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AN APPLICATION OF THE GENERALIZED OPERATORS OF FRACTIONAL INTEGRATION TO DUAL INTEGRAL EQUATIONS INVOLVING MEIJER'S G-FUNCTION

VIRGINIA S. KIRYAKOVA

An unified approach to dual integral equations with Meijer's G-functions as kernels, namely:

$$\int_0^\infty G^{m,n}_{p+n,q+m}\left[xu\left|\begin{matrix}a_1,\dots,a_{n+p}\\ b_1,\dots,b_{m+q}\end{matrix}\right.\right]f(u)\,du=\varphi(x),\quad 0<x<1,$$

$$\int_0^\infty G^{m,n}_{p+n,q+m}\left[xu\left|\begin{matrix}c_1,\dots,c_{n+p}\\ d_1,\dots,d_{m+q}\end{matrix}\right.\right]f(u)du=\psi(x),\quad x>1,$$

is proposed. It is based on a generalization of the Poisson's and Sonine's transmutation operators introduced by I. Dimovski and by the author in their studies of Bessel type opeators. The solution is obtained in a closed form together with the corresponding conditions for the parameters. As a very special case it contains the Peters' and Busbridge's solutions of dual integral equations with Bessel functions as kernels.

There are many techniques available for the solution of mixed boundary value problems arising in mathematical physics [1]. A specific approach to this class of problems is to reduce them to dual integral equations. It seems that the first and simplest example of such a pair of equations is the following:

$$\int_0^\infty u^{-1}J_0(xu)f(u)du=1,\quad 0<x<1,$$

$$\int_0^\infty J_0(xu)f(u)du=0,\quad x>1.$$

In general, the pair of equations

$$\int_0^\infty w(u)K(x,\ u)f(u)du=\varphi(x),\quad 0<x<1,$$

$$\int_0^\infty K(x,\ u)f(u)du=\psi(x),\quad x>1,$$

where the kernel $K(x, u)$, the weight $w(u)$ and the boundary conditions $\varphi(x)$, $\psi(x)$ are known functions and $f(u)$ is to be determined is known as a pair of dual integral equations. Several methods for their solutions in the case $w(u)=u^{\lambda}$, $K(x, u)=J_{\nu}(xu)$ are developed by Weber (1873), Busbridge (1938), Titchmarsh (1948), Noble (1958), Peters (1961), Erdélyi and Sneddon (1962) and others. In a great part of these approaches an important role is played by the operators of fractional integration. More precise description of the history of this method can be found in [2, § 39]. In the papers of Fox (1965), Saxena (1967) and Mathai, Saxena ([3, p. 238-248]) a general method is proposed to obtain the solution of dual integral equations associated with an arbitrary special function having a Mellin-Barnes type integral representation. The kernel function is taken to be H-function of Fox or Meijer's G-function and compositions of Erdélyi – Kober operators of fractional integration are applied to reduce the given equations into two others with one and the same kernel. In this way the solution is given in a closed but rather involved form because these compositions of fractional integrals are not calculated explicitly. Moreover conditions on the parameters of H- and G-functions in the kernel in order the solution to be a nonformal one, are not given in [3].

In this paper we give an explicit solution of a class of dual integral equations whose kernel is Meijer's G-function. It contains as special cases the solutions of dual integral equations involving many special functions of mathematical physics. We illustrate this by the example with Bessel functions as kernels. Our choice of the kernel allows us to include an arbitrary power function $w(u)=u^{\lambda}$ as a weight function. This follows from the simple property of the G-function:

$$u^{\lambda}K(x,\ u)=x^{-\lambda}G^{m,n}_{p,q}\left[xu\left|\begin{matrix}(a_k)\\(b_l)\end{matrix}\right.\right],$$

$$K(x,\ u)=G^{m,n}_{p,q}\left[xu\left|\begin{matrix}(c_k)\\(d_l)\end{matrix}\right.\right],$$

where $a_k=c_k+\lambda,\ b_l=d_l+\lambda,\ k=1,\dots,\ p,\ l=1,\dots,\ q.$

In [4] generalized operators of fractional integration are introduced to study the Bessel type differential operators of order greater than two. In [5, 6] an investigation of them is given in full details. These operators can be characterized as compositions of an arbitrary number of generalized Erdélyi-Kober fractional integrals but they have one-dimensional representation with Meijer's G-function as kernel. In such a way, following the ideas of [3] we give an explicit form of the solution of the dual integral equations considered there. Moreover the conditions on the parameters of G-functions are indicated here.

1. The problem and its reduction by Mellin transform. Let $m\geqq 0$, $n\geqq 0$, $p\geqq 0$, $q\geqq 0$ be integers such that $\delta=(m+n)-(p+q)\geqq 0$. As in [3] we consider the case $\delta=0$, i.e. $m+n=p+q$, although the same approach applies to the general case $\delta\geqq 0$.

