial of degree $\leqq k$ in $\begin{cases} [x_i, x_{i+1}], & i=0, \dots, n-1, \quad k=1, 3, \\ [\bar{x}_i, \bar{x}_{i+1}], & i=0, \dots, n, \quad k=2. \end{cases}$

If in addition S_k satisfies the conditions $S_k(x_i)=g(x_i)$, $i=0, 1, \dots, n$, for a given bounded function g in $[0, 1]$ we say S_k is interpolating spline for g.

For simplicity we assume that the solution y of (1) is 1-periodic function.

Lemma 1. *Let S_k, $k=1, 2, 3$, be 1-periodic spline, defined by* (2). *Then the following estimate* $\|S_k\|_{C[0,\, 1]}\leqq R_k \max_{0\leqq i\leqq n} |S_k(x_i)|$ *holds, where* $R_1=1$, $R_2=R_3=5$.

Proof. For $x \in [x_i, x_{i+1}]$ it follows $S_1(x)=(x_{i+1}-x)/h \,.\, S_1(x_i)+(x-x_i)/h \,.\, S_1(x_{i+1})$ and it is clear that $R_1=1$. In [1, p. 69] the case $k=2$ is proved. In order to find R_3 we shall use the representation of S_3 in $[x_{i-1}, x_i]$, [1, p. 83],

$$(3) \qquad S_3(x)=M_{i-1}\frac{(x_i-x)^3}{6h}+M_i\,\frac{(x-x_{i-1})^3}{6h}+\Big(S_3(x_{i-1})-\frac{M_{i-1}h}{6}\Big)\frac{x_i-x}{h}$$
$$+\Big(S_3(x_i)-\frac{M_i h}{6}\Big)\frac{x-x_i}{h},$$

where $M_i=S_3''(x_i)$. Denoting $\alpha(x)=(x_i-x)/h$, $\beta(x)=(x-x_{i-1})/h$, from (3) and $\alpha(x)+\beta(x)=1$ we get

$$(4) \qquad \begin{aligned} |S_3(x)|&\leqq |S_3(x_{i-1})|\alpha(x)+|S_3(x_i)|\beta(x)+\frac{h^2}{6}\,(|M_i|\alpha^3(x)+|M_{i-1}|\beta^3(x)) \\ &+\frac{h^2}{6}\,(|M_{i-1}|\alpha(x)+|M_i|\beta(x))\leqq \max_{0\leqq i\leqq n}|S_3(x_i)|+\frac{h^2}{3}\max_{0\leqq i\leqq n}|M_i|. \end{aligned}$$

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From the system [1, p. 84]

$$M_{i-1}+4M_i+M_{i+1}=12S_3(x_{i-1},\ x_i,\ x_{i+1}),\quad i=1,\ 2,\ldots,\ n$$
$$M_0=M_n,\ M_1=M_{n+1}$$

and from lemma 1 in [1, p. 29] it follows that

(5) $$\max_{0\le i\le n}|M_i|\le 5\max_{0\le i\le n}|S_3(x_{i-1},\ x_i,\ x_{i+1})|\le\frac{12}{h^2}\max_{0\le i\le n}|S_3(x_i)|.$$

The inequalities (4) and (5) give $R_3=5$.

Let S_k be determined by

(6) $$S_k(x_i)-\lambda\int_0^1 K(x_i,\ t)S_k(t)dt=f(x_i),\quad i=0,\ 1,\ldots,n,$$

and by the periodic conditions (when $k=2,\ 3$)

(7) $$S_k^{(l)}(0)=S_k^{(l)}(1),\quad l=1,\ 2.$$

The conditions (6), (7) determine entirely coefficients $\{a_i\}$, $\{b_i\}$, $\{c_i\}$ in the representation of S_k, $k=1,\ 2,\ 3$, on B-splines,

$$S_1(x)=\sum_{i=0}^{n}a_iB_{1,i}(x),\ S_2(x)=\sum_{i=-1}^{n+1}b_iB_{2,i}(x),\ S_3(x)=\sum_{i=-1}^{n+1}c_iB_{3,i}(x).$$

Let $\bar{S}_k$, $k=1,\ 2,\ 3$, be interpolated spline for the solution y of (1). Then

