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ПЛИСКА

БЪЛГАРСКИ МАТЕМАТИЧЕСКИ СТУДИИ

ON BOCHNER CURVATURE TENSORS IN ALMOST HERMITIAN MANIFOLDS

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A curvature identity characterizing almost Hermitian manifolds with pointwise constant antiholomorphic sectional curvatures is found. An antiholomorphic operator on curvature tensor is introduced and the vector space of the transformed curvature tensors is considered. A decomposition of this space under the action of the unitary group is given using as a base tensors of constant antiholomorphic sectional curvatures. Considering a conformal equivalence to manifolds with zero holomorphic sectional curvatures, zero antiholomorphic sectional curvatures and zero constant type, theorems on the geometric meaning respectively of the generalized Bochner tensor, Bochner tensor and almost Hermitian manifolds of conformal type are proved.

Let V be an n-dimensional real vector space with a positive definite inner product and denote by $\mathcal{R}(V)$ the vector space of all curvature tensors over V. In [6, 7] it is given a decomposition of $\mathcal{R}(V)$ under the action of $O(n)$ into three components which give important classes of Riemannian manifolds. As a base of the decomposition the tensors of constant sectional curvatures are used.

A decomposition under the action of $U(n)$ of the subspace $\mathcal{K}(V)$ of K-curvature tensors in $\mathcal{R}(V)$ is given in [5], [8]. As a base of the decomposition K-curvature tensors with constant holomorphic sectional curvatures are used. Using this decomposition and the holomorphic operator on curvature tensors, we introduced a generalized Bochner curvature tensor, associated with an arbitrary curvature tensor [1, 2].

A complete decomposition of $\mathcal{R}(V)$ over a Hermitian vector space under the action of $U(n)$ is given in [9].

In this paper we introduce an antiholomorphic operator on curvature tensors and using as a base curvature tensors with constant antiholomorphic sectional curvatures we obtain a decomposition of the transformed $\mathcal{R}(V)$ into three components. This gives another interpretation of the decomposition in [9]. The Bochner curvature tensor associated with a curvature tensor R coincides with the Weyl component of the transformed $\mathcal{R}$. We consider an operator on the Kähler difference and using as a base curvature tensors of constant type, we obtain a decomposition of the transformed $\mathcal{R}(V)$ into three components. Further, we consider a conformal equivalence of an almost Hermitian manifold to a manifold with zero holomorphic sectional curvatures, zero antiholomorphic sectional curvatures and zero constant type. In every of these three cases we obtain a theorem conserning the geometric meaning of the generalized Bochner tensor, Bochner tensor and almost Hermitian manifolds of conformal type, respectively.

1. Preliminaries. Let V be an n-dimensional real vector space with a positive definite inner product g. A tensor R of type (1, 3) over V is said to be a curvature tensor over V if it has the following properties for all x, y, z, u in V:

$$R(x, y)z = -R(x, y)z,$$
$$R(x, y)z + R(y, z)x + R(z, x)y = 0, \tag{1.1}$$
$$R(x, y, z, u) = -R(x, y, u, z),$$

where $R(x, y, z, u)=g(R(x, y)z, u)$ is the corresponding tensor of type (0, 4).

The sectional curvature of a 2-plane α in V with respect to R is given by $K(R)(\alpha)=R(x, y, y, x)$ where $\{x, y\}$ is an orthonormal basis of α. The Ricci tensor $\rho(R)$ of type (0, 2) associated with R is defined by

$$\rho(R)(y, z)=\text{trace}(x \in V \rightarrow R(x, y)z).$$

The corresponding tensor of type (1, 1) is given by $g(Q(R)(y), z)=\rho(R)(y, z)$ and the trace of $Q(R)$ is called the scalar curvature $\tau(R)$ of R.

Now, let V be a $2n$-dimensional real vector space with a complex structure J and a Hermitian product g, i. e. $J^2=-I$, where I denotes the identity transformation of V and $g(Jx, Jy)=g(x, y)$ for all x, y in V. With an arbitrary curvature tensor R over V we associate the tensor R^* defined by $R^*(x, y, z, u)=R(x, y, Jz, Ju)$ for all x, y, z, u in V. In general, this tensor is not a curvature tensor, but it has the first and third properties of (1.1). We denote $K^*(R)(\alpha)$ the sectional curvature of a 2-plane α in V with respect to the tensor R^*. The Ricci tensor associated with the tensor R^* is denoted by $\rho^*(R)$ and is given by

$$\rho^*(R)(y, z)=\sum_{i=1}^{2n} R(e_i, y, Jz, Je_i),$$

where $\{e_i\}$ is an arbitrary orthonormal basis of V. The scalar curvature of R^* is denoted by $\tau^*(R)$.

