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ПЛИСКА

БЪЛГАРСКИ МАТЕМАТИЧЕСКИ СТУДИИ

ON THE SECTIONAL CURVATURE OF KÄHLER MANIFOLDS OF INDEFINITE METRICS

OGNIAN KASSABOV, ADRIJAN BORISOV

In this paper we study some curvature properties of Kähler manifolds of indefinite metrics.

1. Introduction. Kulkarni [5] has proved, that if a connected manifold of indefinite metric has bounded from above or from below sectional curvature, then it is of constant sectional curvature. Similar results for indefinite Kählerian metrics are found in [1].

Harris [4] and, independently, Dajczer and Nomizu [3] generalise [5] by using restrictions only on the sectional curvatures for the timelike planes (or for the spacelike planes). The purpose of this paper is to prove analogous results for Kähler manifolds of indefinite metrics.

Let M be a Kähler manifold of indefinite metric tensor g and complex structure J. Let ∇ denote the covariant differentiation with respect to the Levi-Civita connection. Then $g(JX, JY)=g(X, Y)$ for $X, Y\in\mathfrak{X}(M)$, $\nabla J=0$ and the curvature tensor R has the properties:

1) $R(X, Y, Z, U)=-R(Y, X, Z, U)$;
2) $R(X, Y, Z, U)+R(Y, Z, X, U)+R(Z, X, Y, U)=0$;
3) $R(X, Y, Z, U)=-R(X, Y, U, Z)$;
4) $R(X, Y, Z, U)=R(X, Y, JZ, JU)$.

A pair $\{x, y\}$ of tangent vectors at a point $p\in M$ is said to be ortonormal of signature $(+, -)$ if $g(x, y)=0$, $g(x, x)=1$, $g(v, y)=-1$. Analogously are defined pairs of signature $(+, +)$ or $(-, -)$. Similarly, one speaks of the signature of a 2-plane σ, depending on the signature of the restriction of the metric on σ [2].

We recall that the curvature of a nondegenerate 2-plane σ with a basis $\{x, y\}$ is defined by

$$K(\sigma)=\frac{R(x, y, y, x)}{\pi_1(x, y, y, x)},$$

where

$$\pi_1(x, y, z, u)=g(x, u)g(y, z)-g(x, z)g(y, u).$$

A plane σ is said to be holomorphic (resp. antiholomorphic) if $\sigma=J\sigma$ (resp. $\sigma\perp J\sigma$). Then a 2-plane σ is holomorphic (resp. antiholomorphic) if and only if it has a basis x, Jx (resp. x, y with $g(x, y)=g(x, Jy)=0$). We call the pair $\{x, y\}$ antiholomorphic of signature $(+, -)$, if $g(x, x)>0$, $g(y, y)<0$ and $g(x, y)=g(x, Jy)=0$. Similarly we can define antiholomorphic pairs of signature $(+, +)$ or $(-, -)$.

If σ is a nondegenerate 2-plane with an orthonormal basis $\{x, y\}$ we denote

$$K(x, y)=K(\sigma)$$

and in particular, if x is a unit vector, i. e. $g(x, x)=1$ or $g(x, x)=-1$ we denote

$$H(x)=K(x, Jx).$$

The manifold M is said to be of constant holomorphic sectional curvature if for any point $p \in M$ the curvature of an arbitrary holomorphic 2-plane σ in $T_p(M)$ doesn't depend on σ. As in the case of definite metric this occurs when and only when the curvature tensor R has the form

$$R=\frac{\mu}{4}(\pi_1+\pi_2),$$

where

$$\pi_2(x, y, z, u)=g(x, Ju)g(y, Jz)-g(x, Jz)g(y, Ju)-2g(x, Jy)g(z, Ju).$$

In this case μ is a global constant if M is connected.

2. Preliminary considerations. All the manifolds under consideration in this and in the following sections will be assumed to be connected.

Proposition 1. *Let M be a $2m$-dimensional Kähler manifold of indefinite metric, $m \geqq 2$. If*

(1) $$R(x, Jx, Jx, y)=0,$$

whenever the pair $\{x, y\}$ is antiholomorphic of signature $(+, -)$, then M is of constant holomorphic sectional curvature.