(8) $$\bar{S}_k(x_i)=\lambda\int_0^1 K(x_i,\ t)\bar{S}_k(t)dt-\lambda\int_0^1 K(x_i,\ t)(\ y(t)-\bar{S}_k(t))dt=f(x_i),\quad i=0,1,\ldots,n.$$

If we denote $\varphi_k=\bar{S}_k-S_k$, $k=1,\ 2,\ 3$, then subtracting (7) from (8) we have

(9) $$\varphi_k(x_i)=\lambda\int_0^1 K(x_i,\ t)\varphi_k(t)dt+\lambda\int_0^1 K(x_i,\ t)(\ y(t)-\bar{S}_k(t))dt,\quad i=0,\ 1,\ldots,\ n.$$

From Lemma 1 it follows that

(10) $$\frac{1}{R_k}\|\varphi_k\|_{C[0,\ 1]}\le\max_{0\le i\le n}|\varphi_k(x_i)|.$$

Applying Hölder inequality we get from (9)

$$|\varphi_k(x_i)|\le\|\varphi\|_{C[0,\ 1]}|\lambda|\max_{0\le x\le 1}\int_0^1|K(x,t)|dt+|\lambda|\,.\,\|K(x_i,\,.\,)\|_{L_q}\|\,y-\bar{S}_k\|_{L_p[0,\ 1]},\ \frac{1}{p}+\frac{1}{q}=1$$

and in view of (10) follows

$$\|\varphi_k\|_{C[0,\ 1]}\le(R_k|\lambda|\,.\,K_q\|\bar{S}_k-y\|_{L_p[0,\ 1]})/(1-\rho R_k),$$

where $\rho=|\lambda|\max\limits_{0\le x\le 1}\int_0^1|K(x,\ t)|dt$, $K_q=\max\limits_{0\le x\le 1}\|K(x,\,.\,)\|_{L_q[0,\ 1]}$.

From the last inequality and $\|\,y-S_k\|_{L_p[0,\ 1]}\le\|\,y-\bar{S}_k\|_{L_p[0,\ 1]}+\|\varphi_k\|_{L_p[0,\ 1]}$ it follows under the condition $1-\rho R_k>0$

$$(11)\qquad \|y-S_k\|_{L_p[0,1]}\le\Big(1+\frac{|\lambda|R_kK_q}{1-\rho R_k}\Big)\|y-\bar S_k\|_{L_p[0,1]}.$$

In [3] is proved the following

Theorem A. *If 1-periodic, bounded function $f\in L_p[0,1]$, then*

$$\|f-\bar S_k\|_{L_p[0,1]}\le c_1\tau_{k+1}(f;h)_{L_p[0,1]},\quad k=2,3,$$

where S_k is interpolating quadratic or cubic spline for the function f on equidistant set of points.

Without restriction for periodicity in a similar way as in [3] can be proved

$$(12)\qquad \|f-S_1\|_{L_p[0,1]}\le c_2\tau_2(f;h)_{L_p[0,1]}.$$

The modulus $\tau_k(f;\delta)_{L_p}$ in above estimations is defined as [2]: $\tau_k(f;\delta)_{L_p[0,1]}=\|\omega_k(f,\cdot;\delta)\|_{L_p[0,1]}$, where

$$\omega_k(f,x;\delta)=\sup\{|\Delta_h^k f(t)|,\ t,\ t+kh\in[x-k\delta/2,\ x+k\delta/2]\cap[0,1]\}.$$

From (11), (12) and Theorem A we obtain.

Theorem 1. *For the approximate solution S_k of the equation* (1), *where S_k is determined by* (6), (7), *the following estimate*

$$\|y-S_k\|_{L_p[0,1]}\le c(k)\tau_{k+1}(y;h)_{L_p[0,1]},\quad k=1,2,3,$$

holds under the condition $1-\rho R_k>0$.

In the case $k=1$ the condition for periodicity is not necessary.