Let $\mathcal{R}(V)$ denote the vector space of all curvature tensors over V. The inner product g induces a natural inner product on $\mathcal{R}(V)$

$$\langle R', R''\rangle=\sum_{i,j,k} g(R'(e_i, e_j)e_k, R''(e_i, e_j)e_k),$$

where R', R'' are curvature tensors and $\{e_i\}$ is an arbitrary orthonormal basis of V.

The standard representation of the unitary group $U(n)$ in V induces a natural representation of $U(n)$ in $\mathcal{R}(V)$

$$(aR)(x, y, z, u)=R(a^{-1}x, a^{-1}y, a^{-1}z, a^{-1}u)$$

for all x, y, z, u in V, $a \in U(n)$ and $R \in \mathcal{R}(V)$.

A 2-plane α in V is said to be holomorphic (antiholomorphic) if $J\alpha=\alpha$ ($J\alpha \perp \alpha$). The sectional curvatures of the holomorphic (antiholomorphic) 2-planes are called holomorphic (antiholomorphic) sectional curvatures. A curvature tensor R is said to be with constant holomorphic (antiholomorphic) sectional curvatures if $K(R)(\alpha)=\text{const}$ for an arbitrary holomorphic (antiholomorphic) 2-plane α in V.

If S is a symmetric tensor of type (0,2), $\varphi(S)$ denotes the following curvature tensor:

$$\begin{aligned}\varphi(S)(x, y, z, u)=&g(y, z)S(x, u)-g(x, z)S(y, u)\\&+g(x, u)S(y, z)-g(y, u)S(x, z).\end{aligned}$$

If S is a tensor of type (0,2) and $S(Jx, Jy)=S(y, x)$ for all x, y in V, $\psi(S)$ denotes the following curvature tensor:

$$\begin{aligned}\psi(S)(x, y, z, u)=&g(y, Jz)S(x, Ju)-g(x, Jz)S(y, Ju)-2g(x, Jy)S(z, Ju)\\&+g(x, Ju)S(y, Jz)-g(y, Ju)S(x, Jz)-2g(z, Ju)S(x, Jy).\end{aligned}$$

The basic curvature tensors invariant under the action of $U(n)$ are

$$\pi_1(x, y)z=g(y, z)x-g(x, z)y;$$

$$\pi_2(x, y)z=g(Jy, z)Jx-g(Jx, z)Jy-2g(Jx, y)Jz.$$

We shall use the following lemmas [9]:

Lemma 1.1. *For every symmetric tensor S of type* (0,2) *we have*

1) $\rho(\varphi(S))=2(n-1)S+\operatorname{tr} S.g$;
2) $\rho^*(\varphi(S))(x, y)=S(x, y)+S(Jx, Jy)$; $x, y \in V$,
3) $\langle R, \varphi(S)\rangle=4\langle\rho(R), S\rangle$.

Lemma 1.2. *For every tensor S of type* (0,2) *satisfying the condition $S(Jx, Jy)=S(y, x)$ for all x, y in V we have*

1) $\rho(\psi(S))=3(S+S')$;
2) $\rho^*(\psi(S))=2(n+1)S+\operatorname{tr} S.g$;
3) $\langle R, \psi(S)\rangle=8\langle\rho^*(R), S\rangle+4\langle\rho^*(R), S'\rangle$, *where $S'(x, y)=S(y, x)$ for all x, y in V.*

Further, we have

$$\begin{aligned} &\langle aR', aR''\rangle=\langle R', R''\rangle,\\ &\rho(aR)(y, z)=\rho(R)(a^{-1}y, a^{-1}z),\\ &a\varphi(\rho(R))=\varphi(\rho(aR)),\\ &\tau(aR)=\tau(R), \end{aligned} \tag{1.2}$$

where R', R'', R are curvature tensors and $a\in U(n)$.

Analogous formulae hold good for ρ^*, τ^* and ψ as a corollary of $Ja=aJ$, $a\in U(n)$

$$\begin{aligned} &\rho^*(aR)(y, z)=\rho^*(R)(a^{-1}y, a^{-1}z),\\ &a\psi(\rho^*(R))=\psi(\rho^*(aR)),\\ &\tau^*(aR)=\tau^*(R). \end{aligned} \tag{1.3}$$

2. Holomorphic operator on curvature tensors. In [1, 2] we associated with every curvature tensor R a (generalized) curvature tensor HR

$$\begin{aligned} HR(x, y, z, u)=\frac{1}{16}\{&3R(x, y, z, u)+3R(Jx, Jy, z, u)+3R(Jx, Jy, Jz, Ju)\\ &+3R(x, y, Jz, Ju)-R(Jy, Jz, x, u)-R(Jz, Jx, y, u)\\ &+R(y, Jz, Jx, u)+R(Jz, x, Jy, u)-R(y, z, Jx, Ju)\\ &-R(z, x, Jy, Ju)+R(Jy, z, x, Ju)+R(z, Jx, y, Ju)\}. \end{aligned}$$

