# PLISKA
## STUDIA MATHEMATICA BULGARICA

# ПЛИСКА
## БЪЛГАРСКИ МАТЕМАТИЧЕСКИ СТУДИИ

# COMPLEX SUBMANIFOLDS IN ALMOST HERMITIAN MANIFOLDS

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In this paper we prove a number of properties of $\sigma$-hypersurfaces in almost Hermitian manifold with constant pointwise antiholomorphic sectional curvature or a vanishing Bochner curvature tensor. Characteristics of totally quasiumbilical complex submanifold in such almost Hermitian manifolds are given.

**1. Introduction.** The following theorems are basic results in the theory of the complex submanifods in almost Hermitian manifolds.

Theorem A. [10] *Let $M^{2n}$ be a complex hypersurface of Kaehlerian manifold $\tilde{M}^{2n+2}$ of constant holomorphic sectional curvature $\tilde{C}$. If $n\geqq 2$, the following statements are equivalent:*

a) *$M$ is totally geodesic in $\tilde{M}$,*

b) *$M$ is of constant holomorphic sectional curvature $\tilde{c}$,*

c) *$M$ is an Einstein manifold and all sectional curvatures of $M$ at a point of $M$ are $\geqq \tilde{c}/4$ (resp. $\leqq \tilde{c}/4$) when $\tilde{c}\geqq 0$ (resp. $\tilde{c}\leqq 0$).*

Theorem B [11]. *Let $M^{2n}$ be a complex hypersurface of a Kaehlerian manifold $\tilde{M}^{2n+2}$ of constant holomorphic sectional curvature $\tilde{c}$. If $n\geqq 3$ then the following statements are equivalent:*

a) *$M$ has a vanishing tensor of Bochner,*

b) *$M$ is totally geodesic in $\tilde{M}$,*

c) *$M$ is of constant holomorphic sectional curvature $\tilde{c}$.*

Theorem C [8]. *Let $M^{2n}$ be a $\sigma$-hypersurface in the almost Hermitian manifold $\tilde{M}^{2n+2}$ of pointwise constant holomorphic sectional curvature $\tilde{c}(p)$. If $n\geqq 2$, then the following statements are equivalent:*

a) *$M$ is totally geodesic in $\tilde{M}$,*

b) *$M$ is of pointwise constant holomorphic sectional curvature $\tilde{c}(p)$.*

In accordance with a note in [6] we can say that the statements in which the antiholomorphic sections participate, are more natural analogues of the statements from Riemannian geometry.

**2. Preliminaries.** In all our further considerations we suppose that $\tilde{M}^{2n}$ is an almost Hermitian manifold of an almost complex structure $J$, metric tensor $g$, a Levi-Civita connection $\tilde{\nabla}$ and a corresponding curvature tensor $\tilde{R}$. Let $M^{2m}(m<n)$ be a complex submanifold of $\tilde{M}$, the almost complex structure and metric tensor of which are identified with the restrictions of $J$ and $g$, respectively. We denote the curvature tensor by $R$, the connection by $\nabla$, the tangential space to $M$ at an arbitrary point $p\in M$ by $T_p(M)$ and the Lie algebra of $C^\infty$ vector fields by $\mathfrak{X}(M)$. Let $\xi_1, \xi_2, \ldots, \xi_{2n-2m}$ be unit normal vector fields of $M$ and $x, y, z, u$-arbitrary vector fields of $\mathfrak{X}(M)$. The formula of Gauss is given by

(1) $$\tilde{\nabla}_x y = \nabla_x y + \sigma(x, y)$$

as the second fundamental form $\sigma(x, y) = \sum_{i=1}^{2n-2m} h_i(x, y)\xi_i$, where $h_i(x, y)$ for $i=1, 2, \dots, 2n-2m$ are symmetric covariant tensor fields of degree 2 in neighbourhood of point $p \in M$.

The equation of Gauss for the submanifold $M$ is

(2) $$\tilde{R}(x, y, z, u) = R(x, y, z, u) + g(\sigma(x, z), \sigma(y, u)) - g(\sigma(x, u), \sigma(y, z))$$

for $R(x, y, z, u) = g(R(x, y), z, u)$.

