PLISKA

STUDIA MATHEMATICA BULGARICA

ПЛИСКА

БЪЛГАРСКИ МАТЕМАТИЧЕСКИ СТУДИИ

ON THE RADUIS OF ALPHA-STARLIKENESS FOR STARLIKE FUNCTIONS OF ORDER BETA AND FIXED SECOND COEFFICIENT

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Let $K(\alpha)$, $0<\alpha\leqq 1$, denote the class of functions $f(z)=z+a_2z^2+\dots$ which are regular in the unit disc E and for which

$$\operatorname{Re}\left\{p(z)+\frac{\alpha zp'(z)}{1-\alpha+\alpha p(z)}\right\}>0,$$

where $p(z)=zf'(z)/f(z)$ and $f(z)f'(z)\neq 0$ for $z\in E-\{0\}$. The aim of this note is to determine the largest disc centered in the origin for which all starlike functions of order β, $0\leqq\beta<1$, and second coefficient $2b(1-\beta)$ are in class $K(\alpha)$.

Let $f(z)=z+a_2z^2+\dots$ be a regular function in the unit disc and let for $\alpha\in C$:

$$K(\alpha,\ f(z))=p(z)+\frac{\alpha zp'(z)}{1-\alpha+\alpha p(z)},$$

where $p(z)=zf'(z)/f(z)$.

We denote by $K(\alpha)$ the class of functions for which $\operatorname{Re}K(\alpha,\ f(z))>0$ for $z\in E$ and $f(z)f'(z)\neq 0$ for $z\in E-\{0\}$. The elements of the class $K(\alpha)$ are called alpha-starlike functions. It is known [1] that for $\alpha\in\mathscr{D}=\{\alpha:\operatorname{Re}\alpha\geqq|\alpha|^2\}$ there are univalent starlike functions in E. Let $S^*(\beta)$, $0\leqq\beta<1$, be the class of regular and starlike functions $f(z)=z+a_2z^2+\dots$ of order β in E (for such functions $\operatorname{Re}zf'(z)/f(z)>\beta$ in E).

The largest disc centered in the origin in which any starlike function of order β, $0\leqq\beta<1$, is α-starlike, was determined by. Pascu and Podaru [2]. The radius of this disc is called α-starlikeness radius for the starlike functions of order β.

Let $S_b^*(\beta)$ be the class of functions $f(z)=z+2b(1-\beta)z^2+\dots$, $0\leqq b\leqq 1$, which are starlike of order β. For the functions of the class $S_b^*(\beta)$ we shall study the behaviour of the radius of α-starlikeness with respect to the second coefficient of the series expansion.

The radius of α-starlikeness for the class $S_b^*(\beta)$ is defined by

$$R_b(\alpha,\ \beta)=\sup\{r:\operatorname{Re}K(\alpha,\ f(z))>0,\ |z|<r,\ \ f(z)\in S_b^*(\beta)\}.$$

Pascu and Podaru determined the radius of α-starlikeness for the class $S^*(\beta)$ making use of a result due to Robertson which relies on variational techniques, while in this note we shall use a lemma of Dieudonné.

PLISKA Studia mathematica bulgarica, Vol. 10, 1989, p. 157–163.

We introduce the following functional classes:

Let Ω be the class of functions $w(z)$ regular in $E=\{z:|z|<1\}$ and satisfying the conditions $w(0)=0$, $|w(z)|<1$ in E.

For fixed A, B satisfying the conditions $-1\leqq B<A\leqq 1$ let $\mathcal{P}(A, B)$ be the class of functions $p(z)=1+p_1z+\dots$ defined by

$$p(z)=\frac{1+Aw(z)}{1+Bw(z)}$$

for some $w(z)\in\Omega$ and $z\in E$. This class was introduced by Janowski [3].

Let $p(z)=1+p_1z+\dots\in\mathcal{P}(A, B)$ and put $\theta=\arg p_1$. Then $p(e^{-i\theta}z)=1+|p_1|z+\dots$ and for the studying the class $\mathcal{P}(A, B)$ there is no loss of generality if one takes the first coefficient p_1 to be nonnegative. Also it is known that $|p_1|\leqq A-B$ [4].

