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ПЛИСКА

БЪЛГАРСКИ МАТЕМАТИЧЕСКИ СТУДИИ

A GENERALIZATION OF CHEBYSHEV POLYNOMIALS. II

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We extend here the Chebyshev-Markov inequality for the set of polynomials which have fixed multiplicities $\nu_1, \dots \nu_n$ of their zeros $x_1 < \dots < x_n$.

Let $[a, b]$ be a given finite interval. Denote by $\|f\|$ the uniform norm of f in $[a, b]$, $\|f\| = \max \{|f(x)|: x \in [a, b]\}$. We proved in [1] the following result.

Theorem A. *Let $\bar{\nu} = (\nu_k)_1^n$ be a fixed system of arbitrary natural numbers. Given $[a, b]$, there exists a unique system of points $(x_k^*)_1^n$ such that*

$$\|(x-x_1^*)^{\nu_1} \dots (x-x)_n^*)^{\nu_n}\| = \inf_{a \leq x_1 < \dots < x_n \leq b} \|(x-x_1)^{\nu_1} \dots (x-x_n)^{\nu_n}\|.$$

Moreover, $a < x_1^* < \dots < x_n^* < b$. The extremal polynomial $T(\bar{\nu}; x) = (x-x_1^*)^{\nu}$, $\cdots (x-x_n^*)^{\nu_n}$ is uniquely determined by the condition that there exist $n-1$ points $(t_k)_1^{n-1}$, $a = t_0 < t_1 < \dots < t_{n-1} < t_n = b$ such that $T(\bar{\nu}; t_k) = (-1^{N-\nu_1-\dots-\nu_k} \|T(\bar{\nu}: \cdot)\|$, where $N = \nu_1 + \dots + \nu_n$.

Evidently $T(\bar{\nu}; x)$ coincides with the Chebyshev polynomial of first kind $T_n(x)$ in the case $[a, b] = [-1, 1]$, $\nu_1 = \dots = \nu_n = 1$. So, $T(\bar{\nu}; x)$ could be considered as a generalization of the famous Chebyshev polynomials. It is interesting that $T(\bar{\nu}; x)$ preserves some extremal properties of these classical polynomials. For example, it is well-known that

$$(1) \qquad |P^{(\lambda)}(x)| \leq |T_n^{(\lambda)}(x)|, \ |x| \geq 1, \ k = 0, \dots, n,$$

for each polynomial P of degree n such that $\|P\|_{C[-1,1]} \leq 1$. The inequality (1) was proved first for $\lambda = 0$ by Chebyshev (see [2], p. 78) and extended for $0 \leq \lambda \leq n$ by A. A. Markov.

Wa show in this note that $T(\bar{\nu}; x)$ has an analogous extremal property in the set $\Omega(\bar{\nu})$ of all algebraic polynomials f of the form $f(x) = c(x-x_1)^{\nu_1} \cdots (x-x_n)^{\nu_n}$, $a < x_1 < \dots < x_n < b$, where c is a real parameter such that $\|f\| \leq \|T(\bar{\nu}; \cdot)\|$.

First we prove an auxiliary statement. Suppose that $(e_k)_1^{n+1}$ are given positive numbers. If follows from Theorem 1 of [1] that there exist a unique system of points $(x_k)_1^n$, $a = x_0 < x_1 < \dots < x_n < x_{n+1} = b$, and a constant $c > 0$ such that the polynomial $f(x)=c(x-x_1)^{\nu_1}\cdots(x-x_n)^{\nu_n}$ satisfies the conditions $|\int_{x_{k-1}}^{x_k} f(x)dx|=e_k$, $k=1,\ldots,n+1$. Clearly, the above system of equations can be rewriten in the form

$$\varphi_k(c, x_1,\ldots,x_n):=\int_{x_{k-1}}^{x_k} f(x)dx-\varepsilon_k e_k=0,\ k=1,\ldots,n+1, \tag{2}$$

where $\varepsilon_k=(-1)^{N-\nu_1-\cdots-\nu_k}$, $N=\nu_1+\cdots+\nu_n$.

Theorem 1. *The solutions $c, x_1,\ldots,x_n$ of the system* (2) *are differentiable functions of $e_1,\ldots,e_{n+1}$ in the domain $E=\{(e_1\ldots,e_{n+1}): e_i>0,\ i=1,\ldots,n+1\}$ and $f^{(\lambda)}(\xi)$ $(\lambda=0,\ldots,N,\ \xi\notin(a,b))$ is a strictly increasing function with respect to e_j $(j=1,\ldots,n+1)$.*