From Theorem 1 and the properties of the modulus $\tau_k(f;\delta)_{L_p[0,1]}$ [2, 3, 4]

$$(13)\qquad \tau_k(f;\delta)_{L_p[0,1]}\le c(k)\delta\omega_{k-1}(f';\delta)_{L_p[0,1]},\quad \tau_1(f;\delta)_{L_p[0,1]}\le\delta\|f'\|_{L_p[0,1]},$$

$$\tau(f;\delta)_{L_1[0,1]}\le 2\delta\bigvee_0^1 f$$

($\bigvee_0^1 f$ is the variation of f) series of corollaries can be obtained.

In the case when the solution y of (1) has derivatives we shall find estimations for $\|y^{(i)}-S_k^{(i)}\|_{L_p[0,1]}$, $k=1,2,3$, $i=1,\dots,k$.

We introduce the modified Steklov's function [5] $f_{k,h}$ for a given function f defined on $[0,1+kh]$:

$$f_{k,h}(x)=\frac{(-1)^{k-1}}{h^k}\int_0^h\cdots\int_0^h\Big[f(x+t_1+\cdots+t_k)-\binom{k}{1}f\Big(x+\frac{k-1}{k}(t_1+\cdots+t_k)\Big)$$
$$+\cdots+(-1)^{k-1}\binom{k}{k-1}f\Big(x+\frac{t_1+\cdots+t_k}{k}\Big)\Big]dt_1dt_2\cdots dt_k.$$

It is clear that

$$\|f-f_{k,h}\|_{L_p[0,1]}\le\frac{1}{h^k}\int_0^h\cdots\int_0^h\Big(\int_0^h|\Delta^k_{\frac{t_1+\cdots+t_k}{k}}f(x)|^pdx\Big)^{1/p}dt_1\cdots dt_k$$

$$(14)\qquad \le\sup_{0<t\le h}\Big(\int_0^1|\Delta_t^kf(x)|^pdx\Big)^{1/p}\le\sup_{0<t\le h}\Big(\int_0^{1+kh-kt}|\Delta_t^kf(x)|^pdx\Big)^{1/p}\le\omega_k(f;h)_{L_p[0,1+kh]}$$

Analogously, if $f\in L_p[-kh, 1]$, then

(15) $$\|f_{k,-h}-f\|_{L_p[0,1]}\leq\omega_k(f;h)_{L_p[-kh,1]},\quad h>0,$$

and it is evident that if $f^{(k)}\in L_p[0, 1+kh]$ ($f^{(k)}\in L_p[-kh, 1]$, then

(16) $$\|f^{(r)}-f^{(r)}_{k,h}\|_{L_p[0,1]}\leq\omega_k(f^{(r)};h)_{L_p[0,1+kh]},\quad 0\leq r\leq k,$$

(17) $$\|f^{(r)}-f^{(r)}_{k,-h}\|_{L_p[0,1]}\leq\omega_k(f^{(r)};h)_{L_p[-kh,1]},\quad 0\leq r\leq k$$

Lemma 2. *If $f^{(k)}\in L_p[0, 1+kh]$, $h>0$, $0\leq r\leq k$, then*

$$\|f^{(r)}_{k,h}\|_{L_p[0,1]}\leq c(k,r)h^{-r}\|f\|_{L_p[0,1+kh]}.$$

Proof. From

$$f^{(r)}_{k,h}(x)=\frac{(-1)^{k-1}}{h^k}\int_0^h\cdots\int_0^h[f^{(r)}(x+t_1+\cdots+t_k)-\binom{k}{1}f^{(r)}(x+\frac{k-1}{k}(t_1+\cdots+t_k))$$

$$+\cdots(-1)^{k-1}\binom{k}{k-1}f^{(r)}(x)+\frac{t_1+\cdots+t_k}{k})]dt_1\cdots dt_k=\frac{(-1)^{k-1}}{h^k}\int_0^h\cdot$$

$$\cdots\int_0^h[\Delta^r_h f(x+t_1+\cdots+t_{k-r})-\left(\frac{k}{k-1}\right)^r\binom{k}{1}\Delta^r_{\frac{k-1}{k}h}f(x+\frac{k-1}{k}(t_1+\cdots+t_{k-r}))$$