We denote by $\mathcal{P}_b(A, B)$ the following subclass of $\mathcal{P}(A, B)$:

$$\mathcal{P}_b(A, B)=\{p(z)\in\mathcal{P}(A, B):p'(0)=b(A-B),\ 0\leqq b\leqq 1\}.$$

Let $S_b^*(A, B)$ be a class of univalent functions associated with $\mathcal{P}_b(A, B)$ as follows:

$$S_b^*(A, B)=\left\{f(z)=z+b(A-B)z^2+\dots;\ \frac{zf'(z)}{f(z)}\in\mathcal{P}_b(A, B),\ z\in E\right\}.$$

By special choices of A, B this class can be reduced to the class $S_b^*(\beta)$. Precisely, $S_b^*(1-2\beta, -1)\equiv S_b^*(\beta)$. We have also

$$S^*(\beta)=S_1^*(1-2\beta, -1).$$

First we shall determine the expression

$$\min_{|z|=r<1}\operatorname{Re}\left\{\lambda p(z)+\frac{zp'(z)}{p(z)+\mu}\right\},\quad \lambda\geqq 0,\quad \mu\geqq 0,$$

over $\mathcal{P}_b(A, B)$. We need the following lemmas:

Lemma 1 [5]. *If $w(z)\in\Omega$ then for $z\in E$*

$$|zw'(z)-w(z)|\leqq\frac{|z|^2-|w(z)|^2}{1-|z|^2}.$$

Lemma 2 [6]. *If $p(z)\in\mathcal{P}_b(A, B)$ then for each $b\in[0, 1]$ we have that $p(z)$ maps the disc $|z|\leqq r$ onto the disc*

$$\Delta(\zeta)\equiv\{\zeta:|\zeta-a_b|\leqq d_b\},$$

where

$$a_b=\frac{(1+br)^2-ABr^2(r+b)^2}{(1+br)^2-B^2r^2(r+b)^2},\quad d_b=\frac{(B-A)r(r+b)(1+br)}{(1+br)^2-B^2r^2(r+b)^2}$$

and $|z|=r<1$.

From Lemma 2 it follows immediately that if $p(z)\in\mathscr{P}_b(A, B)$, then

$$\frac{1+br-Ar(r+b)}{1+br-Br(r+b)}\leqq \operatorname{Re} p(z)\leqq|p(z)|\leqq\frac{1+br+Ar(r+b)}{1+br+Br(r+b)}$$

on $|z|=r<1$.

The first inequality is sharp for the function

$$p(z)=\frac{1+b(A-1)z-Az^2}{1+b(B-1)z-Bz^2} \quad \text{at} \quad z=-r.$$

The third inequality is sharp for the function

$$p(z)=\frac{1+b(1+A)z+Az^2}{1+b(1+B)z+Bz^2} \quad \text{at} \quad z=-r.$$

Also for a fixed r in (0, 1) we have:

(1) $$a_b-d_b\geqq a_1-d_1,\quad a_b+d_b\geqq a_0+d_0.$$

Theorem 1. *If* $p(z)\in\mathscr{P}_b(A, B)$, $\lambda\geqq 0$, $\mu\geqq 0$, *then*:

$$\operatorname{Re}\left\{\lambda p(z)+\frac{zp'(z)}{p(z)+\mu}\right\}\geqq\begin{cases} N_1+\dfrac{1}{(A-B)(1-r^2)}\dfrac{D_1}{D_4} & \text{for } R_1\leqq R_2,\\ N_1+\dfrac{2}{(A-B)(1-r^2)}D_5 & \text{for } R_2\leqq R_1,\end{cases}$$

on $|z|=r<1$, *where*

$$N_1=\frac{(1-\lambda\mu)A+(\lambda\mu+2\mu+1)B}{A-B},\quad N_2=\mu+1-\mu B-A,$$

$$D_1=N_2[\mu+1+(A+\mu B)r^2]D_2^2+[\lambda(A-B)(1-r^2)+(1-B)(1+Br^2)]D_3^2$$
$$-2[\mu+1-(A+\mu B)Br^2]D_2D_3,$$

$$D_2=-Br^2+b(1-B)r+1,$$

$$D_3=-(A+\mu B)r^2+N_2br+\mu+1,$$

$$D_4=D_2D_3,$$

$$D_5=\sqrt{N_2[\mu+1+(A+\mu B)r^2][\lambda(A-B)(1-r^2)+(1-B)(1+Br^2)]}$$

$$-[\mu+1-(A+\mu B)Br^2],$$

$$R_1=\sqrt{\frac{N_2[\mu+1+(A+\mu B)r^2]}{\lambda(A-B)(1-r^2)+(1-B)(1+Br^2)}},\quad R_2=\frac{D_3}{D_2}.$$

The result is sharp.

