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ПЛИСКА
БЪЛГАРСКИ МАТЕМАТИЧЕСКИ СТУДИИ

FUNCTION SPACES, GENERATED BY THE AVERAGED MODULI OF SMOOTHNESS

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In the paper the connection between the one-sided K-functional (for spaces L_p and W_p^r) and the averaged modulus of r-order is proved. There is proved also the connection between the spaces, generated by the averaged moduli of smoothness, the Bessov's spaces and the spaces, generated by the best one-sided trigonometrical approximation.

In this paper we continue the investigations from [16], where the results are given without proofs.

1. We shall begin with the definition of the notion of averaged moduli of smoothness.

Let f be a function defined and bounded on the finite interval $[a, b]$. The local k-th modulus of the function f at a point $x \in [a, b]$ is given by

$$\omega_k(f, x; \delta) = \sup\{|\Delta_h^k f(t)| : t, t+kh \in [x-k\delta/2, x+k\delta/2] \cap [a, b]\},$$

where, as usually, $\Delta_h^k f(x) = \Sigma_{m=0}^k (-1)^{k+m} \binom{k}{m} f(t+mh)$,

The k-th averaged modulus (or τ_k-modulus) of function f in L_p, $1 \leqq p \leqq \infty$, is given by

$$\tau_k(f; \delta)_{L_p} = \| \omega_k(f, x; \delta) \|_{L_p[a, b]},$$

where

$$\| g \|_{L_p[a, b]} = \| g \|_{L_p} = \{\frac{1}{b-a} \int_a^b | g(x) |^p dx\}^{1/p}, \quad 1 \leqq p < \infty,$$

$$\| g \|_{L_\infty} = \sup\{| g(x) | : x \in [a, b]\}.$$

If we compare the averaged modulus in L_p with the usual k-th modulus of continuity in L_p:

$$\omega_k(f; \delta)_{L_p} = \sup_{0<h\leqq\delta} \{\frac{1}{b-a} \int_a^{b-kh} | \Delta_h^k f(x) |^p dx\}^{1/p},$$

we see at once that

$$(1) \qquad \omega_k(f; \delta)_{L_p} \lessapprox \tau_k(f; \delta)_{L_p}, \quad 1 \leqq p < \infty,$$

and $\omega_k(f; \delta)_{L_\infty} = \omega_k(f; \delta) = \tau_k(f; \delta)_{L_\infty}$.

Some examples show that in general $\omega_k(f; \cdot)_{L_p}$ and $\tau_k(f; \cdot)_{L_p}$ are not equivalent in case $p < \infty$.

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The properties of τ_k are similar to the properties of ω_k. For the history and the properties of τ_k-moduli see [1—3]. We shall mention only the following properties:

P1. $\tau_k(f+g;\delta)_{L_p}\le\tau_k(f;\delta)_{L_p}+\tau_k(g;\delta)_{L_p}$,

P2. $\tau_k(f;\delta)_{L_p}\le c(k)\delta^k\|f^{(k)}\|_{L_p}$, if $f^{(k)}\in L_p$,

P3. $\tau_k(f;\lambda\delta)_{L_d}\le(\lambda+1)^{2k+1}\tau_k(f;\delta)_{L_p}$.

Here and in all the following pages $c(k)$ denotes a constant, depending only on k.

The averaged moduli have many applications:

a) in problems connected with the convergence of sequences of linear positive operators (P. P. Korovkin [4], Bl. Sendov [5], A. Andreev, V. A. Popov [10]);

b) in Hausdorff approximation of functions by means of piecewise monotone functions (E. P. Dolgenko, E. A. Sevastianov [6]);

c) in one-sided approximation of functions (we shall consider these applications more in detail below);

d) in problems connected with estimations of the error of quadrature formulas (V. A. Popov [3], K. Ivanov [7]);

e) in problems of estimations of the error of numerical solution of differential equations (A. Andreev, V. A. Popov, Bl. Sendov [8]), etc.

It is well-known that the usual k-th modulus of continuity $\omega_k(f;\delta)_{L_p}$ is connected with the following K-functional of J. Peetre [9], [10]: $K(f;t)=\inf\{\|f_0\|_{L_p}+t\|f_1^{(k)}\|_{L_p}:f=f_0+f_1\}$.

It is interesting that the k-th averaged moduli are connected with the so-called one-sided K-functional. First we shall give the definition of one-sided K-functional.

Let G be a set and X_i, $i=0,1$ — linear spaces of real-valued functions on G with seminorms $\|\cdot\|_i$, $i=0,1$. We suppose that X_i, $i=0,1$, contain the constant functions. The lower K-functional in X_0+X_1 for seminorms $\|\cdot\|_i$ $i=0,1$, is given by $K_+(f;t)=\inf\{\|f_0\|_0+t\|f_1\|_1:f=f_0+f_1;\ f_0\ge0\}$.

