WELL-POSEDNESS, CONDITIONING AND REGULARIZATION OF MINIMIZATION, INCLUSION AND FIXED-POINT PROBLEMS

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Well-posedness, conditioning and regularization of fixed-point problems are studied in connexion with well-posedness, conditioning and Tikhonov regularization of minimization and inclusion problems. Equivalence theorems are proved. Coupling iteration and well-posedness as well as iteration and regularization are also considered.

Keywords: conditioning, inclusion, maximal monotone, minimization, fixed-point, well-posed, regularization.


1 Introduction

A la Tikhonov well-posedness is introduced for mapping fixed-point problems in connexion with well-posedness of minimization and inclusion problems. This well-posedness leads to strong convergence of the iteration method for nonexpansive mappings.

Conditioning of functions is a useful notion connected with well-posedness in optimization ([28, 7, 5, 14]). An analogue is considered for multivalued operators and mappings in connexion with inclusion and fixed-point well-posedness.

The Tikhonov regularization method for ill-posed problems is well known for minimization and inclusion ([27, 9, 26]). We extend this method to fixed-point. The iteration method suitably combined with regularization allows to select the same solution (fixed-point) than the sole regularization method, akin to recent results ([6, 11, 22], see also [18] for a more general result).

The paper is organized as follows. In section 2 we introduce well-posedness notions for minimization, inclusion and fixed-point problems and we study their connexions. Conditioning for operators and mappings is considered in section 3 in connexion with
conditioning of functions. In section 4, known equivalence results between well-posedness and conditioning for minimization are extended to inclusion and fixed-point problems. Section 5 is devoted to the convergence of the iteration method for a firmly nonexpansive mapping on a Banach space under fixed-point well-posedness. Tikhonov regularization is introduced in section 6 for fixed-point problems in connexion with minimization and inclusion; the well known selection property remains true in this general situation. In section 7 we show that exact regularization (the regularized solution is a solution to the original problem provided the perturbation in the regularization process be small enough) holds true under a special conditioning. Finally, in section 8 we present, in the context of fixed-point, a general framework of recent results on coupling iteration and Tikhonov regularization.

2 Well-posedness

Let \( X \) be a real normed vector space equipped with the norm \( \| \cdot \| \). We note \( \langle \cdot, \cdot \rangle \) the duality pairing between \( X \) and its dual \( X^* \) and \( \| \cdot \|_\ast \), the dual norm on \( X^* \). \( X \) will be often a Hilbert space identified with its dual by the Riesz theorem.

All along this work we consider three classes of problems on \( X \).

**Minimization.**
Data: \( f : X \to \mathbb{R} \), solution set: \( \text{Argmin} f := \{ x \in X ; f(x) = \inf f \} \).

**Inclusion.**
Data: \( Y \) real normed vector space with norm \( \| \cdot \|_Y \), \( T : X \to 2^Y \), solution set: \( T^{-1}(0) := \{ x \in X ; 0 \in T(x) \} \).

**Fixed-point.**
Data: \( P : X \to X \), solution set: \( \text{Fix} P := \{ x \in X ; x = P(x) \} \).

It is worth noting that Fixed-point is reducible to Inclusion taking \( Y := X \) and \( T := I - P \) where \( I \) denotes the identity mapping on \( X \).

A general way to define well-posedness relies on the notion of **asymptotically solving sequence**. Namely, let us consider some class \((P)\) of problems with data set \( D \) and, for \( d \in D \), solution set \( S \) defined by some relation on the cartesian product \( X \times D \). Roughly speaking, an asymptotically solving sequence for \( d \) is a sequence \( \{x_n\} \) in \( X \) such that \( (x_n, d) \) satisfies the relation asymptotically. We will be more precise for the three classes above. Nevertheless the notion of asymptotically solving sequence being well defined, we say that \( d \in D \) is \((P)\) well-posed iff

(i) \( S \) is nonempty,
(ii) any asymptotically solving sequence \( \{x_n\} \) converges to \( S \) in the sense that \( d(x_n, S) \to 0 \).

If any subsequence of an asymptotically solving sequence is also asymptotically solv-
ing (as it is the case in the three situations below) this notion of well-posedness is more
general than the notion of well-posedness in the generalized sense introduced in [8, 19] for
minimization: $S$ is nonempty and any asymptotically solving sequence has a subsequence
converging to some point in $S$. The two notions are equivalent if $S$ is compact.

For the three classes above we will consider the following notions of asymptotically
solving sequence and therefore the corresponding well-posedness notions.

