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SOME MINIMAL ABELIAN GROUPS ARE PRECOMPACT

LUČEZAR N. STOJANOV

The main result of this paper is that if X is an Abelian group, D is the maximal divisible subgroup of X, T is the periodic part of X and card $X/(D+T)<\mathbf{c}$, then all minimal group topologies on X are precompact. Some other results are obtained for precompactness of minimal Abelian groups.

A Hausdorff topological group G is said to be minimal and the topology of G is called minimal group topology, if the topology of G is a minimal (in Zorn sence) element of the set of all Hausdorff group topologies on G. All compact Hausdorff groups are minimal. It is shown in [2] that in non-Abelian case there are non-precompact minimal group topologies. For the time being there are not known examples of non-precompact minimal Abelian groups. Prodanov [6, 7, 8] proves that all elements of some classes of minimal Abelian groups are precompact. Another result of that kind is proved in [9]. Here some new results are obtained for precompactness of minimal Abelian groups.

Prodanov [7] studied minimal group topologies by means of maximal ones. Here we continue the use of this technique. In Section 1 we introduce the notion relatively maximal group topology and study the connection between relatively maximal and maximal group topologies. The results of Section 1 are used in Section 2 to prove that each minimal Abelian group G such that nG is precompact for some natural n, is precompact. In section 3 we prove the main theorem. It is shown in Section 4 that all complete minimal group topologies on torsion-free Abelian groups without non-zero divisible subgroups are compact. As an application of this theorem we prove that $2^{\chi(G)}\leq \operatorname{card} G$ for each infinite complete minimal Abelian group G.

The maximal (in Zorn sence) non-discrete group topologies we call maximal group topologies. By P we denote the set of all primes, by $\mathbf{Z}_p$ — the compact group of p-adic numbers ($p\in P$), by $\mathbf{T}^1$ — the one dimensional torus and $\mathbf{c}=\operatorname{card}\mathbf{T}^1$. The closure of the set A is denoted by $\bar{A}$ and by (x) is denoted the group generated by x. If G is an Abelian group and $\{H_\alpha\}_\alpha$ is a set of subgroups of G, by $\Sigma_\alpha H_\alpha$ we denote the smallest subgroup of G, which contains H_α for each α. The subgroup of the periodic elements g of G such that the period of g is not a multiple of a square of a prime, is called the socle of G. If A and B are subsets of G, A is said to be big with respect to B, if there is a finite subset F of G with $A+F\supset B$. The group G is called bounded, if there exists a natural n with $nG=(0)$. If G is a topological group, by $\chi(G)$ we denote the character of G i. e. the minimum of card β when β runs over the fundamental systems of neighbourhoods of 0 in G. For each prime p by $\operatorname{td}_p(G)$ is denoted the smallest subgroup of G which contains all elements x of G such that the completion $\widehat{(x)}$ of (x) is a compact $\mathbf{Z}_p$-module. Some properties of the groups $td_p(G)$ are established in [10].

1. Relatively Maximal Group Topologies. Everywhere in this section G will be an Abelian group, H will be a subgroup of G and τ_0 — a complete Hausdorff group topology on H. If $\mathcal{T}$ is a group topology on G, by $\mathcal{T}_{|H}$ we denote the corresponding relative topology on H and by $\mathcal{T}/H$ — the corresponding quotient topology on G/H.

Definition. *A Hausdorff group topology $\mathcal{T}$ on G is called maximal with respect to τ_0, if $\mathcal{T}$ is a maximal (in Zorn sense) element of the set of all Hausdorff group topologies $\mathcal{T}'$ on G with $\mathcal{T}'_{|H}=\tau_0$ and $H\notin\mathcal{T}'$.*

By $\mathcal{U}_{G,\tau_0}$ we denote the infimum (in the set of the group topologies on G) of all maximal with respect to τ_0 group topologies on G. If $H=(0)$, $\mathcal{U}_{G,\tau_0}$ coincides with the submaximal topology $\mathcal{U}_G$ on G (see [7]), i. e. the infimum of all maximal group topologies on G. It is shown in [7, 1.3] that $\mathcal{P}_G\subset\mathcal{U}_G$, where $\mathcal{P}_G$ is the biggest precompact group topology on G.

In this section we study the connection between maximal with respect to τ_0 group topologies on G and maximal group topologies on G/H.

The following lemma will be used many times.

Lemma 1.1. *If T and $\mathcal{T}$ are group topologies on G, $\mathcal{T}'=\mathrm{Sup}(T,\mathcal{T})$ and $T_{|H}\subset\mathcal{T}_{|H}$, then $\mathcal{T}'_{|H}=\mathcal{T}_{|H}$.*

Proof. A typical neighbourhood of 0 in $\mathcal{T}'$ is $U\cap V$, where $0\in U\in T$ and $0\in V\in\mathcal{T}$. Since $(U\cap V)\cap H=(U\cap H)\cap(V\cap H)$ and $U\cap H\in T_{|H}\subset\mathcal{T}_{|H}$, we have $(U\cap V)\cap H\in\mathcal{T}_{|H}$, hence $\mathcal{T}'_{|H}\subset\mathcal{T}_{|H}$. The opposite inclusion is obvious. q. e. d.

Lemma 1.2. *For each maximal with respect to τ_0 group topology $\mathcal{T}$ on G, $\mathcal{T}/H$ is a maximal group topology on G/H.*

Proof. Let τ be a non-discrete group topology on G/H and $\mathcal{T}/H\subset\tau$ We show that $\mathcal{T}/H=\tau$.

Denote by φ the canonical epimorphism $G\to G/H$ and by T — the group topology on G with a fundamental system of neighbourhoods of 0 the set of all $\varphi^{-1}(W)$, where $0\in W\in\tau$. Clearly, $T_{|H}=\{H\}$, hence by Lemma 1.1

$$\mathcal{T}'_{|H}=\tau_0, \tag{1}$$

where $\mathcal{T}'=\mathrm{Sup}(T,\mathcal{T})$.

We prove that $H\notin\mathcal{T}'$. Suppose $H\in\mathcal{T}'$, then there exist U and V with $0\in U\in T$, $0\in V\in\mathcal{T}$ and $U\cap V\subset H$. By the definition of T, there is a subset W of G/H with $0\in W\in\tau$ and $U\supset\varphi^{-1}(W)$. Hence

$$\varphi^{-1}(W)\cap V\subset H. \tag{2}$$

Now we have

$$W\cap\varphi(V)=(0). \tag{3}$$

Indeed, if $\xi\in W\cap\varphi(V)$, there is $v\in V$ with $\xi=\varphi(v)$. Since $\xi\in W$, $v\in\varphi^{-1}(W)$ and therefore $v\in\varphi^{-1}(W)\cap V$. By (2) $v\in H$ and $\xi=\varphi(v)=0$.