The lower K-functional is meaningful for all functions in X_0+X_1, for which $\inf\{f(x):x\in G\}>-\infty$.

The upper K-functional in x_0+x_1 for seminorms $\|\cdot\|_i$, $i=0,1$, is given by $K_-(f;t)=\inf\{\|f_0\|_0+t\|f_1\|_1:f=f_0+f_1,\ f_0\le0\}$.

This functional is meaningful for all functions in X_0+X_1, for which $\sup\{f(x):x\in G\}<\infty$.

The one-sided K-functional is defined by $\tilde{K}(f;t)=\max\{K_+(f;t),K_-(f;t)\}$.

In what follows we shall consider for simplicity the 2π-periodical case with L_p-norm

$$\|g\|_{L_p}=\{\frac{1}{2\pi}\int_0^{2\pi}|g(x)|^p dx\}^{1/p}.$$

Let us mention that in 2π-periodic case the sup in the definition of $\omega_k(f,x;\delta)$ is taken over all t, $t+kh$ for which $t,t+kh\in[x-k\delta/2,x+k\delta/2]$.

We shall denote by W_p^k the Sobolev space of all 2π-periodic functions f with absolutely continuous $(k-1)$-th derivative, for which $f^{(k)}\in L_p[0,2\pi]$.

The following theorem is valid:

Theorem 1. *Let* $G=[0,2\pi]$, $X_0=L_p$, $X_1=W_p^k$, $\|f\|_0=\|f\|_{L_p}$, $\|f\|_1=\|f^{(k)}\|_{L_p}$ *and let* $\tilde{K}(f;t)$ *be the one-sided K-functional in* $X_0+x_1(=L_p)$ *for seminorms* $\|\cdot\|_i$, $i=0,1$. *Then there exist constants* $c_i(k)$, $i=0,1$, *depending only on k such that*

$$c_0(k)\tau_k(f;t)_{L_p}\leqq\tilde{K}(f;t^k)\leqq c_1(k)\tau_k(f;t)_{L_p}. \tag{2}$$

Remark. Obviously $\tilde{K}(f;t)$ as well as $\tau_k(f;\delta)_{L_p}$ have a sense only for bounded functions in L_p and only for bounded 2π-periodic functions in L_p theorem 1 has a sense.

Proof of theorem 1. Let f be a 2π-periodic bounded function, $f\in L_p$. It is known (compare with [9; 10; 12]) that for every integer $k>0$ and $h>0$ there exists a function $f_{k,h}$ such that

$$|f(x)-f_{k,h}(x)|\leqq\omega_k(f,x;2h), \tag{3}$$

$$\|f-f_{k,h}\|_{L_p}\leqq\omega_k(f;h)_{L_p}, \tag{4}$$

$$f_{k,h}\in W_p^k \quad\text{and}\quad \|f_{k,h}^{(k)}\|_{L_p}\leqq c'(k)h^{-k}\omega_k(f;h)_{L_p}. \tag{5}$$

For example we can set

$$f_{k,h}(x)=\frac{(-1)^{k-1}}{h^k}\int_0^h\cdots\int_0^h\{f(x+t_1+\dots+t_k)$$

$$-\binom{k}{1}f(x+\frac{k-1}{k}(t_1+\dots+t_k))+\dots+(-1)^{k-1}\binom{k}{k-1}f(x+\frac{t_1+\dots+t_k}{k})\}dt_1\dots dt_k$$

Obviously from (3) it follows

$$f_{k,h}(x)-\omega_k(f.\ x;2h)\leqq f(x). \tag{6}$$

From the results in [11] it follows that there exist two trigonometrical polynomials P and Q of n-th order such that

$$\begin{gathered}P(x)\leqq\omega_k(f,x;2h)\leqq Q(x),\\ \|Q-\omega_k\|_{L_p}\leqq\|Q-P\|_{L_p}\leqq c_2(k)\tau_k(f;h)_{L_p}.\end{gathered} \tag{7}$$

Since Q is a trigonometric polynomial of n-th order we have in view of (7)

$$\|Q^{(k)}\|_{L_p}\leqq n^k\|Q\|_{L_p}\leqq n^k\{\|Q-\omega_k\|_{L_p} \tag{8}$$

$$+\|\omega_k\|_{L_p}\}\leqq n^k(c_2(k)+1)\tau_k(f;2h)_{L_p}.$$

From (6), (7) it follows

$$0\leqq f(x)-f_{k,h}(x)+Q(x) \tag{9}$$

and obviously $f_{k,h}-Q\in W_p^k$ (see (5)).