Proof. Denote by $J=J(c,x_1,\ldots,x_n)$ the Jacobian of (3). We have

$$J=\frac{D(\varphi_1,\ldots,\varphi_{n+1})}{D(c,x_1,\ldots,x_n)}$$

$$=\begin{vmatrix} \int_{x_0}^{x_1} W(x)\omega(x)dx, & -\nu_1\int_{x_0}^{x_1} cW(x)\omega_1(x)dx & \cdots & -\nu_n\int_{x_0}^{x_1} cW(x)\omega_n(x)dx \\ \int_{x_1}^{x_2} W(x)\omega(x)dx, & -\nu_1\int_{x_1}^{x_2} cW(x)\omega_1(x)dx & \cdots & -\nu_n\int_{x_1}^{x_2} cW(x)\omega_n(x)dx \\ \cdots & \cdots & \cdots & \cdots \\ \int_{x_n}^{x_{n+1}} W(x)\omega(x)dx, & -\nu_1\int_{x_n}^{x_{n+1}} cW(x)\omega_1(x)dx & \cdots & -\nu_n\int_{x_n}^{x_{n+1}} cW(x)\omega_n(x)dx \end{vmatrix},$$

where

$$W(x)=\prod_{i=1}^{n}(x-x_i)^{\nu_i-1},\quad \omega(x)=(x-x_1)\cdots(x-x_n),\ \omega_k(x)=\omega(x)/(x-x_k),\ k=1,\ldots,n$$

It is not difficult to verify that

$$\det J(c,x_1,\ldots,x_n)\neq 0 \tag{3}$$

for each $c>0$ and $a<x_1<\cdots<x_n<b$. Indeed, assuming the contrary, there is a linear dependence of the columns of J, i. e., there exist coefficients $b_0,\ldots,b_n$, such that $\sum_{i=0}^{n}|b_i|>0$ and

$$\int_{x_i}^{x_{i+1}} W(x)[b_0\omega(x)+b_1\omega_1(x)+\cdots+b_n\omega_n(x)]dx=0$$

for $i=0,\ldots,n$. Then the polynomial $g(x)=b_0\omega(x)+b_1\omega_1(x)+\cdots+b_n\omega_n(x)$ must change its sign in the intervals $(x_0,x_1),\ldots,(x_n,x_{n+1})$. Thus $g(x)$ must have at least $n+1$ zeros. But g is a non-zero polynomial of degree n, a contradiction.

Now, applying the implicit function theorem we conclude that $c, x_1,\ldots,x_n$ are differentiable functions of $e_1,\ldots,e_{n+1}$ in E. Moreover,

(4) $$\begin{cases} \partial c/\partial e_j = \varepsilon_j J_{j1}/\det J, \\ \partial x_k/\partial e_j = \varepsilon_j J_{j,k+1}/\det J, \quad k=1,\ldots,n, \end{cases}$$

where J_{lm} is the cofactor of the (lm)-th element of the matrix J. Our first observation is that $\partial c/\partial e_j > 0$. Indeed, $J_{j1} \neq 0$ since, otherwise there exists a polynomial g of degree $n-1$, such that $\int_{x_{i-1}}^{x_i} W(x)\,g(x)dx=0$ for $i=1,\ldots, j-1, j+1,\ldots, n+1$, which is impossible. Then c is a monotone function of $e_j (j=1,\ldots, n+1)$. But $c(e_1,\ldots, e_j,\ldots, e_{n+1}) > c(e_1,\ldots, 1,\ldots, e_{n+1})$ for sufficiently large e_j. Therefore

(5) c is a strictly increasing function of e_j $(j=1,\ldots, n+1)$

Let us find the dependence of $f^{(\lambda)}(\xi)$ on e_j. We have

$$\frac{\partial f^{(\lambda)}}{\partial e_j}(\xi) = \frac{\partial^\lambda}{\partial x^\lambda}\frac{\partial}{\partial e_j}\{c(x-x_1)^{\nu_1}\cdots(x-x_n)^{\nu_n}\}\,|_{x=\xi}$$

$$=\{W(\xi)\omega(\xi)\}^{(\lambda)}\frac{\partial c}{\partial e_j} - \nu_1 c\{W(\xi)\omega_1(\xi)\}^{(\lambda)}\frac{\partial x_1}{\partial e_j} - \cdots - \nu_n\{W(\xi)\omega_n(\xi)\}^{(\lambda)}\frac{\partial x_n}{\partial e_j}.$$

After replacing the partial derivatives by the expressions from (4) we see that

$$\varepsilon_j \det J \,.\, \frac{\partial}{\partial e_j} f^{(\lambda)}(\xi) =$$

$$= \begin{vmatrix} \int_{x_0}^{x_1} W(x)\omega(x)dx, & -\nu_1\int_{x_0}^{x_1} cW(x)\omega_1(x)\,dx & \cdots & -\nu_n\int_{x_0}^{x_1} cW(x)\omega_n(x)dx \\ \cdot & \cdot & \cdot & \cdot \\ \int_{x_{j-2}}^{x_{j-1}} W(x)\omega(x)dx, & -\nu_1\int_{x_{j-2}}^{x_{j-1}} cW(x)\omega_1(x)dx & \cdots & -\nu_n\int_{x_{j-2}}^{x_{j-1}} cW(x)\omega_n(x)dx \\ \{W(\xi)\omega(\xi)\}^{(\lambda)}, & -\nu_1 c\{W(\xi)\omega_1(\xi)\}^{(\lambda)} & \cdots & -\nu_n c\{W(\xi)\omega_n(\xi)\}^{(\lambda)} \\ \int_{x_j}^{x_{j+1}} W(x)\omega(x)dx, & -\nu_1\int_{x_j}^{x_{j+1}} cW(x)\omega_1(x)dx & \cdots & -\nu_n\int_{x_j}^{x_{j+1}} cW(x)\omega_n(x)dx \\ \cdot & \cdot & \cdot & \cdot \\ \int_{x_n}^{x_{n+1}} W(x)\omega(x)dx, & -\nu_1\int_{x_n}^{x_{n+1}} cW(x)\omega_1(x)dx & \cdots & -\nu_n\int_{x_n}^{x_{n+1}} cW(x)\omega_n(x)dx \end{vmatrix}.$$