On the other hand, $V\in\mathcal{T}$ and $\mathcal{T}/H\subset\tau$ imply $\varphi(V)\in\tau$. Since $W\in\tau$, by (3) we obtain $\{0\}\in\tau$ which is a contradiction. Hence $H\notin\mathcal{T}'$. Now (1), $\mathcal{T}\subset\mathcal{T}'$ and the maximality of $\mathcal{T}$ give $\mathcal{T}=\mathcal{T}'$. Therefore, $T\subset\mathcal{T}$ and $\tau=T/H$ implies $\tau=\mathcal{T}/H$ q. e. d.

In case τ_0 is compact, the opposite is also true.

L e m m a 1.3. *Let* τ_0 *be a compact Hausdorff group topology on* H. *It* τ *is a group topology on* G/H *with* $\mathscr{P}_{G/H}\subset\tau$, *then there exists a Hausdorff group topology* $\mathscr{T}$ *on* G *such that* $\mathscr{T}_{|H}=\tau_0$ *and* $\mathscr{T}/H=\tau$. *Moreover, if* τ *is a maximal group topology on* G/H, *then* $\mathscr{T}$ *is a maximal with respect to* τ_0 *group topology on* G.

P r o o f. *Let* $\varphi: G\to G/H$ be the canonical epimorphism and T be the group topology on G with a fundamental system of neighbourhoods of 0 the set of all $\varphi^{-1}(W)$, where $0\in W\in\tau$. Hence $T_{|H}=\{H\}$.

Denote by X the group of all characters $\chi: G\to \mathbf{T}^1$ such that $\chi_{|H}$ is continuous with respect to τ_0. Since τ_0 is compact, X separates the points of G. Therefore, the smallest group topology T_X, on G such that all elements of X are continuous with respect to T_X, is Hausdorff. Moreover, T_X is precompact and $(T_X)_{|H}=\tau_0$.

Consider $\mathscr{T}=\operatorname{Sup}(T, T_X)$, obviously $\mathscr{T}$ is a Hausdorff group topology on G. Since $T_{|H}\subset\tau_0=(T_X)_{|H}$, by Lemma 1.1

(4) $$\mathscr{T}_{|H}=\tau_0.$$

We show that $\mathscr{T}/H=\tau$. By $T\subset\mathscr{T}$ we have $\tau=T/H\subset\mathscr{T}/H$. To prove the inclusion $\mathscr{T}/H\subset\tau$ let us consider a typical neighbourhood $U\cap V$ of 0 in $\mathscr{T}$, where $0\in U\in T$ and $0\in V\in T_X$. There is $W\subset G/H$ with $0\in W\in\tau$ and $\varphi^{-1}(W)\subset U$. We have

(5) $$\varphi(U\cap V)\supset\varphi(\varphi^{-1}(W)\cap V)=W\cap\varphi(V).$$

Since T_X/H is precompact, $T_X/H\subset\mathscr{P}_{G/H}$. Now $\mathscr{P}_{G/H}\subset\tau$ implies $T_X/H\subset\tau$ and by $V\in T_X$ we obtain $\varphi(V)\in\tau$. Hence $W\cap\varphi(V)\in\tau$ and by (5) $\mathscr{T}/H\subset\tau$. Therefore $\mathscr{T}/H=\tau$.

Assume now that τ is a maximal group topology on G/H. Since $\mathscr{T}/H=\tau$ and τ is non-discrete,

(6) $$H\notin\mathscr{T}.$$

It remains to prove that $\mathscr{T}$ is maximal with (4) and (6). Suppose $\mathscr{T}'$ is a Hausdorff group topology on G, $\mathscr{T}'_{|H}=\tau_0$, $H\notin\mathscr{T}'$ and $\mathscr{T}\subset\mathscr{T}'$. Then $\mathscr{T}'/H$ is a non-discrete group topology on G/H and $\tau=\mathscr{T}/H\subset\mathscr{T}'/H$, hence $\mathscr{T}/H=\mathscr{T}'/H$. According to $\mathscr{T}_{|H}=\tau_0=\mathscr{T}'_{|H}$ and $\mathscr{T}\subset\mathscr{T}'$ we obtain $\mathscr{T}=\mathscr{T}'$ (see [4]). Therefore, $\mathscr{T}$ is maximal with respect to τ_0 q. e. d.

We shall use the following lemma here and in Section 3.

L e m m a 1.4. *Let* $\mathscr{T}$ *be a maximal with respect to* τ_0 *group topology on* G *and* T *be a Hausdorff group topology on* G *with* $T_{|H}=\tau_0$. *If* $T\not\subset\mathscr{T}$, *then* $\inf(T, \mathscr{T})$ *is Hausdorff.*

P r o o f. Denote $\mathscr{T}'=\operatorname{Sup}(T, \mathscr{T})$, by Lemma 1.1 $\mathscr{T}'_{|H}=\tau_0$. Since $T\not\subset\mathscr{T}$, we have $\mathscr{T}\subset\mathscr{T}'$, $\mathscr{T}\neq\mathscr{T}'$ and therefore $H\in\mathscr{T}'$. Hence there exist U_0 and V_0 with $0\in U_0\in T$, $0\in V_0\in\mathscr{T}$ and

(7) $$U_0\cap V_0\subset H.$$

Consider an element x of G with $x\in U+V$ for each U and V with

(8) $$0\in U\in T \text{ and } 0\in V\in\mathscr{T}.$$

We show that

(9) $$x\in U\cap H+V\cap H$$

for each U and V with (8) and

(10) $$U-U\subset U_0 \text{ and } V-V\subset V_0.$$

Since $x\in U+V$, there exist $x_U\in U$ and $x_V\in V$ such that $x=x_U+x_V$. For each U' and V' with $0\in U'\in T$, $0\in V'\in\mathcal{T}$, $U'\subset U$ and $V'\subset V$ we have $x=x_{U'}+x_{V'}$ for some $x_{U'}\in U'$ and $x_{V'}\in V'$. Therefore, by (10) and (7)

$$x_U-x_{U'}=x_{V'}-x_V\in(U-U)\cap(V-V)\subset U_0\cap V_0\subset H,$$

which implies $x_U\in U'+H$ and $x_V\in V'+H$. Since H is closed with respect to T and $\mathcal{T}$ we obtain $x_U\in H$ and $x_V\in H$. Hence (9) holds. Now $T_{|H}=\mathcal{T}_{|H}=\tau_0$ imply $x=0$. That is why $\inf(T,\mathcal{T})$ is Hausdorff q. e. d.

Consider the map λ, defined by $\lambda(\mathcal{T})=\mathcal{T}/H$. By Lemmas 1.2 and 1.3, λ maps the set of all maximal with respect to τ_0 group topologies on G onto the set of all maximal group topologies on G/H. We are going to show that λ is an injection.