We consider a pair of dual integral equations of the form

$$\text{(1)}\qquad \int_0^{\infty} G^{m,n}_{p+n,q+m}\left[xu\left|\begin{matrix}a_1,\dots,a_{n+p}\\b_1,\dots,b_{m+q}\end{matrix}\right.\right]f(u)du=\varphi(x),\quad 0<x<1,$$

$$\int_0^\infty G^{m,n}_{p+n,q+m}\left[xu\left|\begin{matrix}c_1,\dots,c_{n+p}\\ d_1,\dots,d_{m+q}\end{matrix}\right.\right]f(u)du=\psi(x),\quad x>1, \tag{2}$$

with Meijer's G-functions

$$G_1(x)=G^{m,n}_{p+n,q+m}\left[x\left|\begin{matrix}(a_k)\\ (b_l)\end{matrix}\right.\right]=\frac{1}{2\pi i}\int_L \chi_1(s)x^{-s}ds,$$

$$G_2(x)=G^{m,n}_{p+n,q+m}\left[x\left|\begin{matrix}(c_k)\\ (d_l)\end{matrix}\right.\right]=\frac{1}{2\pi i}\int_L \chi_2(s)x^{-s}ds$$

as kernels, where

$$\chi_1(s)=\frac{\prod_{j=1}^{m}\Gamma(b_j+s)\prod_{j=1}^{n}\Gamma(1-a_j-s)}{\prod_{j=1}^{q}\Gamma(1-b_{m+j}-s)\prod_{j=1}^{p}\Gamma(a_{n+j}+s)}=\frac{\Gamma_{1,1}\Gamma_{2,1}}{\Gamma_{3,1}\Gamma_{4,1}}, \tag{3}$$

$$\chi_2(s)=\frac{\prod_{j=1}^{m}\Gamma(d_j+s)\prod_{j=1}^{n}\Gamma(1-c_j-s)}{\prod_{j=1}^{q}\Gamma(1-d_{m+j}-s)\prod_{j=1}^{p}\Gamma(c_{n+j}+s)}=\frac{\Gamma_{1,2}\Gamma_{2,2}}{\Gamma_{3,2}\Gamma_{4,2}}, \tag{4}$$

and L is a suitably chosen contour in **C** (see [8, p. 203-204]).

The Mellin transform

$$\mathcal{M}[f(u)]=F(s)=\int_0^\infty u^{s-1}f(u)du \tag{5}$$

with its inversion formula

$$\mathcal{M}^{-1}[F(s)]=f(u)=\frac{1}{2\pi i}\int_C F(s)u^{-s}ds \tag{6}$$

is an useful tool in solving the dual integral equations. Applying this transform to the equations (1), (2) we can reduce them to a pair of equations whose kernels are the Mellin transforms of the G-functions

$$\begin{aligned}&\mathcal{M}\left[G^{m,n}_{p+n,q+m}\left[x\left|\begin{matrix}a_1,\dots,a_{n+p}\\ b_1,\dots,b_{m+q}\end{matrix}\right.\right]\right]=\chi_1(s),\\ &\mathcal{M}\left[G^{m,n}_{p+n,q+m}\left[x\left|\begin{matrix}c_1,\dots,c_{n+p}\\ d_1,\dots,d_{m+q}\end{matrix}\right.\right]\right]=\chi_2(s).\end{aligned} \tag{7}$$

As shown in [3] using the Parseval theorem for the Mellin transform, the new pair of equations is

$$(8)\qquad \frac{1}{2\pi i}\int_L \chi_1(s)x^{-s}F(1-s)ds=\varphi(x),\quad 0<x<1,$$

$$(9)\qquad \frac{1}{2\pi i}\int_L \chi_2(s)x^{-s}F(1-s)ds=\psi(x),\quad x>1.$$

Here a new unknown function is the Mellin transform $F(s)$ of $f(u)$. For conditions under which (5), (6) are true, the choice of contour $\mathscr{C}$ and for the details of the reduction of (1), (2) to (8), (9) we refer to [7, p. 31-33] and [3, p. 242-243].

2. Application of generalized operators of fractional integration to equations (8), (9). The solution of the original equations. The main idea in this section is to transform each of the kernels $\chi_i(s)$, $i=1, 2$, of equations (8), (9) to a common kernel $\chi(s)$ of the same form using fractional integration operators. By an application of a generalized Riemann-Liouville fractional integration operator to the equation (8) we try to transform the quotient $\Gamma_{2,1}/\Gamma_{3,1}$ of Γ-functions with $(-s)$ in the argument from the expression (3) for $\chi_1(s)$, into the corresponding quotient $\Gamma_2/\Gamma_3=\Gamma_{2,2}/\Gamma_{3,2}$ from the expression (4) for $\chi_2(s)$. Similarly, by a fractional integration operator of Weyl type applied to (9) we transform the quotient $\Gamma_{1,2}/\Gamma_{4,2}$ of Γ-functions with $(+s)$ in the argument into the corresponding one $\Gamma_1/\Gamma_4=\Gamma_{1,1}/\Gamma_{4,1}$. In this way equations (8), (9) can be reduced to a pair of equation with common kernel

$$(10)\qquad \chi(s)=\frac{\Gamma_1\Gamma_2}{\Gamma_3\Gamma_4}=\frac{\prod_{j=1}^{m}\Gamma(b_j+s)\prod_{j=1}^{n}\Gamma(1-c_j-s)}{\prod_{j=1}^{q}\Gamma(1-d_{m+j}-s)\prod_{j=1}^{p}\Gamma(a_{n+j}+s)}$$

and their solution can be written by one formula.

