PLISKA

STUDIA MATHEMATICA BULGARICA

ПЛИСКА

БЪЛГАРСКИ МАТЕМАТИЧЕСКИ СТУДИИ

ON THE JENKINS INEQUALITY FOR THE PAIRS OF UNIVALENT MAPPINGS WITH QUASICONFORMAL EXTENSION

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The Jenkins inequality for univalent functions is generalized for a class of pairs of meromorphic in the unit disk quasiconformal homeomorphisms of certain bigger disks.

1. Introduction. The given paper is devoted to the spreading of Jenkins inequality ([1]), which is well-known in the theory of univalent functions, to the class $\mathfrak{M}^R_{Q_1,Q_2}$ consisting of pairs of Q_1 and Q_2, respectively, of quasiconformal (qc) homeomorphisms of the disk $U_R=\{z:|z|<R\}$, $1<R$, meromorphic in U_1 and that map U_R onto mutually nonoverlapping domains.

The proof is based on the combination of the variations method for quasiconformal mappings and the area principle for univalent analytic functions (see [4-6] and [2-3], [7-9]).

Let $\mathfrak{M}^R_Q$ be a class of meromorphic univalent functions $f(z)=1/z+b_0+b_1z+\ldots$ in U_1, permitting $Q-$qc extension up to the disk U_R, $1<R$, that do not admit zero in U_R; $\mathfrak{M}^R_{Q_1,Q_2}(\infty,0)$ is a subclass of $\mathfrak{M}^R_{Q_1,Q_2}$ providing the following conditions:

$$f_1(0)=\infty,\quad \lim_{z\to 0} zf_1(z)=1,\quad f_2(0)=0$$

and $\mathfrak{M}(\infty,0)\equiv\mathfrak{M}^1_{Q_1,Q_2}(\infty,0)$.

Note that the combination $\overline{\mathfrak{M}^R_{Q_1,Q_2}(\infty,0)}$ of the classes $\mathfrak{M}^R_{Q_1,Q_2}(\infty,0)$ and $\cup_{F\in\mathfrak{M}^R_{Q_1}}\{F,0\}$ is compact in the topology of locally uniform convergence.

2. The variation of functions of the class $\mathfrak{M}^R_{Q_1,Q_2}$ (0, ∞) and necessary conditions for extremum. Let J be a continuous real functional defined on $\overline{\mathfrak{M}(\infty,0)}$. Assume that J has a real Gateaux derivative at the point $\{F,f\}\in\overline{\mathfrak{M}(\infty,0)}$ relative to the class $\mathfrak{M}(\infty,0)$, i.e. for any pair of functions

$$\{F^*,f^*\}\in\overline{\mathfrak{M}(\infty,0)},$$

where

$$F^*=F+\varepsilon H+o(\varepsilon),\quad f^*=f+\varepsilon h+o(\varepsilon),\quad \varepsilon>0,$$

$$J(F^*,f^*)=J(F,f)+\varepsilon\,\mathrm{Re}\,L(H,h;F,f)+o(\varepsilon).$$

Here $L(\cdot,\cdot, F, f)$ is a continuous bilinear functional on the set of all meromorphic functions on a single disk.

Consider the maximum problem of the functional $J(F|_U, f|_U)$ on the class $\overline{\mathfrak{M}^R_{Q_1,Q_2}}(\infty, 0)$. The existence of extremal functions follows from the compactness of $\overline{\mathfrak{M}^R_{Q_1,Q_2}}(\infty, 0)$ and continuity of the functional J.

Hence forth we are interested in those functionals, the maximum of which is achieved within the class $\overline{\mathfrak{M}^R_{Q_1,Q_2}}(\infty, 0)$. The analysis of another functionals on $\overline{\mathfrak{M}^R_{Q_1,Q_2}}(\infty, 0)$ may be reduced to the analysis of extremal problems for the class $\mathfrak{M}^R_{Q_1}$.