$$+\cdots+(-1)^{k-1}k^{k-1}k^r\binom{k}{k-1}\Delta^r_{\frac{h}{k}}f(x+\frac{t_1+\cdots+t_{k-r}}{k})]dt_1\cdots dt_{k-r},$$

we get

$$\|f^{(r)}_{k,h}\|_{L_p[0,1]}\leq\frac{1}{h^k}\int_0^h\cdots\int_0^h\Big(\int_0^1|\Delta^r_h f(x+t_1+t_{k-r})+\cdots+$$

$$k^r(-1)^{k-1}\binom{k}{k-1}\Delta^r_{\frac{h}{k}}f(x+\frac{t_1+\cdots+t_{k-r}}{k})|^p\,dx\Big)^{1/p}dt_1\cdots dt_{k-r}\leq 2^r h^{-r}(\|f\|_{L_p[0,1+kh]}$$

$$+\left(\frac{k}{k-1}\right)^r\|f\|_{L_p[0,1+(k-1)h]}+\cdots k^r\binom{k}{k-1}\|f\|_{L_p[0,1+h]}\leq c(k,r)h^{-r}\|f\|_{L_p[0,1+kh]}.$$

Lemma 2′. *If $f^{(k)}\in L_p[-kh, 1]$, $h>0$, $0\leq r\leq k$, then*

$$\|f^{(r)}_{k,-h}\|_{L_p[0,1]}\leq c(k,r)h^{-r}\|f\|_{L_p[-kh,1]}.$$

The proof of this Lemma is similar to the proof of Lemma 2.

Lemma 3. *Let $f^{(k)}\in L_p[0, 1+kh]$, $h>0$. For every polynomial $p\in H_{k-1}$ and $0\leq r\leq k$ the following estimate*

$$\|f^{(r)}-P^{(r)}\|_{L_p[0,1]}\leq\omega_k(f^{(r)};h)_{L_p[0,1+kh]}+c(r,k)h^{-r}\|f-P\|_{L_p[0,1+kh]}$$

holds.

Proof. For $x\in[0, 1]$: $f(x)-P(x)=f(x)-f_{k,h}(x)+f_{k,h}(x)-P_{k,h}(x)+P_{k,h}(x)-P(x)$ and since $\|P_{k,h}-P\|_{L_p[0,1]}\leq\omega_k(P;h)_{L_p[0,1+kh]}=0$ it follows

(18) $$\|f^{(r)}-P^{(r)}\|_{L_p[0,1]}\leq\|f^{(r)}-f^{(r)}_{k,h}\|_{L_p[0,1]}+\|f^{(r)}_{k,h}-P^{(r)}_{k,h}\|_{L_p[0,1]}\,.$$

From (16), (18) and Lemma 2 we have

$$\|f^{(r)}-P^{(r)}\|_{L_p[0,1]}\leq\omega_k(f^{(r)};h)_{L_p[0,1+kh]}+c(k,r)h^{-r}\|f-P\|_{L_p[0,1+kh]}.$$

Analogously to Lemma 3 we obtain

Lemma 3'. *Let $f^{(k)}\in L_p[-kh,1]$, $h>0$. For every $P\in H_{k-1}$, $0\leq r\leq k$, the following estimate*

$$\|f^{(r)}-P^{(r)}\|_{L_p[0,1]}\leq\omega_k(f^{(r)};h)_{L_p[-kh,1]}+c(k,r)h^{-r}\|f-P\|_{L_p[-kh,1]}$$

holds.

Lemma 4. *Let $f\in L_p[0,1]$ and $0<t\leq 1/2kn$. Then*

$$\sum_{i=0}^{n-1}\omega_k^p(f;t)_{L_p[\frac{i}{n},\frac{i+1}{n}]}\leq c(k)\omega_k^p(f;t)_{L_p[0,1]}.$$