If in $M$ (when $m>1$) there exist functions $\alpha_i$, $\beta_i$ and unit 1-forms $w_i$ for $i=1, 2, \dots, 2n-2m$ such that

(3) $$h_i = \alpha_i g + \beta_i w_i \otimes w_i,$$

then $M$ is said to be totally quasiumbilical in $\tilde{M}$. In particular when $\alpha_i = \beta_i = 0$ for each $i=1, 2, \dots, 2n-2m$, $M$ is said to be totally geodesic, when $\beta_i = 0$ for each $i=1, 2, \dots, 2n-2m$, $M$ is said to be totally umbilical and when $\alpha_i = 0$ for each $i=1, 2, \dots, 2n-2m$ — $M$ is totally cylindrical [2].

In the case, then $M$ is a complex hypersurface, i. e. $m=n-1$, let $\xi$ be a unit normal vector field of $M$ in neighbourhood of point $p \in M$ then the second fundamental form is

(4) $$\sigma(x, y) = h(x, y)\xi + k(x, y)J\xi$$

in addition

(5) $$\tilde{\nabla}_x \xi = -A^* x + s(x) J\xi, \qquad \tilde{\nabla}_x (J\xi) = -\bar{B}^* x - s(x)\xi$$

If the condition $h(x, y) = k(x, Jy)$ for the complex hypersurface $M$ of $\tilde{M}$ or its equivalents

(6) $$B^* = JA^* \text{ or } \sigma(x, Jy) = \sigma(Jx, y) = J\sigma(x, y)$$

are available then $M$ is said to be $\sigma$-hypersurface [9].

**3.** In this paper some theorems concerning submanifolds $M$ in $\tilde{M}$ with pointwise constant antiholomorphic sectional curvature or with a vanishing Bochner tensor are estabbished.

If $\alpha$ is 2-dimensional linear subspace of $T_p(M)$ and $\{x, y\}$ is orthonormal basis for $\alpha$, then the sectional curvature of $\alpha$ is presented by

(7) $$K_R(\alpha) = K_R(x, y) = R(x, y, y, x).$$

In the case then $\alpha$ is antiholomorphic, i. e. $J\alpha \perp \alpha$ then the curvature is called antiholomorphic sectional curvature.

Bochner curvature tensor of an almost Hermitian manifold is given in [12]. If it is equal to zero, the manifold is called Bochnerian flat.

Theorem 1. *Let $M$ ($2 \leq m < n$) be totally geodesic, or totally cylindrical in $\tilde{M}$. If $\tilde{M}$ is of pointwise constant antiholomorphic sectional curvature $\tilde{c}(p)$, then $M$ is of pointwise constant antiholomorphic sectional curvature $\tilde{c}(p)$.*

Proof. From (2) and (3) follows

(8) $$\tilde{R}(x, y, z, u)=R(x, y, z, u)+\sum_{i=1}^{2n-2m}\{\alpha_i^2(g(x, z)g(y, u)-g(x, u)g(y, z))$$
$$+\alpha_i\beta_i(g(x, z)\, w_i(y)w_i(u)+g(y, u)w_i(x)w_i(z)$$
$$-g(x, u)w_i(y)w_i(z)-g(y, z)\, w_i(x)w_i(u))\}.$$

Let $\{x, y\}$ be an orthonormal basis of antiholomorphic section in (8) and $M$ is totally geodesic or totally cylindrical. So $\tilde{R}(x, y, y, x)=R(x, y, y, x)$ which proves the theorem.

Theorem 2. *Let $M$ be a $\sigma$-hypersurface in $\tilde{M}$ $(n>3)$ with a pointwise constant antiholomorphic sectional curvature $\tilde{c}(p)$. The following statements are equivalent:*

a) *$M$ is totally geodesic in $\tilde{M}$;*

b) *$M$ is with a pointwise constant antiholomorphic sectional curvature;*

c) *$M$ is Bochner flat and $\rho(AR)=\tau(AR)/2(n-1)g$ according to the notations in* [5].

Proof. The statement a)→b) follows from Theorem 1. We shall prove that b)→a) Eqs (2) and (3) give

(9) $$\tilde{R}(x, y, z, u)=R(x, y, z, u)+g(A^*x, z)g(A^*y, u)-g(A^*x, u)g(A^*y, z)$$
$$+g(JA^*x, z)g(JA^*y, u)-g(JA^*x, u)g(JA^*y, z)$$

for arbitrary $x, y, z, u\in T_p(M)$.