We shall show that $\frac{\partial f^{(\lambda)}}{\partial e_j}(\xi) \neq 0$. It suffices to prove that the above matrix is non-singular because of (3). Assuming the contrary, there exists a non-zero polynomial $P(x)$ of degree n (with a positive coefficient in front of x^n) which satisfies the conditions

(6) $$\begin{cases} \int_{x_i}^{x_{i+1}} W(x)P(x)dx = 0, \quad j=0,\ldots, j-2,\ j, j+1,\ldots, n, \\ \{W(\xi)P(\xi)\}^{(\lambda)} = 0. \end{cases}$$

The first n equalities in (6) mean that $P(x)$ must have a zero in (x_i, x_{i+1}) for $i=0,\dots, j-2, j, j+1,\dots, n$. Then the polynomial $q(x)=W(x)P(x)$ which is of degree N has precisely N real zeros and they lie in (a, b). Then, by Rolle's theorem, $q^{(\lambda)}(x)$ does not vanish outside (a, b). This contradicts the last equality in (6). Therefore $\frac{\partial}{\partial e_j} f^{(\lambda)}(\xi)\neq 0$ for each $(e_1,\dots, e_{n+1})\in E$ if $\xi\notin(a, b)$. This implies that $|f^{(\lambda)}(\xi)|$ is a strictly monotone function of e_j in $(0, \infty)$. Now let us assume, for the sake of definiteness, that $b\leq\xi$. Then $f^{(\lambda)}(\xi)>0$. We shall prove that $f^{(\lambda)}(\xi)$ is actually a strictly increasing function of e_j in this case. Indeed, if $b<\xi$, then $f^{(\lambda)}(\xi)\geq c(\xi-b)^{N-\lambda}\cdot\binom{N}{N-\lambda}$ and, according to (5), $f^{(\lambda)}(\xi)$ can be done greater than any positive number for sufficiently large e_j. This shows that $f^{(\lambda)}(\xi)$ is an increasing function of e_j for $b<\xi$, and by continuity, for $b=\xi$ too. The case $\xi\leq a$ is treated similarly. The theorem is proved.

Next we derive as an immediate consequence of Theorem 1 an analog of the Chebyshev-Markov inequality.

T h e o r e m 2. *Let* $\nu_1,\dots,\nu_n$ *be arbitrary natural numbers. Suppose that* $f\in\Omega(\bar{\nu})$. *Then*

$$|f^{(\lambda)}(\xi)|\leq|T^{(\lambda)}(\bar{\nu};\xi)| \tag{7}$$

for each $\xi\notin(a, b)$ *and* $\lambda=0,\dots, N$, $N=\nu_1+\dots+\nu_n$. *The equality is attained if and only if* $f=\pm T(\bar{\nu};\cdot)$.

P r o o f. Denote by $t_1,\dots, t_m$ and $z_1,\dots, z_m$, respectively, the distinct zeros of $f'(x)$ and $T'(\bar{\nu}; x)$. Clearly t_i has the same multiplicity as z_i, $i=1,\dots, m$. Then f' and $T'(\bar{\nu};\cdot)$ are solutions of a system like (2) with parameters $e_i=|\int_{t_{i-1}}^{t_i} f'(x)dx|$ $(i=1,\dots, m+1,\ t_0=a,\ t_{m+1}=b)$ in the first case and $e_i^*=1$ in the second. Since $\|f\|\leq\|T(\bar{\nu};\cdot)\|$ we have

$$e_i\leq e_i^*,\quad i=1,\dots, n+1, \tag{8}$$

with at least one strict inequality if $f\neq T(\bar{\nu};\cdot)$. By Theorem 1, (8) implies $|f^{(\lambda)}(\xi)|\leq T^{(\lambda)}(\bar{\nu};\xi)|$ for $\lambda=1,\dots, N$. It remains to prove (7) for $\lambda=0$. In this case

$$f(\xi)=f(b)+\int_b^{\xi} f'(x)dx\leq T(\bar{\nu}; b)+\int_b^{\xi} T'(\bar{\nu}; x)dx=T(\bar{\nu};\xi)$$

since $f'(x)\leq T'(\bar{\nu}; x)$ for each $x\geq b$, as we have already shown. The theorem is proved.

REFERENCES

1. B. D. B o j a n o v. A generalization of Chebyshev polynomials. *J. Approxim. Theory*, **26**, 1979, 293-300.
2. И. П. Н а т а н с о н. Конструктивная теория функций. М. — Л., 1949.

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