Lemma 1.5. *Let τ_0 be a complete minimal group topology on H, $\mathcal{T}$ be a maximal with respect to τ_0 group topology on G and T — a Hausdorff group topology on G with $T_{|H}=\tau_0$ and $T/H\subset\mathcal{T}/H$. Then $T\subset\mathcal{T}$.*

Proof. Suppose $T\not\subset\mathcal{T}$. Then $H\in\operatorname{Sup}(T,\mathcal{T})$, hence there exist U_0 and V_0 with $0\in U_0\in T$, $0\in V_0\in\mathcal{T}$ and (7). Choose $U_1\in T$ such that $0\in U_1$ and $U_1+U_1\subset U_0$.

By Lemma 1.4 $T'=\inf(T,\mathcal{T})$ is a Hausdorff group topology. Since $T'_{|H}\subset\tau_0$ and τ_0 is minimal, $T'_{|H}=\tau_0$. Now $U_1\cap H\in T_{|H}=\tau_0$ shows that there exist symmetrical U and V with $0\in U\in T$, $0\in V\in\mathcal{T}$ and

(11) $$(U+V)\cap H\subset U_1,\ U\subset U_1 \text{ and } V\subset V_0.$$

We show that

(12) $$(U+H)\cap V\subset H.$$

Suppose $x\in(U+H)\cap V$, then $x=y+h$, where $y\in U$ and $h\in H$. We have $h=x-y\in(U+V)\cap H$ and by (11) $h\in U_1$. Now (11) and (7) imply

$$x=y+h\in(U_1+U_1)\cap V\subset U_0\cap V_0\subset H$$

which proves (12). Since $T/H\subset\mathcal{T}/H$, we have $U+H\in\mathcal{T}$. Then $V\in\mathcal{T}$ and (12) give $H\in\mathcal{T}$ which is a contradiction. Hence $T\subset\mathcal{T}$, q. e. d.

Corollary 1.6. *If τ_0 is compact, the map λ, defined by $\lambda(\mathcal{T})=\mathcal{T}/H$, is a bijection between the set of all maximal with respect to τ_0 group topologies on G and the set of all maximal group topologies on G/H.*

The following theorem gives a description of $\mathcal{U}_{G,\tau_0}$ by means of τ_0 and $\mathcal{U}_{G/H}$.

Theorem 1.7. *Let τ_0 be a compact Hausdorff group topology on H. Then $(\mathcal{U}_{G,\tau_0})_{|H}=\tau_0$ and $\mathcal{U}_{G,\tau_0}/H=\mathcal{U}_{G/H}$. Moreover, $\mathcal{U}_{G,\tau_0}$ is the unique group topology on G with these two properties.*

Proof. Without loss of generality we may assume that G/H is infinite.

Since $\mathcal{P}_{G/H}\subset\mathcal{U}_{G/H}$ (see [7, 1.3]), it follows from Lemma 1.3 that there is a Hausdorff group topology T on G with

(13) $$T_{|H}=\tau_0 \text{ and } T/H=\mathcal{U}_{G/H}.$$

Consider an arbitrary group topology T on G with (13). We prove that $T=\mathcal{U}_{G,\tau_0}$. Clearly, T is Hausdorff. For each maximal with respect to τ_0 group topology $\mathcal{T}$ on G we have $T/H\subset\mathcal{T}/H$ and by Lemma 1.4 $T\subset\mathcal{T}$. Hence

(14) $$T\subset\mathcal{U}_{G,\tau_0}.$$

Since G/H is infinite, there is a maximal group topology on G/H and by Lemma 1.3, there is a maximal with respect to τ_0 group topology $\mathcal{T}$ on G. By (14) and the definition of $\mathcal{U}_{G,\tau_0}$ we have $T\subset U_{G,\tau_0}\subset\mathcal{T}$ and therefore, $\tau_0=T_{|H}\subset(\mathcal{U}_{G,\tau_0})_{|H}\subset\mathcal{T}_{|H}=\tau_0$, which implies $(\mathcal{U}_{G,\tau_0})_{|H}=\tau_0$.

On the other hand for each maximal group topology τ' on G/H, there is a maximal with respect to τ_0 group topology $\mathcal{T}'$ on G with $\mathcal{T}'/H=\tau'$ and therefore $\mathcal{U}_{G,\tau_0}/H\subset\mathcal{T}'/H=\tau'$, because $\mathcal{U}_{G,\tau_0}\subset\mathcal{T}'$. That is why $\mathcal{U}_{G,\tau_0}/H\subset\mathcal{U}_{G/H}$. The opposite inclusion follows from (14) and (13). Hence $\mathcal{U}_{G,\tau_0}/H=\mathcal{U}_{G/H}$. Now $(\mathcal{U}_{G,\tau_0})_{|H}=\tau_0$, (13) and (14) imply $T=\mathcal{U}_{G,\tau_0}$ (see [4]) q. e. d.

It is shown in the following example that in general case if $\{T_\alpha\}_\alpha$ is a set of Hausdorff group topologies on G and $T=\inf_\alpha T_\alpha$ (in the set of all group topologies on G), then $(T_\alpha)_{|H}=\tau_0$ for each α does not imply $T_{|H}=\tau_0$.

Example 1.8. Let G be an infinite countable Abelian group and p be a prime with $pG=(0)$. If $\{T_\alpha\}_\alpha$ is the set of all Hausdorff group topologies on G, then $T=\inf_\alpha T_\alpha$ is not Hausdorff. Indeed, if T is Hausdorff, then T is minimal and by [7, 2.6] T is compact, which is impossible. In fact $T=\{G\}$, since for each non-zero elements x and y of G there is an isomorphism $\psi: G\rightarrow G$ with $\psi(x)=y$. Consider a non-zero element x_0 of G and denote $H=(x_0)$. Then $(T_\alpha)_{|H}=\tau_0$, where τ_0 is the discrete topology on H, and $T_{|H}\neq\tau_0$.

2. Minimal and Relatively Maximal Group Topologies. It is shown in [7, 2.1] that each maximal group topology on an Abelian group G is stronger than each minimal group topology on G. The following proposition specifies this result.

Proposition 2.1. *Let G be an Abelian group, H be a subgroup of G and τ_0 — a complete Hausdorff group topology on H. If T is a minimal group topology on G with $T_{|H}=\tau_0$ and $\mathcal{T}$ is a maximal with respect to τ_0 group topology on G, then $T\subset\mathcal{T}$.*

Proof. Assume $T\not\subset\mathcal{T}$, it follows from Lemma 1.4 that $\inf(T,\mathcal{T})$ is Hausdorff. Since $\inf(T,\mathcal{T})\subset T$ and T is minimal, $\inf(T,\mathcal{T})=T$ and therefore $T\subset\mathcal{T}$, which is a contradiction q. e. d.