This tensor is the uniquely determined curvature tensor having the following properties:

$$\begin{aligned} &HR(x, y, Jz, Ju)=HR(x, y, z, u);\\ &HR(x, Jx, Jx, x)=R(x, Jx, Jx, x) \end{aligned} \tag{2.1}$$

for all x, y, z, u in V. The first identity of (2.1) is called the Kähler identity. The second equality of (2.1) means that HR and R have ones and the same holomorphic sectional curvatures. Now, we call this operator HR: $R\to HR$ the holomorphic operator.

Lemma 2.1. *Let S be a symmetric tensor of type* (0,2) *over V. We have*

$$H\varphi(S)=\frac{1}{8}(\varphi+\psi)(S+\bar{S}),$$

where $\bar{S}(x, y)=S(Jx, Jy)$ for all x, y in V.

The proof is a simple verification.

A curvature tensor with constant holomorphic sectional curvatures is characterized with the following

Lemma 2.2 [1, 2]. *A curvature tensor R over V is with constant holomorphic sectional curvature μ iff*

$$HR=\mu H\pi_1=(\mu/4)(\pi_1+\pi_2).$$

The constant μ *is* $\mu=\tau(HR)/n(n+1)=\{\tau(R)+3\tau^*(R)\}/4n(n+1)$.

In [5, 8] it is proved a decomposition theorem for the vector space of the curvature tensors satisfying the Kähler identity (K-curvature tensors). We shall give this theorem in terms of the holomorphic operator.

Let $\mathcal{HR}(V)$ denote the vector space of all tensors HR, where R is a curvature tensor over V. Then, the following theorem holds.

Theorem 2.1. *Let* $\dim V\geqq 4$. *The following decomposition is orthogonal:*

$$\mathcal{HR}(V)=\mathcal{HR}_1(V)\oplus\mathcal{HR}_2(V)\oplus\mathcal{HR}_w(V),$$

where

$$\mathcal{HR}_1(V)=\{HR\in\mathcal{HR}(V)\mid HR=\mu H\pi_1\},$$

$$\mathcal{HR}_w(V)=\{HR\in\mathcal{HR}(V)\mid\rho(HR)=0\},$$

$\mathcal{HR}_2(V)$ *is the orthogonal complement of* $\mathcal{HR}_w(V)$ *in* $\mathcal{HR}_1(V)^{\perp}$,

$$\mathcal{HR}_2(V)\oplus\mathcal{HR}_w(V)=\{HR\in\mathcal{HR}(V)\mid\tau(HR)=0\};$$

$$\mathcal{HR}_1(V)\oplus\mathcal{HR}_w(V)=\{HR\in\mathcal{HR}(V)\mid\rho(HR)=\frac{\tau(HR)}{2n}g\}.$$

The component of HR in $\mathcal{HR}_w(V)$ (Weyl component) we called the generalized Bochner curvature tensor. This tensor is

$$B(HR)=HR-\frac{2}{n+2}H\varphi(\rho(HR))+\frac{\tau(HR)}{(n+1)(n+2)}H\pi_1,$$

or taking into account Lemma 2.1

$$B(HR)=HR-\frac{1}{2(n+2)}(\varphi+\psi)(\rho(HR))+\frac{\tau(HR)}{4(n+1)(n+2)}(\pi_1+\pi_2).$$

Later this tensor has been obtained as a projection of the tensor R on a subspace of $\mathcal{R}(V)$ [9].

About the geometric meaning of the classical Weyl curvature tensor in a Riemannian manifold we shall recall the following

Proposition. *If a Riemannian manifold* $M(\dim M\geqq 4)$ *is conformally equivalent to a manifold of zero sectional curvatures, then its Weyl curvature tensor vanishes.*

Using holomorphic sectional curvatures instead of all sectional curvatures, we shall prove an analogous proposition for the generalized Bochner tensor.