Definition 1. *Let $m\geqq 1$ be an integer, $\beta>0$, $\{\gamma_k\}_{k=1}^m$ and $\{\delta_k\}_{k=1}^m$ be real numbers. For $\delta_k>0$, $k=1,\dots,m$, we define the generalized operators of fractional integration of Riemann-Liouville type by the integral*

$$(11)\qquad \mathscr{R}f(x)=\mathscr{R}_{\beta,m}^{\{\gamma_k\},\{\delta_k\}}f(x)=\int_0^1 G_{m,m}^{m,0}\left[\sigma\left|\begin{matrix}(\gamma_k+\delta_k)\\(\gamma_k)\end{matrix}\right.\right]f(x\sigma^{1/\beta})d\sigma$$

$$=x^{-\beta}\int_0^x G_{m,m}^{m,0}\left[\left(\frac{t}{x}\right)^{\beta}\left|\begin{matrix}(\gamma_k+\delta_k)\\(\gamma_k)\end{matrix}\right.\right]f(t)d(t^{\beta}).$$

For $m=1$ this is the generalized Erdélyi-Kober operator of fractional integration:

(12) $$I_\beta^{\gamma,\delta} f(x) = \mathscr{R}_{\beta,1}^{\gamma,\delta} f(x) = \int_0^1 \frac{(1-\sigma)^{\delta-1}}{\Gamma(\delta)} \sigma^\gamma f(x\sigma^{1/\beta}) d\sigma$$

$$= \beta \frac{x^{-\beta(\gamma+\delta)}}{\Gamma(\delta)} \int_0^x (x^\beta - t^\beta)^{\delta-1} t^{\beta(\gamma+1)-1} f(t) dt.$$

In [5, 6] these operators are defined on the space of functions

$$\mathscr{C}_\alpha = \{f(x) = x^p \tilde{f}(x);\quad p > \alpha,\ \tilde{f} \in \mathscr{C}[0, \infty)\}$$

with $\alpha \geqq \max_k[-\beta(\gamma_k+1)]$. Their basic properties, an inversion formula and particular cases are given there. Let us note that $\mathscr{R}$ is an invertible map of $\mathscr{C}_\alpha$ into itself and for $p > \alpha$:

(13) $$\mathscr{R}\{x^p\} = c_p x^p, \quad c_p = \prod_{k=1}^m \frac{\Gamma(\gamma_k + 1 + p/\beta)}{\Gamma(\gamma_k + \delta_k + 1 + p/\beta)}.$$

In [5], [6] it is shown that (11) is a composition of m commuting Erdélyi-Kober operators (12) with one and the same $\beta > 0$, namely:

(14) $$\mathscr{R}_{\beta,m}^{\{\gamma_k\},\{\delta_k\}} f(x) = \left(\prod_{k=1}^m I_\beta^{\gamma_k,\delta_k}\right) f(x)$$

$$= \int_0^1 \cdots \int_0^1 \prod_{k=1}^m \left[\frac{(1-\sigma_k)^{\delta_k-1}}{\Gamma(\delta_k)} \sigma_k^{\gamma_k}\right] f[x(\sigma_1 \ldots \sigma_m)^{1/\beta}] d\sigma_1 \ldots d\sigma_m.$$

If α is a given real number, then the operator $\mathscr{R}$ is well defined in $\mathscr{C}_\alpha$ for $\gamma_k > -\alpha/\beta - 1$, $\delta_k > 0$, $k = 1, \ldots, m$.

Definition 2. *Let*

(15) $$r'_j = I_1^{-c_j, c_j - a_j} = \mathscr{R}_{1,1}^{-c_j, c_j - a_j}, \quad j = 1, \ldots, n,$$

and

(16) $$r_k^* = I_1^{-b_{m+k}, d_{m+k} - b_{m+k}} = \mathscr{R}_{1,1}^{-b_{m+k}, d_{m+k} - b_{m+k}}, \quad k = 1, \ldots, q,$$

be Erdélyi-Kober operators of the form (12) *with* $c_j > a_j$, $j = 1, \ldots, n$, *and* $b_{m+k} > d_{m+k}$, $k = 1, \ldots, q$. *We shall use the following notations:* $\mathscr{R}' = r'_1 \ldots r'_n$, $\mathscr{R}^* = r_1^* \ldots r_q^*$ *and*

(17) $$\mathscr{R} = \mathscr{R}^* \mathscr{R}'.$$

Theorem 1. *The operator* (17) *is a generalized Riemann-Liouville fractional integral of the form* (11), *namely:*

(18) $$\mathscr{R}=\mathscr{R}_{1,n+q}^{\{\gamma_k\},\{\delta_k\}}$$

with

$$\gamma_k=-c_k,\ \delta_k=c_k-a_k,\ k=1,\dots,n,$$
$$\gamma_{n+k}=-b_{m+k},\ \delta_{n+k}=b_{m+k}-d_{m+k},\ k=1,\dots,q.$$

For $\varphi(x)\in\mathscr{C}_\alpha$ and $\alpha+1>c_k>a_k$, $k=1,\dots,n$, $\alpha+1>b_{m+k}>d_{m+k}$, $k=1,\dots,q$, this operator transforms the integral equation (8):

$$\frac{1}{2\pi i}\int_L \chi_1(s)x^{-s}F(1-s)ds=\varphi(x),\quad 0<x<1,$$

into an integral equation of the same form but with the function $\chi(s)$ defined by (10) *as a kernel, namely*:

(19) $$\frac{1}{2\pi i}\int_L \chi(s)x^{-s}F(1-s)ds=\mathscr{R}\varphi(x),\quad 0<x<1.$$

Proof. As it is shown in [5]:

$$\mathscr{R}'=\prod_{j=1}^{n}\mathscr{R}_{1,1}^{-c_j,c_j-a_j}=\mathscr{R}_{1,n}^{\{-c_k\},\{c_k-a_k\}}.$$

By applying $\mathscr{R}'$ to (8) and using (13):

$$\mathscr{R}'\{x^{-s}\}=x^{-s}\prod_{k=1}^{n}\frac{\Gamma(1-c_k-s)}{\Gamma(1-a_k-s)},$$

we get

$$\mathscr{R}'\varphi(x)=\mathscr{R}'\left\{\frac{1}{2\pi i}\int_L \chi_1(s)x^{-s}F(1-s)ds\right\}$$

$$=\frac{1}{2\pi i}\int_L \chi_1(s)\mathscr{R}'\{x^{-s}\}F(1-s)ds=\frac{1}{2\pi i}\int_L \frac{\Gamma_{1,1}\Gamma_{2,1}}{\Gamma_{3,1}\Gamma_{4,1}}\frac{\Gamma_{2,2}}{\Gamma_{2,1}}x^{-s}F(1-s)ds.$$