Corollary 2.2. *Let G be a minimal Abelian group and H be a compact subgroup of G. If T is the topology of G, then*

(15) $$T/H\subset\mathcal{U}_{G/H}.$$

Proof. The statement follows from proposition 2.1 and Theorem 1.7 q. e. d.

Corollary 2.3. *If G is a minimal Abelian group and H is a compact subgroup of G, then the socle of G/H is precompact in the quotient topology.*

Proof. Denote by T the topology of G, then (15) holds. Since the socle of G/H is precompact in $\mathcal{U}_{G/H}$ (see [7, 1.3]), by (15) it is precompact in T/H too q. e. d.

The following theorem generalizes the results of the second section of [9].

Theorem 2.4. *Let G be a minimal Abelian group. If there is a natural n such that nG is precompact in the relative topology, then G is precompact.*

P r o o f. Without loss of generality we may assume that G is complete. Denote by n the minimal natural such that nG is precompact. We prove that $n=1$. Suppose $n>1$, then there exist a prime p and a natural m with $n=pm$. Denote $H=\overline{nG}$ and consider the group G' of those $x\in G$ such that $px\in H$. Obviously, G' is a closed (and hence minimal) subgroup of G and H is a compact subgroup of G'. Since $p. G'/H=(0)$, corollary 2.3 shows that G'/H is precompact. Therefore, G' is precompact and $mG\subset G'$ implies that mG is precompact, which is a contradiction with the choice of n. Hence G is precompact, q. e. d.

C o r o l l a r y 2.5. *Let G be a minimal Abelian group and H be a periodic subgroup of G such that for each prime p the p-component of H is bounded. Then H is precompact in the relative topology.*

P r o o f. By Theorem 2.4 and minimality criterion, the p-component H_p of H is precompact for each prime p.

Let U be a neighbourhood of 0 in G. It follows from [7, 1.1 and 2.2] that there is a natural n such that U is big with respect to nG. Let $p_1, p_2, \ldots, p_k$ be all primes which devide n. If p is another prime, then $H_p\subset nG$. Since $\Sigma_{\nu=1}^{k} H_{p_\nu}$ is precompact, U is big with respect to $\Sigma_{\nu=1}^{k} H_{p_\nu}$ and therefore, $U+U$ is big with respect to $\Sigma_{\nu=1}^{k} H_{p_\nu}+nG\supset H$. Hence H is precompact q. e. d.

3. The Main Theorem. The following lemma will play an important role below.

L e m m a 3.1. *Let G be a complete minimal Abelian group,*

$$K=\bigcap_{n=1}^{\infty} \overline{nG} \tag{16}$$

and

$$G_p=\bigcap\{\overline{nG}/n=1, 2, \ldots, (n, p)=1\} \quad (p\in P). \tag{17}$$

Then for each prime p the group G_p/K is a topological $\mathbf{Z}_p$-module,

$$G_p=\mathrm{td}_p(G)+K \tag{18}$$

and there is a continuous isomorphism φ: $\Pi_{p\in P} G_p/K \to G/K$ with

$$\varphi \mid \bigoplus_p G_p/K=\mathrm{id}. \tag{19}$$

Hence, if G_p is compact for each $p\in P$, then G is compact.

P r o o f. Let p be a prime. For each neighbourhood U of 0 in G there is a natural k with

$$p^k G_p\subset U+K. \tag{20}$$

Indeed, by [7, 2.9] there is a natural n such that $\overline{nG}\subset U+K$. There exist k and m with $n=p^k m$ and such that p does not devide m. Then

$$p^k G_p\subset p^k \overline{mG} \subset \overline{p^k mG}=\overline{nG}\subset U+K$$

and (20) holds. It is clear now that G_p/K is a topological $\mathbf{Z}_p$-module. If $\psi_p: G_p \to G_p/K$ is the canonical epimorphism, it follows from [7, 2.7 and 10, 1.5] that $\psi_p(\mathrm{td}_p(G))=G_p/K$, hence (18) is proved.

Denote $G'=\Sigma_{p\in P} G_p$. Algebraically G'/K may be represented in the form $G'/K=\bigoplus_{p\in P} G_p/K$. We show that the product topology on $\bigoplus_{p\in P} G_p/K$ is stronger than the topology of G'/K. For this purpose it is sufficient to prove that for each neighbourhood U of 0 in G there exist primes $p_1, p_2, \dots, p_n$ and a neighbourhood W of 0 in G such that

(21) $$W\cap G_{p_1}+\cdots+W\cap G_{p_n}+\Sigma\{G_p/p\in P\setminus\{p_1,\dots,p_n\}\}\subset K+U.$$

Let U be a neighbourhood of 0 in G. There is a neighbourhood V of 0 in G with $V+V\subset U$. By [7, 2.9] there exists a natural m such that

(22) $$\overline{mG}\subset K+V.$$

Let $p_1, p_2, \dots, p_n$ be all primes which divide m, for each other prime p we have $G_p\subset\overline{mG}$ and therefore,

(23) $$\Sigma\{G_p/p\in P\setminus\{p_1,\dots,p_n\}\}\subset\overline{mG}.$$

Choose a neighbourhood W of 0 in G with

(24) $$\underbrace{W+W+\cdots+W}_{n}\subset V$$

Then (24), (23), (22) and $V+V\subset U$ imply (21). Hence id: $\oplus_{p\in P}(G_p/K)\to G'/K$ is continuous when $\oplus_{p\in P}(G_p/K)$ is provided with the product topology. Since K is compact, $\bar{G}'/K$ is complete. Therefore, there is a continuous homomorphism $\varphi:\Pi_{p\in P}(G_p/K)\to\bar{G}'/K$ with (19).

We show that $\bar{G}'=G$ and φ is an isomorphism. If $x\in G$ then $H=\overline{(x)}$ is compact and therefore $H\cap G_p$ is compact for each $p\in P$. Since $\mathrm{td}_p(G)\subset G_p$ $(p\in P)$, [10, 1.3 and 1.6] show that $\Sigma_{p\in P}H\cap G_p$ is dense in H, which implies

(25) $$\psi(H)=\overline{\sum_{p\in P}\psi(H\cap G_p)},$$

where $\psi: G\to G/K$ is the canonical epimorphism. On the other hand $\Pi_{p\in P}\psi_p(H\cap G_p)$ is a compact subgroup of $\Pi_{p\in P}(G_p/K)$ and

(26) $$\prod_{p\in P}\psi_p(H\cap G_p)=\overline{\bigoplus_{p\in P}\psi_p(H\cap G_p)}.$$

By (19) we have

$$\varphi(\bigoplus_{p\in P}\psi_p(H\cap G_p))=\bigoplus_{p\in P}\psi_p(H\cap G_p)=\sum_{p\in P}\psi(H\cap G_p).$$

Now (25) and (26) imply $\psi(H)=\varphi(\Pi_{p\in P}\psi_p(H\cap G_p))$ and therefore $H\subset\bar{G}'+K=\bar{G}'$. Hence $\bar{G}'=G$ and φ is an epimorphism.