We have from (1), (3)—(9)

$$K_+(f;\, t^k) \leq \| f - f_{k,h} + Q \|_{L_p} + t^k \| f_{k,h}^{(k)} - Q^{(k)} \|_{L_p}$$

$$\leq \| f - f_{k,h} \|_{L_p} + \| Q - \omega_k \|_{L_p} + \| \omega_k \|_{L_p} + t^k \{ \| f_{k,h}^{(k)} \|_{L_p} + \| Q^{(k)} \|_{L_p} \}$$

$$\leq \tau_k(f;\, h)_{L_p} + c_2(k)\tau_k(f;\, h)_{L_p} + \tau_k(f;\, h)_{L_p} + t^k c'(k) h^{-k} \tau_k(f;\, h)_{L_p} + t^k n^k (c_2(k) + 1)\tau_k(f;\, 2h)_{L_p}.$$

If we take $t=h$, $n=[1/t]+1$, we obtain in view of P3 that (we may assume that $t \leq 4\pi$, otherwise (2) is obvious, see also lemma 2 below): $K_+(f;\, t^k) \leq c_1(k)\tau_k(f;\, t)_{L_p}$.

In a similar way we obtain that $K_-(f;\, t^k) \leq c_1(k)\tau_k(f;\, t)_{L_p}$, which proves the right hand side of (2).

To prove the left hand side of (2) we arbitrarily take two functions f_1^+ and f_1^- with the properties

$$f_1^+(x) \leq f(x) \leq f_1^-, \quad f_1^+ \in W_p^k, \quad f_1^- \in W_p^k. \tag{10}$$

Let us set $f_0^\pm = f - f_0^\pm$. Then we have $f_0^+ \geq 0$, $f_0^- \leq 0$.

Let us estimate now $\tau_k(f;\, t)_{L_p}$. We have in view of P1

$$\tau_k(f;\, t)_{L_p} \leq \frac{1}{2}\{ \tau_k(f_0^+;\, t)_{L_p} + \tau_k(f_1^+;\, t)_{L_p} + \tau_k(f_0^-;\, t)_{L_p} + \tau_k(f_1^-;\, t)_{L_p} \}. \tag{11}$$

From (10) we obtain, using P2:

$$\tau_k(f_1^\pm;\, t)_{L_p} \leq c(k) t^k \| (f_1^\pm)^{(k)} \|_{L_p}. \tag{12}$$

It is a little more difficult to estimate $\tau_k(f_0^\pm;\, t)_{L_p}$.

Let x be fixed and $t,\, t+kh \in [x - k\delta/2,\, x + k\delta/2] = \Delta_\delta(x)$.

If $\Delta_h^k f_0^+(t) \geq 0$, we have, since $f_0^+ \geq 0$ and $f_1^- \geq f$:

$$0 \leq \Delta_h^k f_0^+(t) = \sum_{m=0}^{k} (-1)^{k+m} \binom{k}{m} f_0^+(t+mh) \tag{13}$$

$$\leq \sum_{\substack{m=0 \\ m \equiv k (\mathrm{mod}\ 2)}}^{k} \binom{k}{m} f_0^+(t+mh) = \sum_{\substack{m=0 \\ m \equiv k (\mathrm{mod}\ 2)}}^{k} \binom{k}{m} (f - f_1^+(t+mh)$$

$$\leq \sum_{m=0}^{k} \binom{k}{m} (f_1^- - f_1^+)(t+mh).$$

If we set $g = f_1^- - f_1^+ \in W_p^k$, we have from (13)

$$0 \leq \Delta_h^k f_0^+(t) \leq \sum_{m=0}^{k} \binom{k}{m} g(t+mh), \quad t+mh \in \Delta_\delta(x), \quad m=0, \ldots, k. \tag{14}$$

For every function $g \in W_p^k$ we have the Taylor formula

$$g(\theta)=g(x)+\sum_{i=1}^{k-1}\frac{(\theta-x)^{i-1}}{i!}g^{(i)}(x)+\frac{1}{(k-1)!}\int_x^\theta(\theta-s)^{k-1}g^{(k)}(s)ds. \tag{15}$$

Let us set

$$r(\theta)=\sum_{i=1}^{k-1}\frac{(\theta-x)^{i-1}}{i!}g^{(i)}(x).$$

Then r is an algebraic polynomial of $(k-1)$-th degree and we have the Markov's inequality (see for example [17]):

$$\max_{\theta\in\Delta_\delta(x)}|r(\theta)|\leq\frac{2k^2}{k\delta}\max_{\theta\in\Delta_\delta(x)}\left|\int_x^\theta r(y)dy\right|. \tag{16}$$