Theorem 1. *Let for the pair of functions* $\{f_1, f_2\} \in \mathfrak{M}^R_{Q_1,Q_2}(\infty, 0)$ *the functional* J *takes its maximum within the class* $\mathfrak{M}^R_{Q_1,Q_2}(\infty, 0)$ *and*

$$g(\omega, f) = \frac{f}{(f-\omega)\omega}, \tag{1}$$

$$\varphi(\omega) = \int^{\omega} [L(g(t, f_1), g(t, f_2))]^{1/2} dt. \tag{2}$$

Then:

1) $f_F(z)$ *satisfies the differential equation*

$$f_{p\bar{z}} = \frac{|L(g(f_p(z), f_1), g(f_p(z), f_2))|}{L(g(f_p(z), f_1), g(f_p(z), f_2))} q_p \overline{f_{pz}} \tag{3}$$

for almost all (a. a.) z *from* $K_{1,R}/N_p$.

Here

$$N_p = \{z : L(g(f_p(z), f_1), g(f_p(z), f_2)) = 0\},$$

$$K_{1,R} = \{z : 1 < |z| < R\}, \quad q_p = (Q_p - 1)/(Q_p + 1), \quad p = 1, 2.$$

2) *If the analytic function* $L(g(\cdot, f_1), g(\cdot, f_2))$ *is non zero at a neighbourhood of a point* $\omega \in f_p(K_{1,R})$ *then the function*

$$\mathscr{E}_p(z) = \varphi \circ f_p(z) - q_p \overline{\varphi \circ f_p(z)}$$

is analytic at a neighbourhood of a point $z = f_p^{-1}(\omega)$, $p = 1, 2$.

The proof of Theorem 1 is standart and based on the following variation's Lemma (see, e.g., [5, 6] and [10, p. 141]).

Lemma. *Let* $\{f_1, f_2\} \in \mathfrak{M}^R_{Q_1,Q_2}(\infty, 0)$ *has complex characteristics* μ_1 *and* μ_2, *correspondingly. Let* $v_p(z)$, $p = 1, 2$, *be arbitrary functions measurable in* $K_{1,R}$, *such that*

$$\operatorname*{ess\,sup}_{z\in K_{1,R}} |\nu_p(z)| \leqq q_p, \quad p=1,\ 2.$$

Then the class $\mathfrak{M}^R_{Q_1,Q_2}(\infty, 0)$ *for a sufficiently small* $\varepsilon>0$ *contains the pair of functions*

$$(4) \qquad f^*_p(z)=f_p(z)+\frac{\varepsilon}{\pi}f_p(z)\iint_{K_{1,R}}\left[\sum_{l=1}^{2}\frac{\mu_l(t)-\nu_l(t)f^2_{lt}(t)}{f_l(t)-f_p(z)f_l(t)}\right]dm_t+O(\varepsilon^2),$$

$p=1,\ 2$ *and* $O(\varepsilon^2)/\varepsilon^2$ *is uniformly bounded in* $\overline{U_R}$.

The Lemma can be proved by two fundamental results from the theory of quasiconformal mappings, which are connected with the theorems of existence and uniqueness, and by the theorem of representation of a main homeomorphism of Beltrami equation (see [11, p. 80-104] and [12, p. 107-116]).

3. The area theorem. Let $\rho\in(0, R)$. Denote by D_ρ the set $\bar{C}\backslash\overline{\{f_1(U_\rho)\cup f_2(U_\rho)\}}$ and by D'_ρ a simply connected domain obtained by cutting D_ρ along a simple smooth arc.