Proof. Let us denote

$$\omega_k(f,x;\delta)_q=\Big(\frac{1}{2\delta}\int_{-\delta}^{\delta}|\Delta_v^k f(x)|^q dv\Big)^{1/q},$$

where $\Delta_v^k f(x)=0$ if the finite difference $\Delta_v^k f(x)$ is not defined. We define $\tau_k(f;\delta)_{q,p[0,1]}=\|\omega_k(f,\cdot;\delta)_q\|_{L_p[0,1]}$. In [6] it is proved that there exist constants $c_1(k)$ and $c_2(k)$, for which $c_1(k)\tau_k(f;\delta)_{p,p[0,1]}\leq\omega_k(f;\delta)_{L_p[0,1]}\leq c_2(k)\tau_k(f;\delta)_{p,p[0,1]}$, $\delta\leq 1/2k$. Hence

$$\sum_{i=0}^{n-1}\omega_k^p(f;t)_{L_p[\frac{i}{n},\frac{i+1}{n}]}\leq c_2^p\sum_{i=0}^{n-1}\tau_k^p(f;t)_{p,p[\frac{i}{n},\frac{i+1}{n}]}=c_2^p\sum_{i=0}^{n-1}\int_{i/n}^{i+1/n}\omega_k^p(f,x;t)_p dx$$

$$\leq c_2^p\int_0^1\omega_k^p(f,x;t)_p dx=c_2^p\tau_k(f;t)_{p,p[0,1]}\leq(c_1^{-1}c_2)^p\omega_k^p(f;t)_{L_p[0,1]}.$$

Theorem 2. *Let the solution y of the equation* (1) *be 1-periodic and $y^{(r)}\in L_p[0,1]$. Then under the condition $1-\rho R_k>0$ the following estimation*

$$\|y^{(r)}-S_m^{(r)}\|_{L_p[0,1]}=O(\omega_{m+1-r}(y^{(r)};h)_{L_p[0,1]}),\quad r=1,\dots,m,$$

holds, where S_m, $m=1,2,3$, is respectively linear, quadratic or cubic spline, determined by the conditions (6), (7).

Proof. We shall prove the theorem only for $m=3$. In the other two cases the proof is analogous.

For $x\in[x_i,x_i+h/2]$, $i=0,1,\dots,n-1$, $k=4$, $t\leq h/8$, $1\leq r\leq 3$, we get from Lemma 3

$$\|y^{(r)}-S_3^{(r)}\|_{L_p[x_i,x_i+h/2]}\leq\omega_4(y^{(r)};t)_{L_p[x_i,x_i+h/2+4t]}+c(r)t^{-r},$$

(19)

$$\|y-S_3\|_{L_p[x_i,x_i+h/2+4t]}\leq\omega_4(y^{(r)};t)_{L_p[x_i,x_{i+1}]}+c_1(r)h^{-r}\|y-S_3\|_{L_p[x_i,x_{i+1}]}$$

and similary for $x\in[x_i+h/2,x_{i+1}]$, $i=0,1,\dots,n=1$, using Lemma 3' we have

(20) $$\|y^{(r)}-S_3^{(r)}\|_{L_p[x_i+h/2,x_{i+1}]}\leq\omega_4(y^{(r)};t)_{L_p[x_i,x_{i+1}]}+c_2(r)h^{-r}\|y-S_3\|_{L_p[x_i,x_{i+1}]}.$$

From (19) and (20) follows

$$\int_{x_i}^{x_{i+1}} |y^{(r)}(x)-S_3^{(r)}(x)|^p dx \leq$$

$$c(r, p)[\omega_4^p(y^{(r)}; t)_{L_p[i/n, i+1/n]}+h^{-rp}\|y-S_3\|^p_{L_p[x_i, x_{i+1}]}]$$

and consequently

(21) $$\|y^{(r)}-S^{(r)}\|^p_{L_p[0,1]} \leq c(r, p)[\sum_{i=0}^{n-1} \omega_4^p(y^{(r)}; t)_{L_p[x_i, x_{i+1}]}+h^{-rp}\|y-S_3\|^p_{L_p[0,1]}].$$

Using the properties (13) of the modulus $\tau_k(f; \delta)_{L_p}$ and Theorem 1 we get

(22) $$h^{-rp}\|y-S_3\|^p_{L_p[0,1]} \leq c_3(r)\omega^p_{4-r}(y^{(r)}; h)_{L_p[0,1]}.$$

From Lemma 4 it follows

(23) $$\sum_{i=0}^{n-1} \omega_4^p(y^{(r)}; t)_{L_p[x_i, x_{i+1}]} \leq c_4\omega_4^p(y^{(r)}; h)_{L_p[0,1]}.$$