Proof. For $p(z)\in\mathscr{P}_b(A, B)$ we may write:

$$p(z)=\frac{1+Aw(z)}{1+Bw(z)},\quad z\in E,$$

for some $w(z)\in\Omega$.

From this formula we have:

$$\lambda p(z)+\frac{zp'(z)}{p(z)+\mu}=\lambda\frac{1+Aw(z)}{1+Bw(z)}+\frac{(A-B)zw'(z)}{[1+Bw(z)][1+\mu+(A+\mu B)w(z)]}$$

$$=\lambda\frac{1+Aw(z)}{1+Bw(z)}+(A-B)\frac{w(z)}{[1+Bw(z)][1+\mu+(A+\mu B)w(z)]}$$

$$+(A-B)\frac{zw'(z)-w(z)}{[1+Bw(z)][1+\mu+(A+\mu B)w(z)]}.$$

Applying Lemma 1 to the last term of the right-hand side we find

$$\text{(2)}\qquad \operatorname{Re}\left\{\lambda p(z)+\frac{zp'(z)}{p(z)+\mu}\right\}$$

$$\geqq N_1+\frac{1}{A+B}\operatorname{Re}\left\{(\lambda A-\lambda B-B)[p(z)+\mu]-\frac{\mu^2B+\mu(A+B)+A}{p(z)+\mu}\right\}$$

$$-\frac{1}{(A-B)(1-r^2)}\frac{r^2|Bp(z)-A|^2-|1-p(z)|^2}{|p(z)+\mu|}.$$

If we put $p(z)=a_b+u+iv$, $|p(z)|=R=\sqrt{(a_b+u)^2+v^2}$, then

$$r^2|Bp(z)-A|^2-|1-p(z)|^2=-(1-B^2r^2)R^2+2(1-ABr^2)(a_b+u)-(1-A^2r^2)$$

$$=-(1-B^2r^2)R^2+2a_1(1-B^2r^2)(a_b+u)-(1-B^2r^2)(a_1^2-d_1^2).$$

Denoting the right-hand side of (2) by $F(u, v)$ we get:

$$F(u,\ v)=N_1+\frac{1}{A-B}\Bigg\{(\lambda A-\lambda B-B)(a_b+u+\mu)$$

$$-\frac{\mu^2 B+(A+B)\mu+A}{R^2+2\mu(a_b+u)+\mu^2}(a_b+u+\mu)+\frac{1-B^2r^2}{1-r^2}\frac{R^2-2a_1(a_b+u)+a_1^2-d_1^2}{\sqrt{R^2+2\mu(a_b+u)+\mu^2}}\Bigg\}.$$

Now our problem is to determine the absolute minimum of $F(u,\ v)$ on the disc $u^2+v^2\leqq d_b^2$.

$$\frac{\partial F(u,\ v)}{\partial v}=\frac{vS(u,\ v)}{(A-B)[R^2+2\mu(a_b+u)+\mu^2]^2},\tag{3}$$

where

$$S(u,\ v)=2(\mu+1)(A+\mu B)(a_b+u+\mu)+\frac{1-B^2r^2}{1-r^2}[R^2+2(2\mu+a_1)(a_b+u)$$

$$+2\mu^2-a_1^2+d_1^2]\sqrt{R^2+2\mu(a_b+u)+\mu^2}.$$

In view of (1): $S(u,\ v)>0$. Using (3) we see that for every fixed u, $F(u,\ v)$ is an increasing function of v for positive v, it is a decreasing function of v for negative v and $\partial F(u,\ v)/\partial v=0$ for $v=0$. Thus, the minimum of $F(u,\ v)$ inside the disc $u^2+v^2\leqq d_b^2$ is attained on the diameter lying on the real axis. Setting $v=0$ we get:

$$F(u,\ 0)=N_1+\frac{1}{A-B}\Bigg\{(\lambda A-\lambda B-B)(a_b+u+\mu)-\frac{\mu^2 B+\mu(A+B)+A}{a_b+u+\mu}$$

$$+\frac{1-B^2r^2}{1-r^2}[a_b+u+\mu-2(\mu+a_1)+\frac{\mu^2+2a_1\mu+a_1^2-d_1^2}{a_b+u+\mu}\Bigg\},$$

$$\frac{dF(u,\ 0)}{du}=(\lambda A-\lambda B-B)(1-r^2)+1-B^2r^2$$

$$-\frac{\mu^2(1-B)+1-A+\mu(2-A-B)+r^2[\mu^2B(1-B)+\mu(A+B-2AB)+A(1-A)]}{(a_b+u+\mu)^2}.$$