Similarly

$$\mathscr{R}^*=\prod_{k=1}^{q}\mathscr{R}_{1,1}^{-b_{m+k},b_{m+k}-d_{m+k}}=\mathscr{R}_{1,q}^{\{-b_{m+k}\},\{b_{m+k}-d_{m+k}\}}$$

and

$$\mathscr{R}^*\{x^{-s}\}=x^{-s}\prod_{k=1}^{q}\frac{\Gamma(1-b_{m+k}-s)}{\Gamma(1-d_{m+k}-s)}.$$

Therefore

$$\mathcal{R}^*(\mathcal{R}'\varphi(x)) = \frac{1}{2\pi i}\int_L \frac{\Gamma_{1,1}\Gamma_{2,2}}{\Gamma_{3,1}\Gamma_{4,1}}\frac{\Gamma_{3,1}}{\Gamma_{3,2}} x^{-s}F(1-s)ds = \frac{1}{2\pi i}\int_L \chi(s)x^{-s}F(1-s)ds,$$

and hence

$$\frac{1}{2\pi i}\int_L \chi(s)x^{-s}F(1-s)ds = \mathcal{R}\varphi(x), \quad 0<x<1,$$

with $\mathcal{R}\varphi(x)\in\mathcal{C}_\alpha$ for $\varphi(x)\in\mathcal{C}_\alpha$. Here

$$\mathcal{R}=\mathcal{R}^*\mathcal{R}'=\mathcal{R}_{1,q}^{\{-b_{m+k}\},\{b_{m+k}-d_{m+k}\}}\mathcal{R}_{1,n}^{\{-c_k\},\{c_k-a_k\}}=\mathcal{R}_{1,n+q}^{\{\gamma_k\},\{\delta_k\}}.$$

with parameters γ_k and δ_k defined as in (18).

Definition 3. *Let $m\geqq 1$ be an integer, $\beta>0$, $\{\tau_k\}_1^m$ and $\{\sigma_k\}_1^m$ be real numbers. For $\sigma_k>0$, $k=1,\dots, m$, we define the generalized Weyl's operator of fractional integration by the integral*

$$\mathcal{W}f(x)=\mathcal{W}_{\beta,m}^{\{\tau_k\},\{\sigma_k\}}f(x)=\int_1^\infty G_{m,m}^{m,0}\left[\frac{1}{y}\middle|\begin{matrix}(\tau_k+\sigma_k+1)\\(\tau_k+1)\end{matrix}\right]f(xy^{1/\beta})dy \tag{20}$$

$$=x^{-\beta}\int_x^\infty G_{m,m}^{m,0}\left[\left(\frac{x}{t}\right)^\beta\middle|\begin{matrix}(\tau_k+\sigma_k+1)\\(\tau_k+1)\end{matrix}\right]f(t)d(t^\beta)$$

$$=\int_1^\infty\cdots\int_1^\infty\prod_{k=1}^m\left[\frac{(y_k-1)^{\sigma_k-1}}{\Gamma(\sigma_k)}y_k^{-(\tau_k+\sigma_k)}\right]f[x(y_1\dots y_m)^{1/\beta}]dy_1\dots dy_m$$

on the functional space

$$\mathcal{C}_{\alpha*}^*=\{f(x)=x^q\tilde{\tilde{f}}(x);\ q<\alpha^*,\ \tilde{\tilde{f}}\in\mathcal{C}(0,\ \infty),\ |\tilde{\tilde{f}}|\leqq A_{\tilde{\tilde{f}}}\}$$

with $\alpha^*=\min_k(\beta\tau_k)$.

For $m=1$ this is the generalized Erdélyi-Kober operator of Weyl type, namely:

$$K_\beta^{\tau,\sigma}f(x)=\mathcal{W}_{\beta,1}^{\tau,\sigma}f(x)=\int_1^\infty\frac{(y-1)^{\sigma-1}}{\Gamma(\sigma)}y^{-(\tau+\sigma)}f(xy^{1/\beta})dy \tag{21}$$

$$=\frac{x^{\beta\tau}}{\Gamma(\sigma)}\int_x^\infty(t^\beta-x^\beta)^{\sigma-1}t^{-\beta\tau-\beta\sigma}f(t)d(t^\beta).$$

Moreover

$$\mathcal{W}_{\beta,m}^{\{\tau_k\},\{\sigma_k\}}=\prod_{k=1}^m\mathcal{W}_{\beta,1}^{\tau_k,\sigma_k}=\prod_{k=1}^m K_\beta^{\tau_k,\sigma_k}. \tag{22}$$

If α^* is a given real number, then the operator $\mathscr{W}$ is well defined in $\mathscr{C}^*_{\alpha*}$ for $\tau_k>\alpha^*/\beta$, $\sigma_k>0$, $k=1,\dots,m$, and for $q<\alpha^*$:

$$\mathscr{W}\{x^q\}=x^q\prod_{k=1}^{m}\frac{\Gamma(\tau_k-q/\beta)}{\Gamma(\tau_k+\sigma_k-q/\beta)}. \tag{23}$$

Definition 4. *According to* (21) *we define the following Erdélyi-Kober operators of Weyl type*:

$$\omega'_j=K_1^{b_j,d_j-b_j}=\mathscr{W}_{1,1}^{b_j,d_j-b_j},\quad j=1,\dots,m, \tag{24}$$

and

$$\omega^*_k=K_1^{c_{n+k},a_{n+k}-c_{n+k}}=\mathscr{W}_{1,1}^{c_{n+k},a_{n+k}-c_{n+k}},\quad k=1,\dots,p, \tag{25}$$

with $d_j>b_j$, $j=1,\dots,m$, *and* $a_{n+k}>c_{n+k}$, $k=1,\dots,p$. *For* $\mathscr{W}'=\omega'_1\dots\omega'_m$ *and* $\mathscr{W}^*=\omega^*_1\dots\omega^*_p$ *let*

$$\mathscr{W}=\mathscr{W}^*\mathscr{W}' \tag{26}$$

be their composition.