Proof. Let $p \in M$ and x, y be unit vectors in $T_p(M)$, such that $\{x, y\}$ is an antiholomorphic pair of signature $(+, -)$. According to (1), for any α, $|\alpha|<1$

$$R(x+\alpha y, Jx+\alpha Jy, Jx+\alpha Jy, \alpha x+y)=0$$

holds good and hence using (1), we obtain

$$\alpha\{H(x)-K(x, y)-3K(x, Jy)\}$$
$$+\alpha^3\{H(y)-K(x, y)-3K(x, Jy)\}+(3+\alpha^2)\alpha^2 R(y, Jy, Jy, x)=0.$$

Hence we derive

(2) $$R(y, Jy, Jy, x)=0,$$

(3) $$H(x)-K(x, y)-3K(x, Jy)+\alpha^2\{H(y)-K(x, y)-3K(x, Jy)\}=0$$

for $|\alpha|<1$, $\alpha \neq 0$ and hence (3) holds for any α, $|\alpha| \leqq 1$. Now let $\alpha=0$. Then

$$H(x)=K(x, y)+3K(x, Jy).$$

Hence we find

(4) $$K(x, y)=K(x, Jy),$$

(5) $$H(x)=4K(x, y).$$

From (3), (4), (5) it follows

(6) $$H(x)=H(y).$$

If $m=2$ we put $\mu=H(x)$. Then from (1), (2), (4), (5), (6) and the properties of the curvature tensor we obtain $R=(\mu/4)(\pi_1+\pi_2)$, which proves the assertion.

Let $m>2$. We choose a unit vector $z \in T_p(M)$, such that span$\{x, y, z\}$ is antiholomorphic. Then (6) implies $H(x)=H(y)=H(z)$, i. e. (6) holds also if the unit vectors x, y form an antiholomorphic pair of signature $(+, +)$ or $(-, -)$. Now let $\{u, v\}$ be arbitrary unit vectors in $T_p(M)$. We choose a unit vector x in $T_p(M)$ such that span$\{x, u\}$ and span$\{x, v\}$ are antiholomorphic. Then (6) implies

$$H(u)=H(x)=H(v),$$

thus proving Proposition 1.

Remark 1. Obviously, the signature of $\{x, y\}$ in Proposition 1 can be replaced by $(-, +)$.

Proposition 2. *Let M be a $2m$-dimensional Kähler manifold of indefinite metric, $m>2$. If the signature of M is $(-, -, \dots, +, +, +, +, \dots)$ and (1) holds whenever the pair $\{x, y\}$ is antiholomorphic of signature $(+, +)$, then M is of constant holomorphic sectional curvature.*

Proof. In a point $p\in M$ let $\{x, y\}$ be an arbitrary antiholomorphic pair of signature $(+, -)$. We choose a vector z in $T_p(M)$, such that $\{x, z\}$ is an antiholomorphic pair of signature $(+, +)$. Then for sufficiently large α the pair $\{x, y+\alpha z\}$ is antiholomorphic of signature $(+, +)$. Now from

$$R(x, Jx, Jx, y+\alpha z)=0,$$

we obtain the conditions of Proposition 1. Hence Proposition 2 follows.

Remark 2. If the signature of M is $(-, -, -, -, \dots, +, +, \dots)$, the signature of $\{x, y\}$ in Proposition 2 can be replaced by $(-, -)$.

Proposition 3. *Let M be a $2m$-dimensional Kähler manifold of indefinite metric, $m>2$.* If

$$R(x, y, y, z)=0$$

whenever span$\{x, y, z\}$ *is antiholomorphic and the pair $\{x, y\}$ is of signature $(+, -)$ then M is of constant holomorphic sectional curvature.*

Proof. Let e. g. the signature of M be $(-, -, -, -, \dots, +, +, \dots)$ and let $\{x, y\}$ be an arbitrary antiholomorphic pair of signature $(+, -)$ in a point $p\in M$. We choose z in $T_p(M)$ such that span$\{x, y, z\}$ is antiholomorphic and $g(z, z)<0$. Then we can choose $\alpha\neq 0$ such that the antiholomorphic pair $\{x+\alpha z, \alpha Jx+Jz\}$ is of signature $(+, -)$. Then

$$R(x+\alpha z, \alpha Jx+Jz, \alpha Jx+Jz, y)=0 \tag{7}$$

and since (7) holds also if we change α by $-\alpha$, we obtain

$$R(x, Jx, Jx, y)=0.$$

According to Proposition 1, M is of constant holomorphic sectional curvature.

Remark 3. The signature of the pair $\{x, y\}$ in Proposition 3 can be replaced by $(-, +)$ or by $(+, +)$ or $(-, -)$ in the case of an appropriate signature of M.