Theorem 2. *Let* $\{f_1, f_2\}\in\mathfrak{M}^R_{Q_1,Q_2}(\infty, 0)$ *and* $Q(w)$ *be a function with regular derivative in the domain* $D(r_0)$, $r_0<1$. *If the functions*

$$(5) \qquad G_p(z)=Q\circ\varphi_p(z)-q_p\overline{Q\circ\varphi_p(z)}$$

have a regular derivative in the annulus $K_{1,R}$ *then for the expansion coefficients of functions* $Q\circ f_p(z)$ *in the annulus* $K_{r_{0},1}$ *representing into series of type*

$$Q\circ f_p(z)=\sum_{n=1}^{\infty}\beta^{(p)}_n z^{-n}+\sum_{n=0}^{\infty}\alpha^{(p)}_n z^n+\beta^{(p)}\ln z,$$

$p=1,\ 2$ *and* $\beta^{(1)}+\beta^{(2)}=0$, *the following inequality holds*:

$$(6) \qquad \sum_{p=1}^{2}(1-q_p^2)^{-1}\sum_{n=1}^{\infty}n(R^{2n}-q_p^2R^{-2n})\left|\alpha^{(p)}_n-q_p\overline{\beta^{(p)}_n}\,\frac{R^{4n}-1}{R^{4n}-q_p^2}\right|^2$$

$$\leqq\sum_{p=1}^{2}(1-q_p^2)\sum_{n=1}^{\infty}n(R^{2n}-q_p^2R^{-2n})^{-1}|\beta^{(\beta)}_n|$$

$$-2\operatorname{Re}\sum_{p=1}^{2}\overline{\beta^{(p)}}[\alpha^{(p)}_0+(1-q_p^2)^{-1}(\beta^{(p)}+2q_p\overline{\beta^{(p)}}+q_p^2\beta^{(p)})\ln R].$$

The equality is valid only in the case when the area $S(r)$ *of the image domain* $\Delta(r)=Q(D'(r))$ *tends to zero when* $r\to R$.

Remark 1. Theorem 2 assumes that the conditions of analytic functions are fulfilled on $G_p(z)$, $p=1, 2$. Utilizing the variations method there was shown that these conditions are valid for the wide range of extremal problems in p. 2.

Proof of Theorem 2. Using the analyticity of functions $G_p(z)$, $p=1, 2$, we calculate the area $S(r)$ of image domain $\Delta(r)=Q(D'(r))$, $1<r<R$.

Since $S(r)\geqq 0$ $\lim_{z\to R} S'(r)=S\geqq 0$, i.e. the inequality (6) is valid.

4. Main result. Let $\{f_1, f_2\}\in\mathfrak{M}^R_{Q_1,Q_2}(\infty, 0)$, $f_k(0)=a_k$ and

$$\varphi_{k,p}(\tau, z)=\ln\frac{f_k(\tau)-f_p(z)}{(f_k(\tau)-a_p)(a_k-f_p(z))},\quad p\neq k,$$

$$\varphi_{k,p}(\tau, z)=\ln\frac{(f_k(\tau)-f_k(z))\tau z f_k'(0)}{(f_k(\tau)-a_k)(f_k(z)-a_k)(\tau-z)},\quad p=k,$$

$p=1, 2$; $k=1, 2$; moreover the factors containing $a_1=\infty$ must be substituted by the unit.

Note that for $|\tau|<1$, $|z|<1$

$$\varphi_{k,p}(\tau, z)=\sum_{m,n=1}^{\infty}\omega^{k,p}_{m,n}\tau^m z^n$$

and

$$\omega^{k,p}_{m,n}=\omega^{p,k}_{n,m},\quad p=1, 2,\quad k=1, 2.$$

Theorem 3. *Let* $\{f_1, f_2\}\in\mathfrak{M}^R_{Q_1,Q_2}(\infty, 0)$, $x_{m,k}$, $k=1, 2$, $m=1, 2, \dots$, *be arbitrary constants, such that*

$$A=\sum_{k=1}^{2}\sum_{m=1}^{\infty}\frac{1+q_kR^{2m}}{q_k+R^{2m}}\frac{|x_{m,k}|^2}{m}<\infty. \tag{7}$$

Then

$$\sum_{p=1}^{2}\sum_{n=1}^{\infty} n\frac{q_p+R^{2n}}{1+q_pR^{2n}}\left|\sum_{k=1}^{2}\sum_{m=1}^{\infty}\omega^{k,p}_{m,n}x_{m,k}\right|^2\leqq A. \tag{8}$$

Remark 2. Assuming $R=1$ in Theorem 3, we obtain the coefficient inequality of Jenkins type for the class $\mathfrak{M}(\infty, 0)$ ([2, p. 218-219]) which may be transformed into Jenkins inequality when $p=1$ ([1]).