Theorem 2. *The operator* (26) *is a generalized Weyl operator of fractional integration of the form* (20):

$$\mathscr{W}=\mathscr{W}_{1,m+p}^{\{\tau_k\},\{\sigma_k\}} \tag{27}$$

with

$$\tau_k=b_k,\ \sigma_k=d_k-b_k,\ k=1,\dots,m,$$
$$\tau_{m+k}=c_{n+k},\ \sigma_{m+k}=a_{n+k}-c_{n+k},\ k=1,\dots,p.$$

For $\psi(x)\in\mathscr{C}^*_{\alpha*}$ *and* $d_k>b_k>a^*$, $k=1,\dots,m$, $a_{n+k}>c_{n+k}>\alpha^*$, $k=1,\dots,p$, *this operator transforms the integral equation* (9):

$$\frac{1}{2\pi i}\int_L \chi_2(s)x^{-s}F(1-s)ds=\psi(x),\quad x>1,$$

into an integral equation of the same form but with $\chi(s)$ *as the kernel function*:

$$\frac{1}{2\pi i}\int_L \chi(s)x^{-s}F(1-s)ds=\mathscr{W}\psi(x),\ x>1. \tag{28}$$

Proof. By using Mellin transforms

$$\mathscr{M}[K_\beta^{\tau_k,\sigma_k}f]=\frac{\Gamma(\tau_k+s/\beta)}{\Gamma(\tau_k+\sigma_k+s/\beta)}\mathscr{M}[f]$$

and

$$\mathcal{M}[\mathscr{W}_{\beta,m}^{\{\tau_k\},\{\sigma_k\}}f]=\prod_{k=1}^{m}\frac{\Gamma(\tau_k+s/\beta)}{\Gamma(\tau_k+\sigma_k+s/\beta)}\mathcal{M}[f],$$

it is easy to establish (22), or now:

$$\mathscr{W}'=\prod_{j=1}^{m}\mathscr{W}_{1,1}^{b_j,d_j-b_j}=\mathscr{W}_{1,m}^{\{b_k\},\{d_k-b_k\}},$$

$$\mathscr{W}^*=\prod_{k=1}^{p}\mathscr{W}_{1,1}^{c_{n+k},a_{n+k}-c_{n+k}}=\mathscr{W}_{1,p}^{\{c_{n+k}\},\{a_{n+k}-c_{n+k}\}}.$$

Therefore

$$\mathscr{W}'\{x^{-s}\}=x^{-s}\prod_{k=1}^{m}\frac{\Gamma(b_j+s)}{\Gamma(d_j+s)},\quad \mathscr{W}^*\{x^{-s}\}=x^{-s}\prod_{k=1}^{p}\frac{\Gamma(c_{n+k}+s)}{\Gamma(a_{n+k}+s)}.$$

Let us apply $\mathscr{W}'$ to (9):

$$\mathscr{W}'\psi(x)=\mathscr{W}'\left\{\frac{1}{2\pi i}\int_L\chi_2(s)x^{-s}F(1-s)ds\right\}$$

$$=\frac{1}{2\pi i}\int_L\chi_2(s)\mathscr{W}'\{x^{-s}\}F(1-s)ds=\frac{1}{2\pi i}\int_L\frac{\Gamma_{1,2}\Gamma_{2,2}}{\Gamma_{3,2}\Gamma_{4,2}}\frac{\Gamma_{1,1}}{\Gamma_{1,2}}x^{-s}F(1-s)ds.$$

Similarly

$$\mathscr{W}\psi(x)=\mathscr{W}^*(\omega'\psi(x))=\frac{1}{2\pi i}\int_L\frac{\Gamma_{1,1}\Gamma_{2,2}}{\Gamma_{3,2}\Gamma_{4,2}}\frac{\Gamma_{4,2}}{\Gamma_{4,1}}x^{-s}F(1-s)ds$$

$$=\frac{1}{2\pi i}\int_L\frac{\Gamma_{1,1}\Gamma_{2,2}}{\Gamma_{3,2}\Gamma_{4,1}}x^{-s}F(1-s)ds=\frac{1}{2\pi i}\int_L\chi(s)x^{-s}F(1-s)ds,$$

i.e. (28). Here $\mathscr{W}\psi(x)\in\mathscr{C}^*_{\alpha*}$ for $\psi(x)\in\mathscr{C}^*_{\alpha*}$. The theorem is proved.

Now, let us denote by

(29) $$g(x)=\begin{cases}\mathscr{R}\varphi(x), & 0<x<1,\\ \mathscr{W}\psi(x), & x>1,\end{cases}$$

the new right hand side $g(x)\in\mathscr{C}_{\alpha,\alpha*}$, where

$$\mathscr{C}_{\alpha,\alpha*}=\{g(x)\in\mathscr{C}[0,\ 1]\cap\mathscr{C}[1,\ \infty);\ |g|\leqq Ax^{\alpha},\ 0<x<1;\ |g|\leqq Ax^{\alpha^*},\ x>1\}$$

with $\alpha^*\leqq\alpha$. According to Marichev ([7, p. 32, Th. 11]) $\mathscr{C}_{\alpha,\alpha^*}$ is an appropriate space of functions $g(x)$ with Mellin transforms in the strip $\operatorname{Re}s=\gamma,\ -\alpha\leqq\gamma\leqq-\alpha^*$.