It remains to prove that φ is a monomorphism. Suppose $\mathrm{Ker}\,\varphi\neq(0)$, then there exists a non-zero compact subgroup L of $\mathrm{Ker}\,\varphi$. The reasonings of the proof of [5, 3.1] show that $L=\Pi_{p\in P}L_p$, where L_p is a compact subgroup of G_p/K for each $p\in P$. Now $L_p\subset\mathrm{Ker}\,\varphi$ and (19) imply $L_p=(0)$ for each p and therefore $L=(0)$. Contradiction. Hence φ is a monomorphism q. e. d.

It turns out that in some cases the groups G_p are compact.

Lemma 3.2. *Let G be a complete minimal Abelian group, p be a prime and G_p be defined by* (17). *If there is a compact subgroup H of G_p such that G_p/H is periodic, then G_p is compact.*

Proof. Let H be a compact subgroup of G_p and G_p/H be periodic. Without loss of generality we may assume that $K \subset H$, where K is defined by (16). Then by Lemma 3.1. G_p/H is a periodic p-group.

Suppose G_p is not compact. For each natural n denote by H_n the group of those $x \in G_p$ such that $p^n x \in H$. If $p^n G_p \subset H_n$ for some natural n, then $p^{2n} G_p \subset H$ and by 2.4 G_p is compact, which is a contradiction with our assumption. Hence for each natural n there is $x_n \in G_p$ with

(27) $$x_n \in p^n G_p \setminus H_n.$$

Consider the subgroup G' of G_p generated by $x_1, x_2, \ldots, x_n, \ldots$ We show that G' is precompact. If U is a neighbourhood of 0 in G, there is a natural k with (20) (see the proof of Lemma 3.1) and therefore $\Sigma_{\nu=k}^{\infty}(x_\nu) \subset U+K$. On the other hand [7, 2.7] shows that $\Sigma_{\nu=1}^{k-1}(x_\nu)$ is precompact, hence U is big with respect to $K+\Sigma_{\nu=1}^{k-1}(x_\nu)$ and $U+U$ is big with respect to G'. That is why G' is precompact. Then $\bar{G}'$ is compact and if $\mu: G_p \to G_p/H$ is the canonical epimorphism, $\mu(\bar{G}')$ is a compact subgroup of G_p/H. Since G_p/H is periodic, there is a natural n with $p^n \mu(\bar{G}') = (0)$, i. e. $p^n \bar{G}' \subset H$ and $\bar{G}' \subset H_n$. Hence $x_n \in H_n$ which is a contradiction with (27) q. e. d.

Corollary 3.3. *Let G be a complete minimal Abelian group and H be a compact subgroup of G such that G/H is periodic. Then G is compact and there exists a natural n with $nG \subset H$.*

Proof. The statement follows from Lemmas 3.1 and 3.2 q. e. d.

We are going to prove the main result in the paper.

Theorem 3.4. *Let X be an Abelian group, D be the maximal divisible subgroup of X and T be the periodic part of X. If*

(28) $$\operatorname{card} X/(D+T) < \mathbf{c},$$

then all minimal group topologies on X are precompact.

Proof. Let X be provided with a minimal group topology and G be the completion of X with respect to this topology. We have to prove that G is compact, by Lemma 1.3 it will be done, if we show that G_p (defined by (17)) is compact for each $p \in P$.

Let p be a prime and K be defined by (16). Denote by μ the canonical epimorphism $G_p \to G_p/K$. It follows from (28) that there is a subgroup Y of X with

(29) $$\operatorname{card} Y < \mathbf{c}$$

and

(30) $$X = D + T + Y.$$

We show that for each subgroup L of G_p/K such that L is topologically isomorphic to $\mathbf{Z}_p$,

(31) $$L \cap \mu(Y \cap G_p) \neq (0)$$

holds. There is an element u of L with $L = \overline{(u)}$. Let $v \in G_p$ and $\mu(v) = u$. Then $H = \overline{(v)}$ is a compact subgroup of G_p and $\mu(H) = L$. Now [10, 1.5] shows that there exists an element $h \in \operatorname{td}_p(H)$ with $\mu(h) = u$. Obviously, $H' = \overline{(h)}$ is topologically isomorphic to $\mathbf{Z}_p$ and

(32) $$\mu(H')=L.$$

By the minimality criterion [1] $X\cap H'\neq(0)$ and by (30) there exist $d\in D$, $t\in T$ and $y\in Y$ with $0\neq d+t+y\in H'$. Since T is periodic, $mt=0$ for some natural m. Hence

(33) $$0\neq md+my\in H'.$$

We have $my\in Y\cap G_p$. Indeed, $H'\subset H\subset G_p$ and by (33) $my\in H'+D\subset G_p+D$. On the other hand D is divisible, hence $D\subset G_p$ and $my\in G_p$. Now $y\in Y$ implies $my\in Y\cap G_p$. By (32) we have $H'\cap \operatorname{Ker}\mu=(0)$ and (33) gives $\mu(md+my)\neq 0$. Now $md\in D\subset K$ and (32) imply $0\neq\mu(my)=\mu(md+my)\in L$, which proves (31).

Using the idea of the proof of [5, 3.5] we establish that G_p/K does not contain copies of $\mathbf{Z}_p^2$. Assume the contrary. Then there exists a set $\{L_\alpha\}_\alpha$ with cardinality $\mathbf{c}$ of subgroups of G_p/K such that L_α is topologically isomorphic to $\mathbf{Z}_p$ for each α and $L_\alpha\cap L_\beta=(0)$ for $\alpha\neq\beta$. By (31) $L_\alpha\cap\mu(Y\cap G_p)\neq(0)$ for each α and therefore card $Y\cap G_p\geq\mathbf{c}$, which is a contradiction with (29). Hence G_p/K does not contain copies of $\mathbf{Z}_p^2$.

To prove that G_p is compact, by Lemma 3.2 it is sufficient to show that there is a compact subgroup H of G_p such that G_p/H is periodic. If G_p/K is periodic, we set $H=K$. Suppose G_p/K is not periodic, then there is a non-torsion element u of G_p/K. Choose an arbitrary element g of G_p with $\mu(g)=u$ and denote $H=\overline{(g)}+K$. Clearly, H is a compact subgroup of G_p and G_p/H is somorphic to $(G_p/K)/\overline{(u)}$. If v is a non-torsion element of C_p/K, we have

(34) $$\overline{(u)}\cap\overline{(v)}\neq(0).$$

Indeed, $\overline{(u)}$ and $\overline{(v)}$ are topologically isomorphic to $\mathbf{Z}_p$ and $\overline{(u)}\cap\overline{(v)}=(0)$ implies that $\overline{(u)}+\overline{(v)}$ is topologically isomorphic to $\mathbf{Z}_p^2$. Since G_p/K does not contain copies of $\mathbf{Z}_p^2$, (34) holds. Hence there exists a natural n with $p^n v\in\overline{(u)}$. We established that $(G_p/K)/\overline{(u)}$ is periodic, therefore G_p/H is also periodic. Now Lemma 3.2 implies that G_p is compact and this completes the proof of the theorem q. e. d.