Using (15), we obtain from (16) for $\theta\in\Delta_\delta(x)$:

$$|r(\theta)|\leq\frac{2k}{\delta}\max_{\theta\in\Delta_\delta(x)}\left|\int_x^\theta\{g(y)-g(x)-\frac{1}{(k-1)!}\int_x^y(y-s)^{k-1}g^{(k)}(s)ds\}dy\right|$$

$$\leq\frac{2k}{\delta}\int_{x-k\delta/2}^{x+k\delta/2}|g(y)-g(x)|dy+\frac{2k}{\delta(k-1)!}\max_{\theta\in\Delta_\delta(x)}\int_x^\theta\int_x^y|y-s|^{k-1}|g^{(k)}(s)|dsdy \tag{17}$$

$$\leq\frac{2k}{\delta}\int_{-k\delta/2}^{k\delta/2}|g(x+y)-g(x)|dy+\frac{2k}{\delta(k-1)!}\left(\frac{k\delta}{2}\right)^{k-1}k\delta\int_{-k\delta/2}^{k\delta/2}|g^{(k)}(x+y)|dy.$$

Consequently for every $g\in W_p^k$ and $\theta\in\Delta_\delta(x)$ we obtain from (15) and (17)

$$|g(\theta)-g(x)|\leq\frac{2k}{\delta}\int_{-k\delta/2}^{k\delta/2}|g(x+y)-g(x)|dy$$

$$+\frac{2k^{k+1}}{(k-1)!}\left(\frac{\delta}{2}\right)^{k-1}\int_{-k\delta/2}^{k\delta/2}|g^{(k)}(x+y)|dy+\frac{1}{(k-1)!}\left|\int_x^\theta(\theta-s)^{k-1}g^{(k)}(s)ds\right|$$

$$\leq\frac{2k}{\delta}\int_{-k\delta/2}^{k\delta/2}|g(x+y)-g(x)|dy+\Big(\frac{2k^{k+1}}{(k-1)!}\Big(\frac{\delta}{2}\Big)^{k-1}+ \tag{18}$$

$$+\Big(\frac{k\delta}{2}\Big)^{k-1}\frac{1}{(k-1)!}\Big)\int_{-k\delta/2}^{k\delta/2}|g^{(k)}(x+y)|dy$$

$$\leq c_3(k)\Big\{\frac{1}{\delta}\int_{-k\delta/2}^{k\delta/2}|g(x+y)-g(x)|dy+\delta^{k-1}\int_{-k\delta/2}^{k\delta/2}|g^{(k)}(x+y)|dy.$$

From (14) and (18) we obtain

$$0\leq\Delta_h^k f_0^+(t)\leq\sum_{m=0}^{k}\binom{k}{m}(g(t+mh)-g(x))+2^k g(x)$$

$$(19) \qquad \leqq 2^k g(x)+2^k c_3(k)\{\frac{1}{\delta}\int_{-k\delta/2}^{k\delta/2} |g(x+y)-g(x)|\,dy$$

$$+\delta^{k-1}\int_{-k\delta/2}^{k\delta/2} |g^{(k)}(x+y)|\,dy\}=A(x).$$

By analogy, in the case, when $\Delta_h^k f_0^+(t)\leqq 0$, we obtain

$$0\leqq -\Delta_h^k f_0^+(t)\leqq \sum_{\substack{m=0\\ m\equiv k-1 (\mathrm{mod}\ 2)}}^{k} \binom{k}{m} f_0^+(t+mh)\leqq A(x).$$

Consequently for t, $t+kh\in\Delta_\delta(x)$ we have $|\Delta_h^k f_0^+(t)|\leqq A(x)$, what gives us

$$(20) \qquad \omega_k(f_0^+, x;\delta)\leqq A(x).$$

From (19), (20) we obtain

$$(21) \qquad \tau_k(f_0^+;\delta)_{L_p}\leqq \|A(\cdot)\|_{L_p}\leqq 2^k\{\|g\|_{L_p}+c_3(k)(2k\|g\|_{L_p}+k\delta^k\|g^{(k)}\|_{L_p})\}.$$

Since $g=f_1^- - f_1^+ = f_0^+ - f_0^-$ we obtain from (21)

$$(22) \qquad \tau_k(f_0^+;\delta)_{L_p}\leqq c_4(k)\{\|f_0^+\|_{L_p}+\|f_0^-\|_{L_p}+\delta^k(\|(f_1^+)^{(k)}\|_{L_p}+\|(f_1^-)^{(k)}\|_{L_p})\}.$$

By analogy as before we have

$$(23) \qquad \tau_k(f_0^-;\delta)_{L_p}\leqq c_4(k)\{\|f_0^+\|_{L_p}+\|f_0^-\|_{L_p}+\delta^k(\|(f_1^+)^{(k)}\|_{L_p}+\|(f_1^-)^{(k)}\|_{L_p})\}.$$