Using the notation (29) we can write the pair of equations (19), (28) in the following concise form:

$$(30)\qquad \frac{1}{2\pi i}\int_L \chi(s)x^{-s}F(1-s)ds=g(x),\quad 0<x<\infty.$$

By Mellin transform's technique ([3, p. 242, Lemma 7.2.2]) the equation (30) can be rewritten in the form

$$(31)\qquad f(x)=\frac{1}{2\pi i}\int_L \frac{x^{-s}G(1-s)}{\chi(1-s)}ds,$$

where $f(x)$ is the original unknown function and $G(s)=\mathcal{M}[g(x)]$. Denote by

$$(32)\qquad \mathcal{H}(s)=\frac{1}{\chi(1-s)}=\frac{\prod_{j=1}^{q}\Gamma(-d_{m+j}+s)\prod_{j=1}^{p}\Gamma(1-(-a_{n+j})-s)}{\prod_{j=1}^{m}\Gamma(1-(-b_j)-s)\prod_{j=1}^{n}\Gamma(-c_j+s)}.$$

Let $h(x)=\mathcal{M}^{-1}[\mathcal{H}(s)]$ be the original of $\mathcal{H}(s)$. It surely exists for $\delta=(m+n)-(p+q)=0$ under the conditions on the parameters a_i, c_i $(i=1,\dots,n+p)$, b_j, d_j $(j=1,\dots,m+q)$ accepted here and

$$(33)\qquad h(x)=G^{q,p}_{n+p,m+q}\left[x\left|\begin{matrix}(-a_{n+j})_{j=1}^{p}\ ;\ (-c_j)_{j=1}^{n}\\(-d_{m+j})_{j=1}^{q}\ ;\ (-b_j)_{j=1}^{m}\end{matrix}\right.\right].$$

According to Parseval theorem ([3, p. 242, Lemma 7.2.1.]) it follows that the solution $f(x)$ of the original pair of dual integral equations (1), (2) has the form

$$(34)\qquad f(z)=\int_0^\infty h(xu)g(u)du=\int_0^\infty G^{q,p}_{n+p,m+q}\left[xu\left|\begin{matrix}(-a_{n+j}),\ (-c_j)\\(-d_{m+j}),\ (-b_j)\end{matrix}\right.\right]g(u)du$$

with $g(x)$ defined by (29). Thus we obtain that

$$(35)\qquad f(x)=\int_0^1 h(xu)\,\mathrm{R}\varphi(u)du+\int_1^\infty h(xu)\,\mathcal{W}\psi(u)du.$$

We have to bear in mind that the solution (35) is obtained by a formal technique. Therefore we have to verify this solution by reversing all the steps in our procedure by means of the operators $\mathcal{R}^{-1}$ and $\mathcal{W}^{-1}$. In this connection the following inversion formula for generalized Riemann-Liouville fractional integrals (see [5, 6]) is useful:

$$(36)\qquad \mathcal{R}^{-1}=\{\mathcal{R}^{\{\gamma_k\},\{\delta_k\}}_{\beta,m}\}^{-1}=\mathcal{R}^{\{\gamma_k+\delta_k\},\{-\delta_k\}}_{\beta,m}.$$

Here the symbol $\mathscr{R}_{\beta,m}^{\{\gamma_k'\},\{\delta_k'\}}$ with $\delta_k'\leqq 0$ is to be interpreted as the integro-differential operator

$$\mathscr{R}_{\beta,m}^{\{\gamma_k'\},\{\delta_k'\}}=\mathscr{D}_\eta\,\mathscr{R}_{\beta,m}^{\{\gamma_k'\},\{\delta_k'+\eta_k\}} \tag{37}$$

with

$$\eta_k=\begin{cases}[-\delta_k']+1 & \text{for noninteger } \delta_k'<0,\ k=1,\dots,m,\\ -\delta_k' & \text{for integer } \delta_k'\leqq 0,\ k=1,\dots,m,\end{cases}$$

and

$$\mathscr{D}_\eta=\prod_{k=1}^{m}\prod_{j=1}^{\eta_k}\left(\frac{1}{\beta}x\frac{d}{dx}+\gamma_k'+\delta_k'+j\right). \tag{38}$$

A similar inverson formula for Weyl fractional integrals (20) holds. In this way we obtain the following

Theorem 3. *The function*

$$f(x)=\int_0^1 G_{n+p,m+q}^{q,p}\left[xu\left|\begin{matrix}(-a_{n+j})_1^p,\ (-c_j)_1^n\\(-d_{m+j})_1^q,\ (-b_j)_1^m\end{matrix}\right.\right]du \tag{39}$$

$$\int_0^1 G_{n+q,n+q}^{n+q,0}\left[v\left|\begin{matrix}(-a_k)_1^n,\ (-d_{m+k})_1^q\\(-c_k)_1^n,\ (-b_{m+k})_1^q\end{matrix}\right.\right]\varphi(uv)dv$$

$$+\int_1^\infty G_{n+p,m+q}^{q,p}\left[xu\left|\begin{matrix}(-a_{n+j})_1^p,(-c_j)_1^n\\(-d_{m+j})_1^q,(-b_j)_1^m\end{matrix}\right.\right]du\int_1^\infty G_{m+p,m+p}^{m+p,0}\left[\frac{1}{v}\left|\begin{matrix}(d_k+1)_1^m,(a_{n+k}+1)_1^p\\(b_k+1)_1^m,\ (c_{n+k}+1)_1^p\end{matrix}\right.\right]\psi(uv)dv$$