Let M be an almost Hermitian manifold with a metric tensor g and an almost complex structure J; let R denote the Riemann curvature tensor of the metric g. If $\tilde{g}=e^{2\sigma}g$ is a conformal change of the metric g, then the curvature tensor $\tilde{R}$ of $\tilde{g}$ is given by

$$\tilde{R}(x,y,z,u)=e^{2\sigma}\{R(x,y,z,u)+\varphi(Q)(x,y,z,u)\},\tag{2.2}$$

where $Q(x, y)=(\nabla_x\omega)y-\omega(x)\omega(y)+\frac{1}{2}\|\omega\|^2 g(x, y)$, $\omega=d\sigma$ and x, y, z, u are arbitrary vectors in the tangent space T_pM in any point p in M. From (2.2) we obtain

$$HR̃=e^{2\sigma}\{HR+\frac{1}{8}(\varphi+\psi)(Q+\bar{Q})\}, \tag{2.3}$$

where $\bar{Q}(x, y)=Q(Jx, Jy)$ for all vectors x, y in T_pM.

Theorem 2.2. *If an almost Hermitian manifold $M(\dim M\geq 4)$ is conformally equivalent to a manifold with zero holomorphic sectional curvatures, then its generalized Bochner tensor vanishes.*

Proof. Let $\tilde{g}=e^{2\sigma}g$ be the conformal change of the metric g and let the curvature tensor $\tilde{R}$ of $\tilde{g}$ have zero holomorphic sectional curvatures. Lemma 2.2 implies that $H\tilde{R}=0$. From (2.3) we obtain

$$Q+\bar{Q}=-\frac{4}{n+2}\rho(HR)+\frac{\tau(HR)}{(n+1)(n+2)}g$$

and hence $B(HR)=0$.

This theorem gives the following geometric meaning of the Bochner curvature tensor in a Kähler manifold.

Theorem 2.3. *If a Kähler manifold $M(\dim M\geq 4)$ is conformally equivalent to a manifold with zero holomorphic sectional curvatures, then its Bochner curvature tensor vanishes.*

The assertion follows from Theorem 2.2 and the fact that for a Kähler manifold $HR=R$.

Using antiholomorphic sectional curvatures, an analogous scheme can be applied to obtain the Bochner curvature tensor, introduced in [9].

3. Antiholomorphic operator on curvature tensors. Let V be a Hermitian vector space and R be a curvature tensor over V. We denote $\bar{R}(x, y, z, u)=R(Jx, Jy, Jz, Ju)$ for all x, y, z, u in V.

Lemma 3.1 [3]. *Let T be a curvature tensor over $V(\dim V\geq 4)$. If*

1) $T=\bar{T}$;

2) *T has zero holomorphic and antiholomorphic sectional curvatures, then* $T=0$.

Lemma 3.2 [3]. *Let T be a curvature tensor over $V(\dim V\geq 4)$. If*

1) *T has zero holomorphic and antiholomorphic sectional curvatures;*

2) *$T(x, Jx, z, x)=0$, whenever $\{x, z\}$ is an orthonormal antiholomorphic pair in V, then* $T=0$.

Lemma 3.3. *If R is a curvature tensor over $V(\dim V\geq 6)$, the following conditions are equivalent:*

1) *$K(R)(\alpha)=K(\bar{R})(\alpha)$ for an arbitrary antiholomorphic 2-plane α in V.*

2) $R-\bar{R}=\frac{1}{2(n+1)}\psi(\rho^*(R-\bar{R}))$.

Proof. Let T be the curvature tensor $T=R-\bar{R}-\frac{1}{2(n+1)}\psi(\rho^*(R-\bar{R}))$. For every vector x in V we have

$$T(x, Jx, Jx, x)=0. \tag{3.1}$$

Let $\{x, y\}$ be an orthonormal antiholomorphic pair. By the first condition of the lemma we have

(3.2) $$(R-\bar{R})(x, y, y, x)=0,$$

(3.3) $$T(x, y, y, x)=0.$$

Further, if $\{x, y, z\}$ is an orthonormal antiholomorphic triple, the equality (3.2) gives

(3.4) $$(R-\bar{R})(x, y, z, x)=0.$$

Replacing the orthonormal antiholomorphic triples $\{(x+y)/\sqrt{2}, (Jx-Jy)/\sqrt{2}, z\}$ and $\{(x-y)/\sqrt{2}, (Jx+Jy)/\sqrt{2}, z\}$ in (3.4) we get

(3.5) $$(R-\bar{R})(x, Jx, z, x)=\frac{3}{2}(R-\bar{R})(x, z, Jy, y).$$

Let $\{x, z, u_3, \ldots, u_n, Jx, Jz, Ju_3, \ldots, Ju_n\}$ be an adapted basis of V. Substituting $y=u_i (i=3, \ldots, n)$ in (3.5) and summing, we obtain

(3.6) $$(R-\bar{R})(x, Jx, z, x)=-\frac{3}{2(n+1)}\rho^*(R-\bar{R})(x, Jz).$$

The equalities (3.1), (3.2) and (3.6) give that the conditions of Lemma 3.2 are fulfilled for the tensor T. Hence $T=0$ and the condition 1) of the Lemma implies 2). The inverse is a simple verification.