From (11), (12), (22), (23) it follows that for every $f_1^\pm\in W_p^k$, $f_0^+=f-f_1^+\geqq 0$, $f_0^-=f-f_1^-\leqq 0$, we have

$$(24) \qquad \tau_k(f;t)_{L_p}\leqq c_5(k)\{\|f_0^+\|_{L_p}+t^k\|(f_1^+)^{(k)}\|_{L_p}+\|f_0^-\|_{L_p}+t^k\|(f_1^-)^{(k)}\|_{L_p}\}.$$

Since $f_1^\pm\in W_p^k$ are arbitrary functions with the property $f_1^+\leqq f\leqq f_1^-$, from (24) we obtain

$$\tau_k(f,t)_{L_p}\leqq c_5(k)\{K_+(f;t)+K_-(f;t)\}$$

$$\leqq 2c_5(k)\max\{K_+(f;t),\ K_-(f;t)\}=c'(k)\tilde{K}(f;t),$$

that proves the left hand side of (2). Theorem 1 is proved.

2. Using the one-sided K-functional or the averaged moduli of smoothness it is possible to introduce classes of spaces like well-known Bessov spaces.

Let us recall that the Bessov space B_{pq}^θ is the space of all 2π-periodic functions f, for which

$$\{\int_0^\infty (t^{-\theta}\omega_k(f;t)_{L_p})^q \frac{dt}{t}\}^{1/q}<\infty, \quad k>\theta.$$

The norm in B_{pq}^θ is given by

$$\|f\|_{B_{pq}^\theta}=\|f\|_{L_p}+\{\int_0^\infty (t^{-\theta}\omega_k(f;t)_{L_p})^q\frac{dt}{t}\}^{1/q}.$$

In a similar way we can introduce the spaces A^{θ}_{pq} of all 2π-periodic functions f, for which

$$\{ \int_0^{\infty} (t^{-\theta}\tau_k(f:t)_{L_p})^q \frac{dt}{t} \}^{1/q} < \infty$$

with the norm

$$(25) \qquad \|f\|_{A^{\theta}_{pq}} = \|f\|_{L_p} + \{ \int_0^{\infty} (t^{-\theta}\tau_k(f;t)_{L_p})^q \frac{dt}{t} \}^{1/q}.$$

Let $\tilde{K}(f;t)$ be the one-sided K-functional for the function $f \in L_p$ for the seminorms $\|f\|_{L_p}$, $\|f^{(k)}\|_{L_p}$. By theorem 1 we see that the spaces A^{θ}_{pq} have an equivalent norm

$$(26) \qquad \|f\|_{\tilde{K}^{\theta}_q} = \|f\|_{L_p} + \{ \int_0^{\infty} (t^{-\theta}\tilde{K}(f;t))^q \frac{dt}{t} \}^{1/q}.$$

Using the best one-sided trigonometrical approximation, we can define the spaces A^{θ}_{pq} in another way. Let us remember (see [1]—[3]) that the best one-sided approximation of the 2π-periodic bounded function f in the metric L_p, by means of trigonometrical polynomials of n-th order is given by

$$(27) \quad \tilde{E}_n(f)_{L_p} = \inf \{ \|P-Q\|_{L_p} : P, Q \in T_n;\ P(x) \le f(x) \le Q(x) \text{ for every } x \},$$

where T_n denotes the set of all trigonometrical polynomials of n-th order.

For the best one-sided trigonometrical approximation the following direct and converse theorem holds:

Theorem A ([1]—[3], [11]). *There exist constants $c_6(k)$, $c_7(k)$, depending only on k such that for every 2π-periodic bounded function f we have*

$$\tilde{E}_n(f)_{L_p} \le c_6(k)\tau_k(f;n^{-1})_{L_p}, \quad \tau_k(f;n^{-1})_{L_p} \le \frac{c_7(k)}{n^k} \sum_{s=0}^{n} (s+1)^{k-1}\tilde{E}_s(f)_{L_p}.$$

By means of Theorem A it is easy to prove the following theorem:

Theorem 2. *The following norms are equivalent:*

i) $$\|f\|_{A^{\theta}_{pq}};$$

ii) $$|||f||| = \|f\|_{L_p} + \tilde{E}_0(f)_{L_p} + \{ \sum_{n=0}^{\infty} (2^{\theta n}\tilde{E}_{2^n}(f)_{L_p})^q \}^{1/q}.$$

For the proof of this theorem we shall use some lemmas.