Proof of Theorem 3. Let the pair of the functions $\{f_1, f_2\}$ maximizes a real continuous functional

$$J(f_1, f_2)=\operatorname{Re}\sum_{k,p=1}^{2} L_k\circ L_p(\varphi_{k,p}(\tau, z)) \tag{9}$$

defined on $\overline{\mathfrak{M}^R_{Q_1,Q_2}(\infty, 0)}$. Here L_k is a linear continuous functional such that $L_k(\tau^m)=x_{m,k}$. It is easy to see that $\{f_1, f_2\}\in\mathfrak{M}^R_{Q_1,Q_2}(\infty, 0)$.

Concretizing the statements of Theorem 1 with respect to the functional of type (9) one can verify that the functions $\{f_1, f_2\}$ and

$$Q(w)=L_1\left(\ln\left(1-\frac{w}{f_1(\tau)}\right)\right)+L_2\left(\ln\left(1-\frac{f_2(z)}{w}\right)\right)$$

satisfy the conditions of Theorem 2.

Applying the inequality (6) to the extremal pair $\{f_1, f_2\}$ with the given function $Q(w)$ and performing the necessary simple transformations we obtain the expression

$$\text{(10)} \qquad \operatorname{Re}\left\{\sum_{k,p=1}^{2}\sum_{m,n=1}^{\infty}\omega^{k,p}_{m,n}x_{m,k}x_{n,p}\right\}\leqq A,$$

which is valid for the class $\mathfrak{M}^R_{Q_1,Q_2}(\infty, 0)$.

The inequality (8) can be obtained from (10) with the standard transformation by Cauchy inequality (see, for example, [3, p. 174-176]).

REFERENCES

1. J. A. Jenkins. Some area theorems and a special coefficient theorem. *Ill. J. Math.*, **8**, 1964, 80-99.
2. N. A. Lebedev. An application of the area principle to nonoverlapping domains. *Trudy Mat. Inst. AN SSSR*, **60**, 1961, 211-231 (in Russian).
3. N. A. Lebedev. The area principle in the theory of univalent functions. Moskow, 1975 (in Russian).
4. P. P. Belinskiy. The use of the variation method in solving extremum problems of quasiconformal mappings. *Dokl. Akad. Nauk SSSR*, **121**, 1958, No 2, 199-201 (in Russian).
5. M. Schiffer, G. Schober. Coefficient problems and generalized Grunsky inequality for schlicht functions with quasiconformal extensions. *Arch. Rational Mech. and Anal.*, **60**, 1976, No 3, 205-228.
6. V. Ya. Gutlyans'kiy. Variation method for univalent analytical functions with quasiconformal extension. *Sib. Math. J.*, **21**, 1980, No 2, 61-78 (in Russian).
7. O. Lehto. Schlicht functions with a quasiconformal extension. *Ann. Acad. Sci. Fenn., Ser. A*1, 1971, No 500, 3-10.
8. V. Ya. Gutlyans'kiy. On the principle of areas for one class of quasiconformal mappings. *Dokl. Acad. Nauk SSSR*, **212**, 1973, No 3, 540-543 (in Russian).
9. R. Kühnau. Der Flächensatz in einer klasse schlichter Abbildungen. *Rev. Roum. Math., Pure et Appl.*, **26**, 1981, No 8, 1119-1121.
10. G. Schober. Univalent Functions — Selected Topics. Berlin — Heidelberg — New York, 1975.
11. L. Ahlfors. Lectures on quasiconformal mappings. Moskow, 1969 (in Russian).
12. J. N. Vekua. Generalized analytical functions. Moskow, 1959 (in Russian).

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Received 12.8.1985