For the case $\dim V=4$ we shall use

Lemma 3.4 [9]. *Let* $\dim V=4$ *and* R *be a curvature tensor over* V. *Then the following identity holds good*

$$R-\bar{R}=\frac{1}{2}\varphi(\rho(R-\bar{R}))+\frac{1}{6}\psi(\rho^*(R-\bar{R})).$$

Lemma 3.5. *Let* $\dim V\geq 4$ *and* R *be a curvature tensor over* V. *If* $K(R)(\alpha)=K(\bar{R})(\alpha)$ *for an arbitrary antiholomorphic 2-plane* α *in* V, *then* $\rho(R)=\rho(\bar{R})$.

The lemmas 3.3, 3.4 and 3.5 imply

Proposition 3.1. *Let* R *be a curvature tensor over* V ($\dim V\geq 4$). *The following conditions are equivalent:*

1) $K(R)(\alpha)=K(\bar{R})(\alpha)$ *for an arbitrary antiholomorphic 2-plane* α *in* V.

2) $R-\bar{R}=\frac{1}{2(n+1)}\psi(\rho^*(R-\bar{R}))$.

Proposition 3.2. *Let* R *be a curvature tensor over* V ($\dim V\geq 4$). R *has zero antiholomorphic sectional curvatures iff*

$$R-\frac{1}{2(n+1)}\psi(\rho^*(R))+\frac{\tau^*(R)}{(2n+1)(2n+2)}\pi_2=0.$$

Proof. Let $\{x, y\}$ be an orthonormal antiholomorphic pair. Then

(3.7) $$R(x, y, y, x)=0.$$

From here we get for an arbitrary unit vector x in V

(3.8) $$R(x, Jx, Jx, x)-\rho(R)(x, x)=0.$$

According to the Lemma 3.1, the equalities (3.8) and (3.7) give that the following curvature tensor is zero:

(3.9) $$R+\bar{R}-\frac{1}{6}\psi(\rho(R+\bar{R}))=0.$$

The next two equalities directly follow from (3.9).

(3.10) $$\rho^*(R+\bar{R})-\frac{n+1}{3}\rho(R+\bar{R})-\frac{\tau}{3}g=0,$$

(3.11) $$3\tau^*(R)-(2n+1)\tau(R)=0.$$

The equalities (3.9), (3.10) and (3.11) imply

$$R+\bar{R}-\frac{1}{2(n+1)}\psi(\rho^*(R+\bar{R}))+\frac{\tau^*(R)}{(2n+1)(n+1)}\pi_2=0.$$

Taking into account Proposition 3.1 and the last equality, we obtain

$$R-\frac{1}{2(n+1)}\psi(\rho^*(R))+\frac{\tau^*(R)}{(2n+1)(2n+2)}\pi_2=0.$$

The inverse is an easy verification.

We define an antiholomorphic operator $A\colon \mathscr{R}(V)\to\mathscr{R}(V)$ as follows

$$AR=R-\frac{1}{2(n+1)}\psi(\rho^*(R))+\frac{\tau(R)}{(2n+1)(2n+2)}\pi_2.$$

From Proposition 3.2 we obtain

Proposition 3.3. *Let R be a curvature tensor over V(dim $V\geqq 4$). R has constant antiholomorphic sectional curvatures ν iff*

$$AR=\nu A\pi_1.$$

The constant ν is $\nu=\tau(AR)/\tau(A\pi_1)=\{(2n+1)\tau(R)-3\tau^*(R)\}\,8n(n^2-1)$.

Now, let M be an almost Hermitian manifold with a curvature tensor R. The manifold is said to be with pointwise constant antiholomorphic sectional curvatures if in every point p in $M\,K(R)(\alpha;\,p)=\nu(p)$, where α is an arbitrary antiholomorphic 2-plane in the tangential space T_pM and $\nu(p)$ does not depend on α.

Applied to almost Hermitian manifolds, Proposition 3.3 gives

Theorem 3.1. *Let M(dim $M\geqq 4$) be an almost Hermitian manifold with a curvature tensor R. The manifold M has pointwise constant antiholomorphic sectional curvatures iff*

$$AR=\nu(p)\,A\pi_1.$$

The function ν is $\nu(p)=\{(2n+1)\tau(p)-3\tau^*(p)\}/8n(n^2-1)$.

Let $\mathscr{A}\mathscr{R}(V)$ denote the vector space of all tensors AR, where R is a curvature tensor over V.

Proposition 3.4. *A tensor AR is orthogonal to $A\pi_1$ iff $\tau(AR)=0$.*

Proof. We have $\rho^*(AR)=0$ and hence $\tau^*(AR)=0$. Using the Lemmas 1.1 and 1.2 we obtain $\langle AR, A\pi_1\rangle=2\tau(AR)$ and this gives the assertion.