Lemma 1. *Let f be a bounded 2π-periodic function. Let $h = 2\pi m + \alpha$, where m is an integer and $|\alpha| < \pi$. Then $\Delta^k_h f(t) = \Delta^k_{\alpha} f(t)$.*

Proof.

$$\Delta^k_h f(t) = \sum_{s=0}^{k} (-1)^{k+s} \binom{k}{s} f(t+sh)$$

$$= \sum_{s=0}^{k} (-1)^{k+s} \binom{k}{s} f(t+2\pi ms+s\alpha) = \sum_{s=0}^{k} (-1)^{k+s} \binom{k}{s} f(t+s\alpha) = \Delta^k_{\alpha} f(t).$$

Lemma 2. *Let f be a bounded 2π-periodic function. Then for $t\geq 4\pi$ we have $\tau_k(f;t)_{L_p}=\tau_k(f;4\pi)_{L_p}=\omega_k(f,0;2\pi)$.*

Proof. Obviously the statement of the lemma follows from the following equality: for $t\geq 4\pi$ we have

$$(28)\qquad \omega_k(f,x;t)=\omega_k(f,x;4\pi)=\omega_k(f,0;2\pi).$$

Let us prove (28). By definition

$$\omega_k(f,x;t)=\sup\{|\Delta_h^k f(y)|:y,\ y+kh\in[x-kt/2,\ x+kt/2]\}.$$

From Leumma 1 it follows that for every h such that $y+kh\in[x-kt/2,\ x+kt/2]$ there exists α, $|\alpha|<\pi$, such that

$$(29)\qquad |\Delta_h^k f(y)|=|\Delta_\alpha^k f(y)|.$$

If $\alpha\geq 0$, since f is 2π-periodic, there exists y', $-2\pi\leq y'\leq 0$, such that

$$(30)\qquad |\Delta_\alpha^k f(y)|=|\Delta_\alpha^k f(y')|.$$

If $\alpha<0$, again since f is 2π-periodic, there exists y', $0\leq y'\leq 2\pi$, such that

$$(31)\qquad |\Delta_\alpha^k f(y)|=|\Delta_\alpha^k f(y')|.$$

From (20)—(31) it follows that there exist y', α such that $|\Delta_h^k f(y)|=|\Delta_\alpha^k f(y')|$ and y', $y'+k\alpha\in[-k\pi,k\pi]$ for $k\geq 2$, y', $y'+k\alpha\in[-2\pi,2\pi]$ for $k=1$. Consequently

$$(32)\qquad \begin{aligned}&\omega_k(f,x;t)\leq\omega_k(f,0;2\pi),\quad k\geq 2;\\ &\omega_1(f,x;t)\leq\omega_1(f,0;4\pi).\end{aligned}$$

We have for $x\in[-\pi,\pi]$

$$\begin{aligned}&\omega_k(f,0;2\pi)\leq\omega_k(f,x;4\pi),\quad k\geq 2;\\ &\omega_1(f,0;4\pi)=\omega_1(f,x;4\pi)\end{aligned}$$

From here and (32) follows (28).

Lemma 3 (Whitney [18]). *We have*

$$\begin{aligned}E_0(f)_C&=\inf\{\|f-\lambda\|_{C[0,2\pi]}:\lambda=\text{const}\}\\ &\leq c_8(k)\sup\{|\Delta_h^k f(x)|:x\in(-\infty,\infty),\quad h\in(-\infty,\infty)\},\end{aligned}$$

where $c_8(k)$ is a constant, depending only on k.

Lemma 4. *Let f be a bounded 2π-periodic function. There exists a constant $c_9(k)$ depending only on k such that $\tilde{E}_0(f)_{L_p}\leq c_9(k)\tau_k(f;2\pi)_{L_p}$.*

Proof. We have

$$(33)\qquad \tilde{E}_0(f)_{L_p}=\{\frac{1}{2\pi}\int_0^{2\pi}(\sup f-\inf f)^p dx\}^{1/p}=\sup f-\inf f=\frac{1}{2}E_0(f)_C.$$

From lemma 2 it follows

$$\sup\{|\Delta_h^k f(x)|:h\in(-\infty,\infty)\}=\omega_k(f,x;4\pi)=\omega_k(f,0;2\pi)$$

$$\sup\{|\Delta_h^k f(x)|:\ x\in(-\infty,\infty),\quad h\in(=\infty,\infty)\}=\omega_k(f,0;2\pi).$$

From here, lemma 2, lemma 3 and P3 it follows

$$\frac{1}{2} E_0(f)_C \leqq \frac{1}{2} c_8(k)\omega_k(f, 0; 2\pi) \tag{34}$$

$$= \frac{1}{2} c_8(k)\tau_k(f; 4\pi)_{L_p} \leqq c_9(k)\tau_k(f; 2\pi)_{L_p}.$$

From (33) and (34) follows the lemma.