The above theorem generalizes some results from [7, 8, 9]. In particular we obtain that all minimal group topologies on periodic Abelian groups are precompact. Using the reasonings of the proof of [3, 3.7] we are able to describe all periodic Abelian groups which admit minimal group topologies. Here we mention only the following.

Corollary 3.5. *If p is a prime and G is an unbounded periodic Abelian p-group, then G does not admit minimal group topologies.*

Proof. Assume that there is a minimal group topology on G and denote by $\widehat{G}$ the completion of G with respect to this topology. By Theorem 3.4, $\widehat{G}$ is compact and [1] implies that the periodic part of $\widehat{G}$ is a p-group.

By [3, 2.4], there exists an exact sequence

$$0\longrightarrow F_p\longrightarrow\widehat{G}\longrightarrow\mathbf{T}^n\longrightarrow 0,$$

where F_p is a compact p-group and n is a non-negative integer. There is a natural k with $p^kF_p=(0)$, hence $p^k\widehat{G}$ is isomorphic to $\mathbf{T}^n$. Therefore the periodic part of T^n (as a subgroup of $\widehat{G}$) is a p-group which implies $n=0$. Now we have $p^kG=(0)$. Contradiction q. e. d.

4. Minimal Group Topologies on Torsion-free Abelian Groups. This section deals with complete minimal torsion-free Abelian groups. Let us mention that by the minimality criterion [1], if G is a minimal torsion-free Abelian group, then the completion $\widehat{G}$ of G is also a minimal torsion-free Abelian group.

Proposition 4.1. *If G is a complete minimal torsion-free Abelian group, then for each natural n the homomorphism $\varphi: G \to nG$, defined by $\varphi(x)=nx$, is a topological isomorphism and nG is a closed subgroup of G.*

Proof. Obviously, φ is a continuous isomorphism and by the minimality of G, φ is a topological isomorphism. Now the completeness of G gives that nG is a closed subgroup of G q. e. d.

Lemma 4.2. *Let G be a complete minimal torsion-free Abelian group and p be a prime with $\bigcap_{n=1}^{\infty} p^nG=(0)$. Then:*

(i) *for each neighbourhood U of 0 in G there is a natural n with $p^nG \subset U$;*

(ii) *for each neighbourhood U of 0 in G there is a neighbourhood V of 0 in G such that $x \in G$ and $px \in V$ imply $x \in U$;*

(iii) *for each neighbourhood U of 0 in G there is a neighbourhood W of 0 in G with $pW \subset W \subset U$;*

(iv) *for each natural n, if p does not divide n, then $nG=G$.*

Proof. (i) If $p^nG \not\subset U$ for each natural n, then the sets $V+p^kG$, where V runs over the neighbourhoods of 0 in G and $k=1, 2, \ldots$, form a fundamental system of neighbourhoods of 0 for a Hausdorff group topology on G strictly weaker than the topology of G. Contradiction with the minimality of G.

(ii) By proposition 4.1 the homomorphism $\varphi: G \to pG$, defined by $\varphi(x)=px$, is a topological isomorphism and if U is a neighbourhood of 0 in G, then $\varphi(U)$ is a neighbourhood of 0 in pG. Hence there is a neighbourhood V of 0 in G with $V \cap pG=\varphi(U)$. Suppose $x \in G$ and $px \in V$, then $px \in V \cap pG$ and therefore $px \in \varphi(U)$. I. e. there is $u \in U$ with $px=\varphi(u)=pu$. Since G is torsion-free, $x=u$ and $x \in U$.

(iii) There is a neighbourhood V of 0 in G with $V+V \subset U$. By (i) $p^nG \subset V$ for some natural n. Let V' be a symmetrical open neighbourhood of 0 in G with

$$\underbrace{V'+V'+\cdots+V'}_{k} \subset V, \tag{35}$$

where $k=\Sigma_{j=0}^{n-1} p^j$. Denote

$$W=V'+pV'+\cdots+p^{n-1}V'+p^nG.$$

Obviously, W is a symmetrical open neighbourhood of 0 in G. By (35) we have

$$W \subset \underbrace{V'+\cdots+V'}_{k}+p^nG \subset V+V \subset U.$$

Moreover,

$$pW=pV'+p^2V'+\cdots+p^nV'+p^{n+1}G \subset pV'+\cdots+p^{n-1}V'+p^nG \subset W.$$

(iv) Let $x \in G$. For each natural k there exist integers s and t with $sn+tp^k=1$. Then $x=snx+tp^kx \in nG+p^kG$. Hence $x \in nG+p^kG$ for each natural k and by (i) $x \in \overline{nG}=nG$ q. e. d.

The following lemma is fundamental for this section.

Lemma 4.3. *Let G be a complete minimal torsion-free Abelian group and p be a prime with $\bigcap_{n=1}^{\infty} p^n G=(0)$. Then G is compact.*

Proof. We show that each neighbourhood of 0 in G is big with respect to G.

Let U_0 be a neighbourhood of 0 in G. We shall establish first that U_0+pG is big with respect to G. There is a symmetrical open neighbourhood V of 0 in G such that $sV\subset U_0$ for $s=1, 2, \dots, p-1$. Denote

$$U_1=\left(\bigcup_{s=1}^{p-1} sV\right)+pG.$$

It follows from Proposition 4.1 and Lemma 4.2 (iv) that sV is an open neighbourhood of 0 in G $(s=1, 2, \dots, p-1)$, hence U_1 is a symmetrical open neighbourhood of 0 in G and $U_1\subset U_0+pG$. It is easy to see that

(36) $$U_1+pG=U_1$$

and

(37) $$nU_1\subset U_1, \quad n=0, \pm 1, \pm 2, \dots$$

We prove that U_1 is big with respect to G. Denote by M a maximal (in Zorn sense) subset of G such that for each n, if $x_1, x_2, \dots, x_n$ are different elements of M and $r_1, r_2, \dots, r_n$ are integers, then $r_1x_1+r_2x_2+\dots+r_nx_n\in U_1$ implies that p devides r_j for $j=1, 2, \dots, n$. The existence of M follows from Zorn's lemma. We show that