satisfies the dual integral equations (1), (2):

$$\int_0^\infty G_{p+n,q+m}^{m,n}\left[xu\left|\begin{matrix}(a_j)_1^{n+p}\\(b_j)_1^{m+q}\end{matrix}\right.\right]f(u)du=\varphi(x),\quad 0<x<1,$$

$$\int_0^\infty G_{p+n,q+m}^{m,n}\left[xu\left|\begin{matrix}(c_j)_1^{n+p}\\(d_j)_1^{m+q}\end{matrix}\right.\right]f(u)du=\psi(x),\quad x>1.$$

Remark 1. The result obtained in [3, p. 247, (7.2.35)] is the same, but it contains compositions of fractional integrals

$$\mathscr{R}\varphi(u)=r_1^*r_2^*\dots r_q^*(r_1'\dots r_n'\varphi(u)),\quad \mathscr{W}\psi(u)=\omega_1^*\dots\omega_p^*(\omega_1'\dots\omega_m'\psi(u))$$

written only symbolically instead of their explicit representation

$$\mathscr{R}\varphi(u)=\int_0^1 G_{n+q,n+q}^{n+q,0}\left[v\left|\begin{matrix}(-a_k)_1^n,\ (-d_{m+k})_1^q\\(-c_k)_1^n,\ (-b_{m+k})_1^q\end{matrix}\right.\right]\varphi(uv)\,dv$$

$$=\int_0^1\cdots\int_0^1 \prod_{k=1}^{n+q}\left[\frac{(1-t_k)^{\delta_k-1}}{\Gamma(\delta_k)}\, t_k^{\gamma_k}\right]\varphi[u(t_1\dots t_{n+q})]dt_1\dots dt_{n+q}$$

and the similar one about $\mathscr{W}\psi(u)$ given by our Theorem 3. Moreover the conditions

$$\alpha+1>c_k>a_k,\quad \alpha+1>b_{m+k}>d_{m+k},\quad k=1,\dots,q.$$
$$\text{(40)}\qquad a_{n+k}>c_{n+k}>\alpha^*,\quad d_k>b_k>\alpha^*,\quad k=1,\dots,p,$$

assure the correct application of the fractional integration operators $\mathscr{R}$, $\mathscr{W}$ to the right hand sides $\varphi(x)\in\mathscr{C}_\alpha$, $\psi(x)\in\mathscr{C}^*_{\alpha*}$, $\alpha^*\leqq\alpha$. These conditions on the parameters of G-functions are not given in [3].

Remark 2. The solutions of dual integral equations involving many special functions used in applied mathematics can be derived as special cases from (39).

Now we illustrate above results on the example of dual integral equations whose kernels are Bessel functions of arbitrary order.

3. Example. Dual integral equations of Titchmarsh with Bessel functions as kernels. Let $J_\nu(x)$ be the Bessel function

$$\text{(41)}\qquad J_\nu(x)=\frac{(x/2)^\nu}{\Gamma(\nu+1)}\,{}_0F_1\left(\nu+1;\ -\frac{x^2}{4}\right)=\sum_{j=0}^{\infty}\frac{(-1)^j(x/2)^{2j+\nu}}{j!\,\Gamma(\nu+j+1)}.$$

Let us consider the dual integral equations

$$\text{(42)}\qquad \int_0^\infty u^{-\gamma}J_\mu(2\sqrt{xu})f(u)du=g_1(x),\quad 0<x<1,$$

$$\text{(43)}\qquad \int_0^\infty u^{-\delta}J_\nu(2\sqrt{xu})f(u)du=g_2(x),\quad x>1,$$

with $f(u)$ as unknown function and with given parameters γ, δ, μ, ν and right hand sides $g_1(x)$, $g_2(x)$.

Since

$$\text{(44)}\qquad J_\nu(2\sqrt{x})=x^{\nu/2}G^{1,0}_{0,2}[x|0,\ -\nu]$$

(see [8, p. 211, (3)]) the kernels of equations (42), (43) are the following G-functions

$$u^{-\gamma}J_\mu(2\sqrt{xu})=u^{-\gamma}(xu)^{\mu/2}\,G^{1,0}_{0,2}[xu|0,\ -\mu]=x^\gamma G^{1,0}_{0,2}[xu|\mu/2-\gamma,\ -\mu/2-\gamma],$$

$$u^{-\delta}J_\nu(2\sqrt{xu})=x^\delta G^{1,0}_{0,2}[xu|\nu/2-\delta,\ -\nu/2-\delta]$$

satisfying the condition $(m+n)-(p+q)=(1+0)-(0+1)=0$.