Theorem 3.2. *Let* dim $V\geqq 6$. *The following decomposition is orthogonal:*

$$\mathscr{A}\mathscr{R}(V)=\mathscr{A}\mathscr{R}_1(V)\oplus\mathscr{A}\mathscr{R}_2(V)\oplus\mathscr{A}\mathscr{R}_w(V),$$

where

1) $\mathscr{A}\mathscr{R}_1(V)=\{AR\in\mathscr{A}\mathscr{R}(V)/AR=\nu A\pi_1\}$;

2) $\mathscr{A}\mathscr{R}_w(V)=\{AR\in\mathscr{A}\mathscr{R}(V)/\rho(AR)=0\}$;

$\mathscr{A}\mathscr{R}_2(V)$ is the orthogonal complement of $\mathscr{A}\mathscr{R}_w(V)$ in $\mathscr{A}\mathscr{R}_1(V)^\perp$.

3) $\mathscr{A}\mathscr{R}_2(V)\oplus\mathscr{A}\mathscr{R}_w(V)=\{AR\in\mathscr{A}\mathscr{R}(V)/\tau(AR)=0\}$;

4) $\mathscr{A}\mathscr{R}_1(V)\oplus\mathscr{A}\mathscr{R}_w(V)=\{AR\in\mathscr{A}\mathscr{R}(V)/\rho(AR)=\frac{\tau(AR)}{2n}g\}$.

The subspaces are invariant under the action of $U(n)$.

Proof. Taking into account Proposition 3.4, it remains to prove 4). Let $AR=(AR)_1+(AR)_2+(AR)_w$. From Proposition 3.4 it follows that $(AR)_1=\{\tau(AR)/\tau(A\pi_1)\}A\pi_1$. If $(AR)_2=0$, then $\rho(AR)=\{\tau(AR)/2n\}g$, and vice versa. The last assertion of the theorem follows from the formulae (1.2), (1.3) and the fact that $A(aR)=a(AR)$, $a\in U(n)$.

Using Theorem 3.2 we find the projections of a tensor AR.

Proposition 3.5. *The projections of a tensor AR on the subspaces of* $\mathcal{AR}(V)$ *are*

$$(AR)_1=\frac{(2n+1)\operatorname{tr}S}{8n(n^2-1)}A\pi_1;$$

$$(AR)_2=\frac{n+1}{2(n^2-4)}A\varphi\left(\frac{S+\bar{S}}{2}\right)+\frac{1}{2(n-1)}A\varphi\left(\frac{S-\bar{S}}{2}\right)-\frac{(n+1)\operatorname{tr}S}{2n(n^2-4)}A\pi_1;$$

$$(AR)_w=AR-\frac{n+1}{2(n^2-4)}A\varphi\left(\frac{S+\bar{S}}{2}\right)-\frac{1}{2(n-1)}A\varphi\left(\frac{S-\bar{S}}{2}\right)+\frac{(2n^2+3n+4)\operatorname{tr}S}{8(n^2-1)(n^2-4)}A\pi_1,$$

where

$$S=\rho(AR)=\rho(R)-\frac{3}{2(n+1)}\rho^*(R+\bar{R})+\frac{3\tau^*(R)}{(2n+1)(2n+2)}g,$$

$$\bar{S}(x,y)=S(Jx,Jy);\quad x,y\in V, \tag{3.12}$$

$$\operatorname{tr}S=\{(2n+1)\tau(R)-3\tau^*(R)\}/(2n+1).$$

Studying detaily the vector space $\mathcal{R}(V)$ F. Tricerri and L. Vanhecke introduced Bochner tensor $B(R)$, associated with a curvature tensor R [9]. Comparing $B(R)$ with the component $(AR)_w$ (Weyl component) of AR we have

Corollary 3.1. *The Weyl component of the tensor AR coinsides with the Bochner tensor* $B(R)$, *associated with* R.

From Theorem 3.2 we obtain

Proposition 3. 6. *A curvature tensor R over* $V(\dim V\geq 6)$ *has constant antiholomorphic sectional curvatures iff* $B(R)=0$ *and* $\rho(AR)=\{\tau(AR)/2n\}g$.

Our interpretation of the Bochner curvature tensor gives the following geometric meaning of $B(R)$.