(38) $$F_M+U_1=G,$$

where F_M is the set of all sums $\Sigma_{j=1}^{k} r_jx_j$, where k is a natural; $x_1, x_2, \dots, x_k$ are different elements of M and for each $j=1, 2, \dots, k$; r_j is an integer with $|r_j|<p$. Let $x\in G$. If $x\in M$, then x belongs to F_M+U_1. Suppose $x\notin M$, then the maximality of M implies that there exist different elements $x_1, x_2, \dots, x_k$ of M and integers $r, r_1, \dots, r_k$ such that p does not divide r and

(39) $$rx+r_1x_1+\dots+r_kx_k\in U_1.$$

There exist integers m and n with

(40) $$mr+np=1.$$

By (39) and (37) $mrx+mr_1x_1+\dots+mr_kx_k\in mU_1\subset U_1$ and by (40)

(41) $$x-np+mr_1x_1+\dots+mr_kx_k\in U_1.$$

For each $j=1, 2, \dots, k$ there exist integers t_j and r'_j with $|r'_j|<p$ and $mr_j=pt_j-r'_j$. Now (36) and (41) imply $x\in r'_1x_1+\dots+r'_kx_k+U_1\subset F_M+U_1$ which proves (38).

To prove that U_1 is big with respect to G by (38) it is enough to show that M is finite. Assume the contrary. Then there is an infinite sequence $x_1, x_2, \dots, x_n, \dots$ of different elements of M. For each natural n denote by L_n the subgroup of G generated by $x_n, x_{n+1}, \dots, x_{n+k}, \dots$ It is easy to see that the sets $U+L_n$, where U runs over the neighbourhoods of 0 in G and

$n=1, 2, \dots$, form a fundamental system of neighbourhoods of 0 for a group topology τ on G. Moreover, τ is strictly weaker than the initial topology on G, because $x_n \in L_n \setminus U_1$ and therefore $U+L_n \not\subset U_1$ for each neighbourhood U of 0 in G and each natural n.

We show that τ is Hausdorff, this will be a contradiction with the minimality of G. Let x be an arbitrary element of G such that

(42) $$x \in \bigcap \{U+L_n / U \text{ is a neighbourhood of } 0 \text{ in } G;\ n=1,2,\dots\}.$$

In order to prove that $x=0$ we shall need some technical preparation.

Using Lemma 4.2 (ii) and (iii) we construct a sequence $U_2, U_3, \dots, U_n, \dots$ of symmetrical open neighbourhoods of 0 in G such that

(43) $$pU_n \subset U_n, \quad n=2, 3, \dots,$$

and for each $y \in G$, $py \in U_n$ implies $y \in U_{n-1}$, $n=2, 3, \dots$ It is easy to see that

(44) $$U_n \subset U_{n-1}, \quad n=2, 3, \dots,$$

and

(45) $$p^{n-1} y \in U_n \text{ implies } y \in U_1, \quad n=2, 3, \dots$$

For the element x, satisfying (42), we prove

(46) $$L_m \cap (x+U_n) \subset p^n L_m, \quad n=1, 2, \dots;\ m=1, 2, \dots$$

To prove (46) we shall use an induction with respect to n. We omit the case $n=1$, because its poof is similar to the proof of the general case.

Suppose $k>1$ and (46) is true for $n=k-1$ and each natural m. Let m be an arbitrary natural. We show that (46) holds for $n=k$ and m. Take an element y of $L_m \cap (x+U_k)$. By (44) $x+U_k \subset x+U_{k-1}$ and by the inductive hypothesis $y \in p^{k-1} L_m$. Hence there exist integers $t_m, t_{m+1}, \dots, t_s$ $(s \geqq m)$ and $u \in U_k$ with

(47) $$y = p^{k-1}(t_m x_m + \dots + t_s x_s) = x+u.$$

Since $u \in U_k$ and U_k is open, there is a symmetrical neighbourhood V of 0 in G such that $V \subset U_{k-1}$ and

(48) $$u+V \subset U_k.$$

Since $x \in V+L_{s+1}$ (see (42)), there exists $z \in L_{s+1}$ with $z \in x-V$. By the inductive hypothesis we have

$$L_{s+1} \cap (x+U_{k-1}) \subset p^{k-1} L_{s+1}$$

and therefore,

$$z \in L_{s+1} \cap (x-V) \subset L_{s+1} \cap (x+U_{k-1}) \subset p^{k-1} L_{s+1}.$$

Hence there exist integers $t_{s+1}, \dots, t_r$ $(r \geqq s+1)$ such that

$$z = p^{k-1}(t_{s+1} x_{s+1} + \dots + t_r x_r).$$

Now $x \in V+z$ implies

(49) $$x \in V + p^{k-1}(t_{s+1} x_{s+1} + \dots + t_r x_r)$$

and by (47) and (49)

$$p^{k-1}(t_m x_m + \cdots + t_s x_s) = x + u \in u + V + p^{k-1}(t_{s+1} x_{s+1} + \cdots + t_r x_r).$$

According to (48) we obtain

$$p^{k-1}(t_m x_m + \cdots + t_s x_s - t_{s+1} x_{s+1} - \cdots - t_r x_r) \in u + V \subset U_k$$

and by (45)

(50) $$t_m x_m + \cdots + t_s x_s - t_{s+1} x_{s+1} - \cdots - t_r x_r \in U_1.$$

Since $x_m, \ldots, x_r$ are different elements of M, the definition of M and (50) give that p divides t_j for $j = m, m+1, \ldots, r$. Hence by (47)

$$y = p^{k-1}(t_m x_m + \cdots + t_s x_s) \in p^k L_m$$

which proves (46).

Now we are able to prove that $x = 0$. For this purpose it is enough to show that $x \in U$ for each neighbourhood U of 0 in G. Let U be an arbitrary neighbourhood of 0 in G and V be a neighbourhood of 0 in G with $V + V \subset U$. Then $p^n G \subset V$ for some natural n. It follows from (42) that $x \in V \cap U_n + L_1$, hence there is $y \in L_1$ with

(51) $$x \in V \cap U_n + y.$$

Then $y \in L_1 \cap (x + U_n)$ and by (46) $y \in p^n L_1 \subset p^n G$. According to (51) and the choice of V we have

$$x \in V + y \subset V + p^n G \subset V + V \subset U.$$

Hence $x \in U$ for each neighbourhood U of 0 in G and $x = 0$. In this way we have established that τ is a Hausdorff group topology on G strictly weaker than the initial one, which is a contradiction with the minimality of G. Hence M is finite and by (38), U_1 is big with respect to G. Since $U_1 \subset U_0 + pG$, we obtain that $U_0 + pG$ is big with respect to G.