It is known [8] that there exists an orthonormal basis $\{l_1, l_2, \ldots, l_{n-1}, Jl_1, Jl_2, \ldots Jl_{n-1}\}$ of $T_p(M)$ with respect to which the matrix of $A^*$ has a diagonal form

$$\left\|\begin{matrix}\lambda_1 &&&&&&& \\ & \lambda_2 &&&&&& \\ && \ddots &&&& 0 & \\ &&& \lambda_{n-1} &&&& \\ &&&& -\lambda_1 &&& \\ &&&&& -\lambda_2 && \\ 0 &&&&&& \ddots & \\ &&&&&&& -\lambda_{n-1}\end{matrix}\right\|$$

as $A^*l_i=\lambda_i l_i$ and $A^*Jl_i=-\lambda_i Jl_i$, $i=1, 2, \ldots, n-1$.

Let $x=l_i$, $y=l_j$ for $i\mp j$, so $\{x, y\}$ is orthonormal basis of antiholomorphic section. From (9) we get $A^*=0$, i. e. $M$ is totally geodesic in $\tilde{M}$.

The equivalence of b) and c) follows from some statements in [5].

Theorem 3. *Every totally quasiumbilical complex submanifold $M$ $(3<m<n)$ in Bochner flat almost Hermitian manifold $\tilde{M}$ is Bochner flat.*

Proof. Let $x, y, z, u$ be orthonormal antiholomorphic quadruple from $T_p(M)$. From the condition that $\tilde{M}$ is Bochner flat and (8) follows that $M$ is also Bochner flat [3].

Definition. *We shall say that the manifold $\tilde{M}$ $(n>4)$ satisfies the axiom of Bochner flat totally quasiumbilical complex submanifold, if for every $p\in\tilde{M}$ and any 2m-dimensional section $H$ in $T_p(M)$, $4\leq m<n$, there exists a Bochner flat totally quasiumbilical complex submanifold $M\ni p$ and $H=T_p(M)$.*

Theorem 4. *If an almost Hermitian manifold $\tilde{M}$ $(n>4)$ satisfies the axiom of Bochner flat totally quasiumbilical complex submanifolds, then it is Bochner flat.*

P r o o f. Let $M$ be totally quasiumbilical. Then for every $x, y, z, u \in T_p(M)$ (8) is valid. Let us suppose that $\tilde{M}$ satisfies the axiom. Then using (8) when $x, y, z, u$ is orthonormal antiholomorphic quadruple and [3] follows the correctness of the theorem.

T h e o r e m 5. *Let $M$ be a $\sigma$-hypersurface in a Bochner flat manifold $\tilde{M}$ and $n>4$. The following statements are equivalent:*

a) *$M$ is totally geodesic in $\tilde{M}$,*

b) *$M$ is Bochner flat.*

P r o o f. The implication of a)→b) follows from the theorems mentioned above.

Conversely, let $B(R)=0$ and $B(\tilde{R})=0$. Then from [3] follows that Bochner generalized tensors for $M$ and $\tilde{M}$ are also zeros. Using the notations and concepts in [1, 4] about the holomorphic sections curvatures of $M$ and $\tilde{M}$ in the direction of the unit vector $x \in T_p(M)$, we get

$$H(x)=\frac{4}{n+1}S^*(x,x)-\frac{2}{n(n+1)}\tau^*(R),\quad \tilde{H}(x)=\frac{4}{n+2}\tilde{S}^*(x,x)-\frac{2}{(n+1)(n+2)}\tau^*(\tilde{R}),$$

where $S^*$, $\tilde{S}^*$ and $\tau^*(R)$, $\tau^*(\tilde{R})$ are the generalized Ricci curvatures and the generalized scalar curvatures of $M$ and $\tilde{M}$, respectively. Using some results from [7], we get

$$\frac{4}{n+1}g(A^*x, A^*x)-g^2(A^*x, x)-g^2(JA^*x, x)-\frac{2}{(n+1)n}\sum_{i=1}^{n-1}g(A^*l_i, A^*l_i)=0.$$

Let $x=l_i$, where $l_i$ is an arbitrary vector of the basis which has been introduced in theorem 2. Then from the last equation we get

$$\frac{n-2}{4}\sum_{i=1}^{n-1}g^2(A^*l_i, l_i)=0$$

from where follows that $A^*\equiv 0$, i. e. $M$ is totally geodesic in $\tilde{M}$.

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