Theorem 3.3. *If an almost Hermitian manifold* $M(\dim M\geq 6)$ *is conformally equivalent to a manifold with zero antiholomorphic sectional curvatures, its Bochner curvature tensor vanishes.*

Proof. Let $\bar{g}=e^{2\sigma}g$ be a conformal change of the metric g in M. For the curvature tensor $\tilde{R}$ of $\tilde{g}$ we have (2.2). From this equality we obtain $A\tilde{R}=e^{2\sigma}\{AR+A\varphi(Q)\}$. Theorem 3.1 implies $A\tilde{R}=0$ and hence $AR+A\varphi(Q)=0$. From here we calculate

$$Q=-\frac{n+1}{2(n^2-4)}\cdot\frac{S+\bar{S}}{2}-\frac{1}{2(n-1)}\cdot\frac{S-\bar{S}}{2}+\frac{2n^2+3n+4}{16(n^2-1)(n^2-4)}\operatorname{tr}S\,.\,g,$$

where S, $\bar{S}$ and $\operatorname{tr}S$ are given in (3.12). Hence $B(R)=(AR)_w=0$.

Taking into account that for a Kähler manifold $B(R)$ is the usual Bochner curvature tensor, we obtain

Theorem 3.4. *If a Kähler manifold* $M(\dim M\geq 6)$ *is conformally equivalent to a manifold with zero antiholomorphic sectional curvatures, then its Bochner curvature tensor vanishes.*

4. Almost Hermitian manifolds of conformal type. Let R be a curvature tensor over $V(\dim V\geq 4)$. The tensor R is said to be of constant type λ [10] if

$$K(R)(\alpha)-K^*(R)(\alpha)=\lambda, \tag{4.1}$$

where α is an arbitrary antiholomorphic 2-plane in V and λ does not depend on α. This constant is $\lambda=\{\tau(R)-\tau^*(R)\}/4n(n-1)$.

For an arbitrary curvature tensor R we define $\Delta R(x, y, z, u)=R(x, y, z, u)-R(x, y, Jz, Ju)$ for all x, y, z, u in V.

The following lemma gives a curvature identity equivalent to (4.1):

Lemma 4.1 [10]. *Let R be a curvature tensor over V (dim $V\geq 4$) with the property $R=\bar{R}$. Then, R is of constant type iff*

$$\Delta R=\lambda\Delta\pi_1.$$

We shall find a curvature identity equivalent to (4.1) in the general case.

Proposition 4.1. *Let R be a curvature tensor over V (dim $V\geq 4$). R is of constant type iff*

$$\Delta R=\frac{1}{4(n+1)}\Delta\psi(\rho^*(R-\bar{R}))+\lambda\Delta\pi_1.$$

Proof. For an arbitrary antiholomorphic 2-plane α in V we always have $K^*(R)(\alpha)=K^*(\bar{R})(\alpha)$. Then, (4.1) is equivalent to the conditions

$$K(R+\bar{R})(\alpha)-K^*(R+\bar{R})(\alpha)=2\nu; \quad K(R)(\alpha)=K(\bar{R})(\alpha)$$

for every antiholomorphic 2-plane α in V. Applying Lemma 4.1 and Proposition 3.1, we obtain the assertion.

We define the operator D as follows:

$$DR=\Delta R-\frac{1}{4(n+1)}\Delta\psi(\rho^*(R-\bar{R})).$$

Taking into account that $D\pi_1=\Delta\pi_1$, we get from Proposition 4.1

Proposition 4.2. *Let R be a curvature tensor over V (dim $V\geq 4$). R is of constant type λ iff*

$$DR=\lambda D\pi_1.$$

It is not difficult to check

Proposition 4.3. *A tensor DR is orthogonal to $D\pi_1$ iff $\tau(DR)=0$.*

Let $\mathcal{DR}(V)$ denote the vector space of all tensors DR, where $R\in\mathcal{R}(V)$.

Theorem 4.1. *Let dim $V\geq 6$. The following decomposition is orthogonal:*

$$\mathcal{DR}(V)=\mathcal{DR}_1(V)\oplus\mathcal{DR}_2(V)\oplus\mathcal{DR}_w(V),$$

where

$$\mathcal{DR}_1(V)=\{DR\in\mathcal{DR}(V)/DR=\lambda D\pi_1\};$$

$$\mathcal{DR}_w(V)=\{DR\in\mathcal{DR}(V)/\rho(DR)=0\};$$

$\mathcal{DR}_2(V)$ is the orthogonal complement of $\mathcal{DR}_w(V)$ in $\mathcal{DR}_1(V)^\perp$;

$$\mathcal{DR}_2(V)\oplus\mathcal{DR}_w(V)=\{DR\in\mathcal{DR}(V)/\tau(DR)=0\};$$

$$\mathcal{DR}_1(V)\oplus\mathcal{DR}_w(V)=\{DR\in\mathcal{DR}(V)/\rho(DR)=\frac{\tau(DR)}{2n}g\}.$$

The subspaces are invariant under the action of $U(n)$.