Remark 4. We can define in an obvious manner triples of signature $(+, +, +)$, $(+, +, -)$, etc. Then in the case of an appropriate signature of M Proposition 3 can be formulated for antiholomorphic triples $\{x, y, z\}$ of signature $(+, +, +)$ or $(+, +, -)$, etc.

3. Main results.

Theorem 1. *Let M be a Kähler manifold of indefinite metric, $m\geq 2$. If for each point $p\in M$ there exists a constant $c(p)$, such that for any vector $x\in T_p(M)$ with $g(x, x)=1$ the holomorphic sectional curvature $H(x)$ satisfies $|H(x)|\leq c(p)$, then M is of constant holomorphic sectional curvature.*

Proof. Let $p\in$ M and $\{x, y\}$ be arbitrary orthonormal pair of signature $(+, -)$ in p, such that $g(x, y)=g(x, Jy)=0$. Then from

$$|H(\frac{x+\alpha y}{\sqrt{1-\alpha^2}})|\leq c(p)$$

for $|\alpha|<1$ we obtain

(8)
$$|H(x)+4\alpha R(x, Jx, Jx, y)-2\alpha^2\{K(x, y)+3K(x, Jy)\} +4\alpha^3 R(x, Jy, Jy, y)+\alpha^4 H(y)|\leq c(p)(1-\alpha^2)^2.$$

Hence we obtain easily

(9)
$$|H(x)-2\alpha^2\{K(x, y)+3K(x, Jy)\}+\alpha^4 H(y)|\leq c(p)(1-\alpha^2)^2.$$

On the other hand, from (8) it follows by continuity

$$H(x)+4\alpha R(x, Jx, Jx, y)-2\alpha^2\{K(x, y)+3K(x, Jy)\}+4\alpha^3 R(x, Jy, Jy, y)+\alpha^4 H(y)=0$$

for $\alpha=\pm 1$, which implies

(10)
$$R(x, Jx, Jx, y)+R(x, Jy, Jy, y)=0.$$

From (8) and (10) we find

$$(H(x)-2\alpha^2\{K(x, y)+3K(x, Jy)\}+4\alpha(1-\alpha^2)R(x, Jx, Jx, y)+\alpha^4 H(y)|\leq c(p)(1-\alpha^2)^2$$

and hence and (9)

$$|\alpha R(x, Jx, Jx, y)|\leq\frac{1}{2}c(p)(1-\alpha^2).$$

Now by continuity we obtain $R(x, Jx, Jx, y)=0$ and the Theorem follows from Proposition 1.

Remark 5. In Theorem 1 the requirement $g(x, x)=1$ can be changed by $g(x, x)=-1$.

Theorem 2. *Let M be a Kähler manifold of indefinite metric, $m>2$. If for each point $p\in M$ there exists a constant $c(p)$, such that for any antiholomorphic 2-plane σ in $T_p(M)$ $K(\sigma)\leq c(p)$ holds, then M is of constant holomorphic sectional curvature.*

Proof. Let $p\in M$ and $\{x, y\}$ be an arbitrary orthonormal pair of signature $(+, -)$ and $g(x, Jy)=0$. We choose a unit vector $z\in T_p(M)$ such that $\operatorname{span}\{x, y, z\}$ is antiholomorphic. Then

$$K(\operatorname{span}\{x+\alpha y, z\})\leq c(p)$$

for $\alpha\neq\pm 1$ implies

$$K(x, z)+2\varepsilon\alpha R(x, z, z, y)-\alpha^2 K(y, z)\leq(1-\alpha^2)c(p)$$

for $|\alpha|<1$ and

$$K(x, z)+2\varepsilon\alpha R(x, z, z, y)-\alpha^2 K(y, z)\geq(1-\alpha^2)c(p)$$

for $|\alpha|>1$, where $\varepsilon=g(z, z)$. By continuity

$$K(x, z)+2\varepsilon\alpha R(x, z, z, y)-\alpha^2 K(y, z)=0$$

holds good for $\alpha=\pm 1$ and hence $R(x, z, z, y)=0$. So the theorem follows from Remark 4.

Remark 6. In Theorem 2 the requirement $K(\sigma)\leq c(p)$ can be replaced by $K(\sigma)\geq c(p)$.

Remark 7. The conclusion of Theorem 2 rests true if $|K(\sigma)|\leq c(p)$ for any antiholomorphic 2-plane σ of signature $(+, -)$. Analogously for antiholomorphic 2-planes of signature $(+, +)$ or $(-, -)$ in the case of an appropriate signature of M.

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Centre for Mathematics and Mechanics
1090 Sofia P. O. Box 373

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