Using (21), (22), (23) and the property $\omega_4^p(y^{(r)}; h)_{L_p[0,1]} \leq c_5(p)\omega^p_{4-r}(y^{(r)}; h)_{L_p[0,1]}$ we finally obtain

$$\|y^{(r)}-S_3^{(r)}\|_{L_p[0,1]} = O(\omega_{4-r}(y^{(r)}; h)_{L_p[0,1]}).$$

When the solution y of (1) is not a periodic function, then in the estimations in Theorems 1, 2 additional terms of the form ch^k appear. For example we shall consider the case when the solution y of (1) has the property $y''' \in L_p[0,1]$ and we solve the equation (1) by cubic collocation splines.

Lemma 5. *If S_3 is cubic spline for which $S_3''(0)=A_n$, $S_3''(1)=B_n$. Then the estimate* $\|S_3\|_{C[0,1]} \leq 5 \max_{0\leq i\leq n} |S_3(x_i)|+h^2 \max[|A_n|, |B_n|]/3$ *holds true.*

Proof. From (4) we have

(24) $$\|S_3\|_{C[0,1]} \leq \max_{0\leq i\leq n} |S_3(x_i)|+h^2 \max_{0\leq i\leq n} |M_i|/3,$$

and from the system [1, p. 84]

(25) $$\begin{aligned} &4M_1+M_2=6\Delta^2 S_3(x_0)/h^2-A_n,\ M_0=A_n \\ &M_{i-1}+4M_i+M_{i+1}=6\Delta^2 S_3(x_{i-1}),\quad i=2, 3, \dots, n-2, \\ &M_{n-2}+4M_{n-1}=6\Delta^2 S_3(x_{n-2})/h^2-B_n,\ M_n,\ M_n=B_n, \end{aligned}$$

using Lemma 1 in [1, p. 29] we find

(26) $$\max_{1\leq i\leq n} |M_i| \leq 12 \max_{0\leq i\leq n} |S_3(x_i)|/h^2+\max[|A_n|, |B_n|]/2.$$

The inequalities (24) and (26) prove the lemma.

Lemma 6. *Let* $x_i+\sum_{j=1,\, j\neq i}^{n} a_{ij}x_j=y_i$, $i=1, 2, \ldots, n$, *be a system of linear equations and* $\sum_{j=1,\, j\neq i}^{n}|a_{ij}|\leq q<1$ $\sum_{i=1,\, i\neq j}^{n}|a_{ij}|\leq q<1$. *Then the estimation* $\|x\|_{l_p}\leq\|y\|_{l_p}/(1-q)$ *holds, where* $\|x\|_{l_p}=(\sum_{i=1}^{n}|x_i|^p)^{1/p}$.

This lemma is proved in [1] for $p=2$. For arbitrary $p\geq 1$ the proof is the same, replacing everywhere 2 by p.

Lemma 7. *Let the function* y *have a bounded second derivative in* $[0, 1]$ *Then* $|y_i''-\Delta_h^2 y_{i-1}/h^2|\leq\omega_2(y'', x_i; h)/2$, $h=1/n$, $y_i=y(i/n)$.

The proof is given in [7].

Lemma 8. *If* $f'''\in L_p[0, 1]$ *and* f'' *is absolutely continuous function, then*

$$\|f^{(r)}-S_3^{(r)}\|_{L_p[0,1]}\leq c_1h^{3-r}\omega(f''';h)_{L_p[0,1]}+c_2h^{2+1/p-r}\max[|f_0''-A_n|, |f_n''-B_n|],$$

where S_k *is interpolated cubic spline for* f *under boundary conditions* $S_3''(0)=A_n$, $S_3''(1)=B_n$, $h=1/n$, $0\leq r\leq 3$.