The proof of the theorem is similar to that of Theorem 3.2 applying the corresponding notions to the tensors DR with a slight modification.

Proposition 4.4. *The projections of a tensor DR on the subspaces of $\mathcal{DR}(V)$ are*

$$(DR)_1=\frac{\operatorname{tr} S}{4n(n-1)}D\pi_1;$$

$$(DR)_2=\frac{1}{2(n-2)}D\varphi\Big(\frac{S+\bar{S}}{2}\Big)+\frac{1}{2(n-1)}D\varphi\Big(\frac{S-\bar{S}}{2}\Big)-\frac{\operatorname{tr} S}{2n(n-2)}D\pi_1;$$

$$(DR)_w=DR-\frac{1}{2(n-2)}D\varphi\Big(\frac{S+\bar{S}}{2}\Big)-\frac{1}{2(n-1)}D\varphi\Big(\frac{S-\bar{S}}{2}\Big)+\frac{\operatorname{tr} S}{4(n-1)(n-2)}D\pi_1;$$

where

$$S=\rho(DR)=\rho(R)-\frac{1}{2}\rho^*(R+\bar{R});$$

(4.1)
$$\bar{S}(x,y)=S(Jx,Jy);\quad x,y\in V,$$

$$\operatorname{tr} S=\tau(R)-\tau^*(R).$$

The component $(DR)_w$ (Weyl component) can be written in the form

$$(DR)_w=\Delta R-\frac{1}{4(n-2)}\Delta\varphi((\rho-\rho^*)(R+\bar{R}))-\frac{1}{4(n-1)}\Delta\varphi(\rho(R-\bar{R}))$$

$$-\frac{1}{4(n+1)}\Delta\psi(\rho^*(R-\bar{R}))+\frac{\tau(R)-\tau^*(R)}{4(n-1)(n-2)}\Delta\pi_1.$$

Comparing this formula with the Bochner tensor $B(R)$ we find

Corollary 4.1. *The Weyl component $(DR)_w$ of DR is*

$$(DR)_w=\Delta B(R).$$

We call the curvature tensor R over $V(\dim V\geq 6)$ to be of conformal type if $(DR)_w=0$.

The next proposition follows directly from Theorem 4.1.

Proposition 4.5. *Let $\dim V\geq 6$. A curvature tensor R is of constant type iff R is of conformal type and $\rho(DR)=\{\tau(DR)/2n\}g$.*

Let $M(\dim M\geqq 6)$ be an almost Hermitian manifold with a curvature tensor R. We call M to be of conformal type if $(DR)_w=\Delta B(R)=0$.

In [4] we proved

Proposition 4.6. *Let R be a curvature tensor over $V(\dim V\geq 8)$. R is of conformal type iff $\Delta R(x,y,z,u)=0$ for an arbitrary orthonormal antiholomorphic quadruple $\{x,y,z,u\}$ in V.*

Proposition 4.7. *Let $M(\dim M\geq 8)$ be an almost Hermitian manifold with a curvature tensor R. The following conditions are equivalent:*

1) *$R(x,y,z,u)=0$ for an arbitrary orthonormal antiholomorphic quadruple of tangent vectors in every point in M;*

2) $B(R)=0$;

3) *$B(HR)=0$ and M is of conformal type.*

The Bochner curvature tensor $B(R)$ was shown to be conformally invariant [9]. Using Corollary 4.1, we obtain

Corollary 4.2. *The tensor $(DR)_w$ is conformally invariant.*

About the geometric meaning of an almost Hermitian manifold of conformal type we have

Theorem 4.2. *Let $M(\dim M\geq 6)$ be an almost Hermitian manifold. If M is conformally equivalent to a manifold with zero constánt type, then M is of conformal type.*

Proof. Let $\tilde{g}=e^{2\sigma}g$ be a conformal change of the metric in M. The curvature tensor $\tilde{R}$ of $\tilde{g}$ is given by (2.2), from where we find $D\tilde{R}=e^{2\sigma}(DR+D\varphi(Q))$. If $\tilde{R}$ is with zero constant type, Proposition 4.2 implies that $DR+D\varphi(Q)=0$. From this equality we check

$$Q=-\frac{1}{4(n-2)}(S+\bar{S})-\frac{4}{4(n-1)}(S-\bar{S})+\frac{\operatorname{tr} S}{8(n-1)(n-2)}g,$$

where S, $\bar{S}$ are given in (4.1). Hence $(DR)_w=0$.

Corollary 4.3. *Every almost Hermitian manifold* M ($\dim M \geqq 6$) *conformally equivalent to a Kähler manifold is of conformal type.*

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