Proof of Theorem 2. Let us prove first that $|||f||| \leqq c\|f\|_{A^\theta_{pq}}$. From theorem A we have

$$\tilde{E}_n(f)_{L_p} \leqq c_6(k)\tau_k(f; n^{-1})_{L_p}, \quad n \geqq 1.$$

Consequently

$$|||f||| = \|f\|_{L_p} + \tilde{E}_0(f)_{L_p} + \{\sum_{n=0}^{\infty} (2^{n\theta}\tilde{E}_{2^n}(f)_{L_p})^q\}^{1/q} \tag{35}$$

$$\leqq \|f\|_{L_p} + \tilde{E}_0(f)_{L_p} + \{\sum_{n=0}^{\infty} (2^{n\theta}c_6(k)\tau_k(f; 2^{-n})_{L_p})^q\}^{1/q}.$$

Since

$$(2^{n\theta}\tau_k(f; 2^{-n})_{L_p})^q \leqq 2^{1+\theta q} \int_{2^{-n}}^{2^{-n+1}} (t^{-\theta}\tau_k(f; t)_{L_p})^q \frac{dt}{t},$$

we obtain from (35)

$$|||f||| \leqq \|f\|_{L_p} + \tilde{E}_0(f)_{L_p} + c_6(k)2^{1/q+\theta}(\sum_{n=0}^{\infty} \int_{2^{-n}}^{2^{-n+1}} (t^{-\theta}\tau_k(f; t)_{L_p})^q \frac{dt}{t})^{1/q} \tag{36}$$

$$= \|f\|_{L_p} + \tilde{E}_0(f)_{L_p} + c_6(k)2^{1/q+\theta}\{\int_0^2 (t^{-\theta}\tau_k(f; t)_{L_p})^q \frac{dt}{t}\}^{1/q}.$$

From lemma 2 and lemma 4 we have $\tilde{E}_0(f)_{L_p} \leqq c_9(k)\tau_k(f; 4\pi)_{L_p} = c_9(k)\tau_k(f; t)_{L_p}$ for $t \geqq 4\pi$, consequently

$$\tilde{E}_0(f)_{L_p} \leqq c'(k, \theta)\{\int_{4\pi}^{\infty} (t^{-\theta}\tau_k(f; t)_{L_p})^q \frac{dt}{t}\}^{1/q}$$

what, together with (36), gives us $|||f||| \leqq c\|f\|_{A^\theta_{pq}}$, where the constant c depends on θ, k, q ($\theta > k$).

Let us prove now that $\|f\|_{A^\theta_{pq}} \leqq c|||f|||$. We have

$$\|f\|_{A^\theta_{pq}} = \|f\|_{L_p} + \{\int_0^{\infty} (t^{-\theta}\tau_k(f; t)_{L_p})^q \frac{dt}{t}\}^{1/q} \tag{37}$$

$$= \|f\|_{L_p} + \{\sum_{n=0}^{\infty} \int_{2^{-n}}^{2^{-n+1}} (t^{-\theta}\tau_k(f; t)_{L_p})^q \frac{dt}{t}\}^{1/q} + \{\int_2^{\infty} (\cdot)\}^{1/q}.$$

Since

$$\int_{2^{-n}}^{2^{-n+1}} (t^{-\theta}\tau_k(f;\ t)_{L_p})^q \frac{dt}{t} \leqq 2^{(2k+1)q}(2^{n\theta}\tau_k(f;\ 2^{-n})_{L_p})^q,$$

we obtain from (37)

(38) $$\|f\|_{A^\theta_{pq}} \leqq \|f\|_{L_p} + 2^{2k+1}\{\sum_{n=0}^{\infty}(2^{n\theta}\tau_k(f;\,2^{-n})^q_{L_p}\}^{1/q} + \{\int_2^{\infty}(\cdot)\}^{1/q}.$$

From theorem A we have for $n \geqq 1$

(39) $$\tau_k(f;\ n^{-1})_{L_p} \leqq \frac{c_7(k)}{n^k}\sum_{m=0}^{n}(m+1)^{k-1}\tilde{E}_m(f)_{L_p}.$$

From (38) and (39) it follows

(40) $$\|f\|_{A^\theta_{pq}} \leqq \|f\|_{L_p} + 2^{2k+1}\{\sum_{n=1}^{\infty}(\frac{c_7(k)2^{n\theta}}{2^{nk}}\sum_{m=0}^{2^n}(m+1)^{k-1}\tilde{E}_m(f)_{L_p})^p\}^{1/q}$$

$$+\{\int_2^{\infty}(t^{-\theta}\tau_k(f;\ t)_{L_p})^q\frac{dt}{t}\}^{1/q}.$$

For $2 \leqq t < \infty$ we have in view of lemma 2 $\tau_k(f;t)_{L_p} \leqq \tau_k(f;\ 4\pi)_{L_p}$.