Denote by $b_1=\mu/2-\gamma$, $b_2=-\mu/2-\gamma$, $d_1=\nu/2-\delta$, $d_2=-\nu/2-\delta$. Now the above equations get the form

$$\int_0^\infty G_{0,2}^{1,0}[xu|b_1,\ b_2]f(u)du=x^{-\gamma}g_1(x),\quad 0<x<1, \tag{45}$$

$$\int_0^\infty G_{0,2}^{1,0}[xu|d_1,\ d_2]f(u)du=x^{-\delta}g_2(x),\quad x>1, \tag{46}$$

and their solution $f(x)$ can be obtained as a particular case from the solution (39) of general dual integral equations (1), (2). By Theorem 3 we get

$$f(x)=\int_0^1 G_{0,2}^{1,0}[xu|-d_2,\ -b_1]du\int_0^1 G_{1,1}^{1,0}\left[v\left|\begin{matrix}-d_2\\-b_2\end{matrix}\right.\right]\varphi(uv)dv$$
$$+\int_1^\infty G_{0,2}^{1,0}[xu|-d_2,\ -b_1]du\int_1^\infty G_{1,1}^{1,0}\left[\frac{1}{v}\left|\begin{matrix}d_1+1\\b_1+1\end{matrix}\right.\right]\psi(uv)dv,$$

where $\varphi(x)=x^{-\gamma}g_1(x)$ and $\psi(x)=x^{-\delta}g_2(x)$, i.e.

$$f(x)=x^{\gamma}\int_0^1 G_{0,2}^{1,0}[xu|-d_2-\gamma,\ -b_1-\gamma]du\int_0^1 G_{1,1}^{1,0}\left[v\left|\begin{matrix}-d_2-\gamma\\-b_2-\gamma\end{matrix}\right.\right]g_1(uv)dv \tag{47}$$
$$+x^{\delta}\int_1^\infty G_{0,2}^{1,0}[xu|-d_2-\delta,\ -b_1-\delta]du\int_1^\infty G_{1,1}^{1,0}\left[\frac{1}{v}\left|\begin{matrix}d_1+1+\delta\\b_1+1+\delta\end{matrix}\right.\right]g_2(uv)dv$$

with the abbreviated notations of the parameters given above. If we bear in mind (44) and

$$G_{1,1}^{1,0}\left[x\left|\begin{matrix}\alpha+\beta\\\alpha\end{matrix}\right.\right]=\frac{x^{\alpha}(1-x)^{\beta-1}}{\Gamma(\beta)},\quad 0<x<1,$$

(see [3, p. 54]), then the solution of (42), (43) gets the form

$$f(x)=x^{(\mu+3\nu)/4+(3\delta-\gamma)/2}\left\{\int_0^1 u^{(\mu+3\nu)/2+3(\delta-\gamma)/2}J_\lambda(2\sqrt{xu})du\int_1^\infty\frac{v^{\mu/2}(1-v)^{\lambda-\mu-1}}{\Gamma(\lambda-\mu)}g_1(uv)dv\right. \tag{48}$$
$$\left.+\int_1^\infty u^{(\mu+3\nu)/2+(\delta-\gamma)/2}J_\lambda(2\sqrt{xu})du\int_1^\infty\frac{v^{-\nu/2}(v-1)^{\nu-\lambda-1}}{\Gamma(\nu-\lambda)}g_2(uv)dv\right\},$$

where

$$\lambda=\frac{\mu+\nu}{2}+(\delta-\gamma).$$

If the right hand sides are $g_1(x)\in\mathscr{C}_\alpha$, $g_2(x)\in\mathscr{C}^*_{\alpha*}$, then the corresponding conditions (40) on the parameters are equivalent to the conditions

$$\nu>\lambda>\mu,\quad \mu/2+\alpha+1>0,\quad \mu/2+(\delta-\gamma)-\alpha^*>0.$$

After a change of the variable we obtain the solution of the classical dual integral equations of Titchmarsh ([1, p. 84]), namely:

$$\int_0^\infty u^{-2\gamma}J_\mu(xu)f(u)du=G_1(x),\quad 0<x<1, \tag{49}$$

$$\int_0^\infty u^{-2\delta}J_\nu(xu)f(u)du=G_2(x),\quad x>1. \tag{50}$$

For $\gamma=-\omega/2$, $\delta=0$ and $\lambda=(\mu+\nu+\omega)/2$ this solution has the form

$$f(x)=2^{\lambda-\nu}x^{1-\lambda+\nu}\left\{\int_0^1 t^{1-\lambda-\nu}J_\lambda(xt)h_1(t)dt+\int_1^\infty t^{1-\lambda+\nu}J_\lambda(xt)h_2(t)dt\right\}, \tag{51}$$

given by Peters (1961) (see [1, p. 85-86]), where

$$h_1(t)=\frac{2^{1-\omega}t^{\nu+2\omega}}{\Gamma(\lambda-\mu)}\int_0^t(t^2-\tau^2)^{\lambda-\mu-1}\tau^{\mu+1}G_1(\tau)d\tau$$

and

$$h_2(t)=\frac{2t^{\mu+\omega}}{\Gamma(\nu-\lambda)}\int_t^\infty(\tau^2-t^2)^{\nu-\lambda-1}\tau^{1-\nu}G_2(\tau)d\tau.$$

Let us assume that $G_1(x)\in\mathscr{C}_{-1}$, $G_2(x)\in\mathscr{C}^*_{-1}$, i.e. $\alpha=\alpha^*=-1$ and therefore

$$G(x)=\begin{cases}G_1(x), & 0<x<1,\\ G_2(x), & x>1,\end{cases}$$

belongs to the space $\mathscr{C}_{-1,-1}\subset L_1$. This case is the most natural. The new right hand sides to which the fractional integrals $\mathscr{R}$ and $\mathscr{W}$ are to be applied are

$$\varphi(x)=x^{\omega/2}2^{-\omega}G_1(\sqrt{x})\in\mathscr{C}_{-1/2+\omega/2}$$

and

$$\psi(x)=G_2(\sqrt{x})\in\mathscr{C}^*_{-1/2}.$$

The corresponding conditions (40) get the form

$$\lambda>\mu>-1,\quad \mu+\omega>-1,\quad \nu>\lambda$$

and coincide with the conditions for the parameters in Peters' solution, considered in [1].

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Institute of Mathematics with Computer Centre
1090 Sofia *Bulgaria*

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