The solution providing the minimum of $F(u,\ 0)$ is

$$u_0=\sqrt{\frac{\mu^2(1-B)+1-A+\mu(2-A-B)+r^2[\mu^2B(1-B)+\mu(A+B-2AB)+A(1-A)]}{(\lambda A-\lambda B-B)(1-r^2)+1-B^2r^2}}$$

$$-a_b-\mu.$$

It is clear that the absolute minimum of $F(u,\ v)$ is attained in the point u_0, if u_0 lies in $[-d_b,\ d_b]$ and its value is $N_1+2D_5/(A-B)(1-r^2)$. In view of (1) and from the conditions $-1 \leqq B < A \leqq 1$, $\lambda \geqq 0$, $\mu \geqq 0$, $0 < r < 1$, we see that $u_0 < d_b$. For the case $u_0 \leqq -d_b$, that is, if $R_1 \leqq R_2$, the absolute minimum of $F(u, v)$ is attained at the point $u = -d_b$, the value of which is

$$F(-d_b, 0) = N_1 + \frac{1}{(A-B)(1-r^2)} \frac{D_1}{D_4}.$$

The result is sharp for the function

$$p_1(z) = \frac{1+\mu+b[A-1+\mu B-\mu]z-(A+\mu B)z^2}{(1+\mu)[1+b(B-1)z-Bz^2]}.$$

An application of Theorem 1 gives:

Theorem 2. *The radius of α-starlikeness for the class $S_b^*(\beta)$ is:*

$$R_b(\alpha, \beta) = \begin{cases} r_1 & \text{if } R_1 \leqq R_2, \\ r_2 & \text{if } R_2 \leqq R_1, \end{cases}$$

where R_1 and R_2 are as in Theorem 1 *with $\lambda = 1$, $\mu = (1-\alpha)/\alpha$, $A = 1-2\beta$, $B = -1$.*

The radius r_1 is given by the smallest root in (0, 1] *of the equation*

$$(r^2+2br+1)[(1+2\alpha\beta-2\alpha)r^2+(2\alpha\beta-2\alpha+2)br+1][(2a-2\alpha\beta+\beta-3)$$
$$-r^2(4\alpha-4\alpha\beta+\beta-3)]+(\alpha\beta-\alpha+1)[1+(2\alpha-2\alpha\beta-1)r^2](r^2+2br+1)^2$$
$$+(2-\beta)(1-r^2)[(1+2\alpha\beta-2\alpha)r^2+(2\alpha\beta-2\alpha+2)br+1]^2=0.$$

The radius r_2 is given by the smallest root in (0, 1] *of the equation*

$$(4\alpha-4\alpha\beta+\beta-3)r^2+2\alpha-2\alpha\beta+\beta-3$$
$$+2\sqrt{(2-\beta)(\alpha\beta-\alpha+1)(1-r^2)[1+(2\alpha-2\beta\alpha-1)r^2]}=0.$$

The result is sharp.

Proof. Let $f(z) \in S_b^*(\beta)$. Then $zf'(z)/f(z) = p(z)$, where $p(z) \in \mathscr{P}_b(1-2\beta, -1)$.

Now $K(\alpha, f(z)) = p(z) + zp'(z)/p(z) + \mu$, where $\mu = (1-\alpha)/\alpha$ and $p(z) \in \mathscr{P}_b(1-2\beta, -1)$ and hence the radius of α-starlikeness $R_b(\alpha, \beta)$ for the class $S_b^*(\beta)$ is the smallest positive root of the equation $Q_b(r) = 0$, where

$$Q_b(r) = \min\left\{\mathrm{Re}\left[p(z) + \frac{zp'(z)}{p(z)+\mu}\right] : |z| = r < 1, \quad p(z) \in \mathscr{P}_b(1-2\beta, -1)\right\}.$$

Applying Theorem 1 with $\lambda = 1$, $\mu = (1-\alpha)/\alpha$, $A = 1-2\beta$, $B = -1$ we complete the proof of Theorem 2.

The result is sharp for the function $f_1(z)=\int_0^z p_1(\xi)d\xi$, where $p_1(z)$ is an extremal function for Theorem 1.

By the substituting $b=1$ we get the result of Pascu and Podaru [2].

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Received 3. 6. 1986