Using (33), we have $\omega_k(f,\ x;\ 4\pi) \leqq 2^k(\sup f - \inf f) = 2^k\tilde{E}_0(f)_{L_p}$, what gives us

(41) $$\tau_k(f;\ 4\pi)_{L_p} \leqq 2^k\tilde{E}_0(f)_{L_p}.$$

From (41) it follows

(42) $$(\int_2^{\infty}(t^{-\theta}\tau_k(f;t)_{L_p})^q\frac{dt}{t})^{1/q}$$

$$\leqq 2^k\tilde{E}_0(f)_{L_p}(\int_2^{\infty}t^{-\theta q}\frac{dt}{t})^{1/q} \leqq c(k,\ \theta,\ q)\tilde{E}_0(f)_{L_p}.$$

On the other hand, we have

$$\sum_{m=0}^{2^n}(m+1)^{k-1}\tilde{E}_m(f)_{L_p} \leqq \tilde{E}_0(f)_{L_p} + \sum_{m=0}^{n}2^{(m+1)k}\tilde{E}_{2^m}(f)_{L_p},$$

what, together with (40), (42), give us

(43) $$\|f\|_{A^\theta_{pq}} \leqq \|f\|_{L_p} + c'(k,\ \theta,\ q)\tilde{E}_0(f)_{L_p}$$

(43) $$+c_8(k)\{\sum_{n=1}^{\infty}(2^{n(\theta-k)}\sum_{m=0}^{n}2^{mk}\tilde{E}_{2^m}(f)_{L_p})^q\}^{1/q}.$$

Since

(44) $$\sum_{n=1}^{\infty}2^{nq(\theta-k)}\{\sum_{m=0}^{n}2^{mk}\tilde{E}_{2^m}(f)_{L_p}\}^q \leqq c''(k,\ \theta,\ q)\sum_{m=0}^{\infty}2^{n\theta q}(\tilde{E}_{2^m}(f)_{L_p})^q$$

(compare with S. M. Nikolskii [17], p. 260]), (43) and (44) give us

$$\|f\|_{A^\theta_{pq}} \leq c\{\|f\|_{L_p} + \tilde{E}_0(f)_{L_p} + (\sum_{m=0}^{\infty} (2^{m\theta}\tilde{E}_{2^m}(f)_{L_p})^q)^{1/q}\},$$

where the constant c depends on k, θ, q, $k>\theta$.

3. The following connection exists between the spaces A^θ_{pq} and the Besov spaces B^θ_{pq}:

Theorem 3. *For* $\theta > 1/p$ *we have* $A^\theta_{pq} = B^\theta_{pq}$ *(by equivalent norms).*

Proof. Obviously is $f \in A^\theta_{pq}$ then $f \in B^\theta_{pq}$ and $\|f\|_{B^\theta_{pq}} \leq \|f\|_{A^\theta_{pq}}$. Let now $f \in B^\theta_{pq}$, and let $\theta > 1$. Then it is well-known that f is absolute continuous and the norm

$$\|f\|_{L_p} + (\int_0^\infty (t^{-\theta+1}\omega_{k-1}(f'; t)_{L_p})^q \frac{dt}{t})^{1/q}, \quad k>\theta,$$

is equivalent to $\|\cdot\|_{B^\theta_{pq}}$.

For f absolutely continuous the following inequality holds: $\tau_k(f; t)_{L_p} \leq c(k) t\omega_{k-1}(f'; t)_{L_p}$ (K. Ivanov, for the case $k=2$ see [13]).

Consequently

$$(\int_0^\infty (t^{-\theta}\tau_k(f; t)_{L_p})^q \frac{dt}{t})^{1/q} \leq c(\int_0^\infty (t^{-\theta+1}\omega_{k-1}(f'; t)_{L_p})^q \frac{dt}{t})^{1/q},$$

$$f \in A^\theta_{pq} \quad \text{and} \quad \|f\|_{A^\theta_{pq}} \leq c\|f\|_{B^\theta_{pq}}.$$

In the case when $1/p < \theta \leq 1$ we must use fraction derivatives. The proof is the same, if we use the following inequality of K. Ivanov [18]:

If $\alpha > 1/p$ and f has a fraction derivative $f^{(\alpha)}$ of order α, then

$$\tau_k(f; \delta)_{L_p} \leq c(\alpha, p)\delta^\alpha \omega_{k-\alpha}(f^{(2)}; \delta)_{L_p}.$$

For $\theta < 1/p$ the spaces A^θ_{pq} are not equal to the Besov spaces (see the example in the above-mentioned paper of K. Ivanov).

Let us mention at the end that the spaces A^θ_{pq} are obviously Banach spaces and that the usual imbedding theorems are valid for them.

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