Proof. Denoting $M_i=S_3''(x_i)$, $i=0, 1, \ldots, n$, for $x\in[x_i, x_{i+1}]$ we have

$$|S_3'''(x)-f'''(x)|^p=\left|\frac{M_{i+1}-f_{i+1}''}{h}-\frac{M_i-f_i''}{h}+\frac{f_{i+1}''-f_i''}{h}-f'''(x)\right|^p$$

$$\leq c(p)\left(\left|\frac{M_{i+1}-f_{i+1}''}{h}\right|^p+\left|\frac{M_i-f_i''}{h}\right|^p+\left|\frac{f_{i+1}''-f_i''}{h}-f'''(x)\right|^p\right)$$

and consequently

$$(27)\quad \|S_3'''-f'''\|_{L_p[0,1]}^p\leq c_1(p)\left(h^{1-p}\sum_{i=0}^{n}|M_i-f_i''|^p+\sum_{i=0}^{n}\int_{x_i}^{x_{i+1}}\left|\frac{f_{i+1}''-f_i''}{n}-f'''(x)\right|^p dx\right).$$

If we write the system (25) in matrix form $AM=B$, $M=(M_1, M_2, \ldots, M_{n-1})$ $B=(6\Delta^2 f_0/h-A_n, 6\Delta^2 f_1/h^2, \ldots, 6\Delta^2 f_{n-2}/h^2-B_n)$, then setting $\bar{f}''=(f_1'', f_2'', \ldots, f_{n-1}'')$ we get $A(M-\bar{f}'')=B-A\bar{f}''$. Applying Lemma 6 to the last system we have

$$\sum_{i=1}^{n-1}|M_i-f_i''|^p\leq\sum_{i=1}^{n-1}|6\Delta^2 f_{i-1}/h^2-f_{i-1}''-4f_i''-f_{i+1}''|^p+|f_0''-A_n|^p+$$

$$|f_n''-B_n|^p\leq 2^p\sum_{i=1}^{n-1}[6^p|\Delta^2 f_{i-1}/h^2-f_i''|^p+|f_{i-1}''-2f_i''+f_{i+1}''|^p]+|f_0''-A_n|^p+|f_n''-B_n|^p$$

aud using now Lemma 7 we receive

$$(28)\qquad \sum_{i=1}^{n-1}|M_i-f_i''|^p\leq 2^p(1+3^p)\sum_{i=1}^{n-1}\omega_2^p(f'', x_i; h)+|f_0''-A_n|^p+|f_n''-B_n|^p.$$

On the other hand,

$$(29)\qquad h^{1-p}\sum_{i=1}^{n-1}\omega_2^p(f'', x_i; h)=\frac{h^{-p}}{2}\sum_{i=1}^{n-1}\int_{x_{i-1}}^{x_{i+1}}\omega_2^p(f'', x_i; h)dx$$

$$\leq \frac{h^{-p}}{2}\sum_{i=1}^{n-1}\int_{x_{i-1}}^{x_{i+1}}\omega_2^p(f'',x;2h)dx=\frac{h^{-p}}{2}\tau_2^p(f'';2h)_{L_p[0,1]}\leq c^p\omega^p(f''';h)_{L_p[0,1]},$$

$$(30)\quad \sum_{i=0}^{n-1}\int_{x_i}^{x_{i+1}}|(f''_{i+1}-f''_i)/h-f'''(x)|^pdx\leq\sum_{i=0}^{n-1}\int_{x_i}^{x_{i+1}}\Big(\frac{1}{h}\int_{x_i}^{x_{i+1}}|f'''(t)-f'''(x)|dt\Big)^pdx$$

$$\leq h^{p/q-p}\sum_{i=0}^{n-1}\int_{x_i}^{x_{i+1}}\int_{x_i}^{x_{i+1}}|f'''(t)-f'''(x)|^pdtdx,$$

where $1/p+1/q=1$.

Let us define the functions

$$\alpha(x)=\begin{cases}-x, & 0\leq x\leq h,\\ -h, & h\leq x\leq 1,\end{cases}\qquad \beta(x)=\begin{cases}h, & 0\leq x\leq 1-h,\\ 1-x, & 1-h\leq x\leq 1.\end{cases}$$