We are going to prove that $U + p^n G$ is big with respect to G for each neighbourhood U of 0 in G and each natural n. Up to here we have proved this for $n = 1$ and arbitrary U. Suppose $n > 1$ and $V + p^{n-1} G$ is big with respect to G for each neighbourhood V of 0 in G. Let U be a neighbourhood of 0 in G and V be a neighbourhood of 0 in G with $V + V \subset U$. Then $W = \{x \in G / px \in V\}$ is a neighbourhood of 0 in G and by the inductive hypothesis there is a finite set $F \subset G$ such that $W + p^{n-1} G + F = G$. Since $pW \subset V$, we have

$$V + p^n G + pF \supset pW + p^n G + pF = p(W + p^{n-1} G + F) = pG$$

and therefore,

(52) $$V + V + p^n G + pF \supset V + pG.$$

On the other hand $V + pG$ is big with respect to G, hence there is a finite set $E \subset G$ with

(53) $$V + pG + E = G.$$

Now $V + V \subset U$, (52) and (53) imply

(54) $$U + p^n G + pF + E \supset V + V + p^n G + pF + E \supset V + pG + E = G,$$

Since F and E are finite, (54) shows that $U+p^nG$ is big with respect to G.

Let U be a neighbourhood of 0 in G. There exists a neighbourhood V of 0 in G with $V+V\subset U$. By Lemma 4.2 (i) there is a natural n such that $p^nG\subset V$. Since $V+p^nG$ is big with respect to G and $V+p^nG\subset V+V\subset U$, we have that U is big with respect to G. Hence G is precompact and the completeness of G implies that G is compact q. e. d.

The following theorem is the main result in this section.

Theorem 4.4. *Let G be a torsion-free Abelian group without non-zero divisible subgroups. Then all complete minimal group topologies on G are compact.*

Proof. Since G is torsion-free, $K=\bigcap_{n=1}^{\infty} nG$ is divisible and therefore $K=(0)$. For each $p\in P$ denote $G_p=\cap\{nG/n=1, 2, \ldots; (n, p)=1\}$.

Let G be provided with a complete minimal group topology. Then by Proposition 4.1 $\overline{nG}=nG$ for each natural n and therefore G_p is a closed (hence minimal) subgroup of G. Moreover,

$$\bigcap_{n=1}^{\infty} p^n G_p = \bigcap_{m=1}^{m} mG=(0)$$

and by Lemma 4.3 G_p is compact in the relative topology. Now Lemma 3.1 implies that G is compact q. e. d.

Corollary 5. *Let G be a complete minimal torsion-free Abelian group without non-zero compact connected subgroups. Then G is compact.*

Proof. By Proposition 4.1 and [7, 2.9] $K=\bigcap_{n=1}^{\infty} nG$ is a compact subgroup of G. On the other hand K is divisible and therefore K is connected. Hence $K=(0)$, which shows that G is a group without divisible subgroups. By Theorem 4.4, G is compact q. e. d.

Let us mention that on the assumption of Theorem 4.4 (or Corollary 4.5) G is topologically isomorphic to $\Pi_{p\in P}\mathbf{Z}_p^{\tau_p}$ for an appropriate sequence of cardinals $\{\tau_p\}_p$.

Now we shall consider an application of Theorem 4.4.

Theorem 4.6. *Let G be an infinite complete minimal Abelian group Then there exists a compact subgroup H of G such that $\chi(H)=\chi(G)$. Hence $2^{\chi(G)}\leqq \operatorname{card} G$.*

Proof. Denote by S the socle of G. By [7, 1.3, 2.2] S is precompact in the relative topology, hence $\bar{S}$ is compact. If $\chi(\bar{S})=\chi(G)$, we set $H=\bar{S}$. Suppose $\chi(\bar{S})<\chi(G)$. Then there is a set $\mathscr{F}$ of open neighbourhoods of 0 in G with $\operatorname{card}\mathscr{F}<\chi(G)$ and

$$\bar{S}\cap\cap\{U/U\in\mathscr{F}\}=\{0\}. \tag{55}$$

Moreover, we can assume that for each $U\in\mathscr{F}$ there is $V\in\mathscr{F}$ with $V-V\subset U$. Indeed, for each $U\in\mathscr{F}$ there is a sequence $U_1, U_2, \ldots, U_n, \ldots$ of neighbourhoods of 0 in G with $U_1-U_1\subset U$ and $U_{n+1}-U_{n+1}\subset U_n$ $(n=1, 2, \ldots)$. The set $\mathscr{F}_1$ of all U_n, where $U\in\mathscr{F}$ and $n=1, 2, \ldots$ has the properties of $\mathscr{F}$ and if $W\in\mathscr{F}_1$, then there exists $W'\in\mathscr{F}_1$ with $W'-W'\subset W$. Hence we can assume that $G'=\cap\{U/U\in\mathscr{F}\}$ is a closed subgroup of G and by (55) $G'\cap S=(0)$. Therefore G' is torsion-free. Except that we have

$$\chi(G')=\chi(G). \tag{56}$$

Indeed, if $\chi(G')<\chi(G)$, then there is a set $\mathcal{F}'$ of open neighbourhoods of 0 in G with card $\mathcal{F}'<\chi(G)$ and

$$G'\cap\{V/V\in\mathcal{F}'\}=\{0\}. \tag{57}$$

Now (55), (57) and the definition of G' imply $\cap\{U\cap V/U\in\mathcal{F},\ V\in\mathcal{F}'\}=\{0\}$, hence by the minimality of G, $\chi(G)\leq\text{card}\,\mathcal{F}.\text{card}\,\mathcal{F}'<\chi(G)$ which is a contradiction. Therefore (56) holds.

Consider $K=\bigcap_{n=1}^{\infty}\overline{nG'}$. By [7, 2.9] K is a compact subgroup of G'. If $\chi(K)=\chi(G')$, we set $H=K$ and by (56) $\chi(H)=\chi(G)$. Suppose $\chi(K)<\chi(G')$. As above we find a closed subgroup H of G' such that $H\cap K=(0)$ and $\chi(H)=\chi(G')$. It is clear now that in the relative topology H is a complete minimal torsion-free Abelian group without non-zero divisible subgroups. By Theorem 4.4, H is compact. Since $\chi(H)=\chi(G')$, (56) implies $\chi(H)=\chi(G)$.

For each compact Abelian group L we have $2^{\chi(L)}=\text{card}\,L$. Hence $2^{\chi(H)}=\text{card}\,H$ and therefore, $2^{\chi(G)}=\text{card}\,H\leq\text{card}\,G$ q. e. d.

Since card $X\leq 2^{w(X)}$ for each Hausdorff topological space X ($w(X)$ is the weight of X), it follows from Therem 4.6 that $2^{\chi(G)}\leq\text{card}\,G\leq 2^{w(G)}$ for each infinite complete minimal Abelian group G. It is interesting to see whether card $G=2^{w(G)}$ or card $G=2^{\chi(G)}$ for each infinite complete minimal Abelian group G.

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Centre for Mathematics and Mechanics
1090 Sofia P. O. Box 373

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