Then

$$(31)\quad \sum_{i=0}^{n-1}\int_{x_i}^{x_{i+1}}\int_{x_i}^{x_{i+1}}|f'''(t)-f'''(x)|^pdtdx\leq\int_0^1\int_x^{x+\beta(x)}|f'''(t)-f'''(x)|^pdtdx$$

$$+\int_0^1\int_{x+\alpha(x)}^{x}|f'''(t)-f'''(x)|^pdtdx=\int_0^1\int_0^{\beta(x)}|f'''(x+u)-f'''(x)|^pdudx$$

$$+\int_0^1\int_{\alpha(x)}^{0}|f'''(x+u)-f'''(x)|^p\,dudx=\int_0^h\int_0^{1-u}|f'''(x+u)-f'''(x)|^pdxdu$$

$$+\int_{-h}^{0}\int_{-u}^{1}|f'''(x+u)-f'''(x)|^p\,dxdu\leq\int_0^h\Big(\sup_{0\leq u\leq h}\int_0^{1-u}|f'''(x+u)-f'''(x)|^p\,dx\Big)du$$

$$+\int_{-h}^{0}\Big(\max_{-h\leq u\leq 0}\int_{-u}^{1}|f'''(x+u)-f'''(x)|^pdx\Big)du\leq 2h\omega(f''';h)_{L_p}^{[0,1]}.$$

From (27)—(31)

$$\|f'''-S_3'''\|_{L_p[0,1]}\leq c\omega(f''';h)_{L_p[0,1]}+c_1h^{1/p-1}\max(|f''_0-A_n|,\ |f'_n-B_n|).$$

For completion of the proof we must use the well-known inequality

$$\|f^{(i)}-S_3^{(i)}\|_{L_p[0,1]}\leq h\,\|f^{(i+1)}-S_3^{(i+1)}\|_{L_p[0,1]},\quad i=0,\ 1,\ 2.$$

Theorem 3. *Let the solution y of the equation* (1) *have the third derivative $y'''\in L_p[0,1]$ and y'' is absolutely continuous function. Then under the condition $1-5|\lambda|\,\|\int_0^1|K(\,.\,,t)|\,dt\,\|_{c[0,1]}>0$ the following estimate for $r=0,\ 1,\ 2,\ 3$*

$$\|y^{(r)}-S_3^{(r)}\|_{L_p[0,1]}=0(h^{3-r}\omega(y''';h)_{L_p[0,1]}+h^{2+1/p-r}\max[\,|y''_0-A_n|,\ |y''_n-B_n|\,])$$

holds, where S_3 is a cubic spline, which is determined by (6) *and by the boundary condition $S''_3(0)=A_n$, $S''_3(1)=B_n$.*

The proof of this theorem can be done in the same way as the proof of Theorems 1, 2, using Lemmas 3, 5, 8, choosing intermediate interpolated cubic spline $\bar{S}_3$ for the solution y (in Theorem 1) with boundary condition $\bar{S}''_3(\theta)=A_n$, $\bar{S}''_3(1)=B_n$. So the spline $\varphi(x)=S_3(x)-\bar{S}_3(x)$ (used in Theorem 1) will be with boundary condition $\varphi''(0)=0$, $\varphi''(1)=0$.

REFERENCES

1. С. Б. Стечкин, Ю. Н. Суботин. Сплайны в вычислительной математике. М. 1976.
2. V. A. Popov, A. S. Andreev. Steckin's type theorems for one-sided triginometrical and spline approximation. *C. R. Acad. bulg. Sci.*, **31**, 1978, 151-154.
3. A. Andreev, V. A. Popov. Approximation of functions by means of linear summation operators in L_p. *Acta matematica* (in print).
4. Е. П. Долженко, Е. А. Севастьянов. О приближениях функций в хаусдорфовой метрике поср. кусочно монотонных (в частности рациональных) функций. *Матем. сб.*, **101**, 1976, 508—541.
5. Г. Фройд, В. А. Попов. Некоторые вопросы, связанные с аппроксимацией сплайн-функциями и многочленами. *Studia Sci. Math. Hung.* **5**, 1970, 161—171.
6. K. Ivanov. On a new characteristic of functions. *Pliska*, **5**, 1983, 151—163.
7. А. С. Андреев, В. А. Попов, Бл. Сендов. Оценки погрешности численного решения обыкновенных дифференциальных уравнений. *Ж. вычисл. мат. и мат, физ.*, **21**, 1981, 635—650.

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