**Minimization.**

$f$-minimizing: $f(x_n) \to \inf f$.

**Inclusion.**

$(Y,T)$-stationary: $d_Y(0,T(x_n)) \to 0$ or equivalently:

\[ \forall n \in \mathbb{N}, \exists y_n \in T(x_n), \| y_n \|_Y \to 0. \]

**Fixed-point.**

$P$-asymptotically regular: $x_n - P(x_n) \to 0$.

In case of minimization the corresponding notion of well-posedness is nothing but
the (generalized to nonuniqueness) Tikhonov one, and in case of inclusion we recover the
notion of well asymptotical behaviour introduced in [3] for the subdifferential of a convex
function and in [2] for a general maximal monotone operator. It is proved in [17] that
for variational inequalities (subclass of Inclusion), a sequence is asymptotically solving
in the sense of [20] for a given variational inequality iff it is asymptotically solving for
the equivalent inclusion problem.

Of course, with $Y := X$, fixed-point well-posedness for $P$ is nothing but inclusion
well-posedness for $I - P$.

If, in addition to well-posedness, the problem with data $d$ has a unique solution $\mathbf{r}$,
$d$ will be said (P) Tikhonov well-posed. It is worth noting that this implies: “there
exists $\mathbf{r}$ in $X$ such that any asymptotically solving sequence converges to $\mathbf{r}$”, the
converse being true if there exists an asymptotically solving sequence, if the limit of any
convergent asymptotically solving sequence is in $S$, and if any solution defines a (con-
stant) asymptotically solving sequence, which is the case in the three considered classes
if, respectively,

- $f$ is lower-semi-continuous (minimization),
- $T$ has a closed graph and 0 belongs to the closure of the image of $T$ (inclusion),
- $P$ is continuous and 0 belongs to the closure of the image of $I - P$.

In the following we give examples of Tikhonov well-posed problems if the solution
set $S$ is not empty.

**Minimization.**

$f$ is $\alpha$-strongly convex. Indeed this implies $S = \{ \mathbf{r} \}$ and

\[ \forall x \in X, \quad f(x) \geq \min f + \alpha \| x - \mathbf{r} \|^2, \]
$S$ being nonempty if, in addition, $X$ is a reflexive Banach space and $f$ is closed proper.

**Inclusion.**

$Y := X^*$ and $T$ is $\gamma$-strongly monotone. Indeed this implies $S = \{x\}$ and 
\[ \forall (x, y) \in T, \quad \|y\|_* \geq \gamma \|x - x\|, \]
$S$ being nonempty if, in addition, $X$ is a Hilbert space and $T$ is maximal monotone.

**Fixed-point.**

$P$ is $\sigma$-strongly nonexpansive. Indeed this implies $S = \{x\}$ and 
\[ \forall x \in X, \quad \|x - P(x)\| \geq (1 - \sigma)\|x - x\|, \]
$S$ being nonempty if, in addition, $X$ is a Banach space.

It is well known that, if $f$ is $\alpha$-strongly convex, then its subdifferential $\partial f$ is $2\alpha$-strongly monotone and ($X$ being a Hilbert space) if $T$ is maximal monotone and $\gamma$-strongly monotone, then its resolvent $J^T : (I + \lambda T)^{-1}$ ($\lambda > 0$) is $1/(1 + \lambda \gamma)$-strongly nonexpansive. So ($X$ being a Hilbert space) if $f$ closed proper convex is strongly convex, then $f$ is minimization Tikhonov well-posed, its subdifferential is inclusion Tikhonov well-posed and its proximal mapping $\text{prox}_{\lambda f}$ is fixed-point Tikhonov well-posed. Moreover the three problems: minimization for $f$, inclusion for $\partial f$, fixed-point for $\text{prox}_{\lambda f}$ are known to be equivalent in the sense that they have the same solution set ([21, 25]):

\[ S = \text{Argmin } f = (\partial f)^{-1}(0) = \text{Fix } \text{prox}_{\lambda f}. \]

More generally, the connexion between the three notions of well-posedness for equivalent problems is given in the two following propositions.

**Proposition 2.1** ([14, 4]).

Let $X$ be a real Banach space and $f$ be a closed proper convex function on $X$. Then $f$ is minimization well-posed iff $\partial f$ is inclusion well-posed. Of course, $S = \text{Argmin } f = (\partial f)^{-1}(0)$.

**Proposition 2.2** Let $X$ be a real Hilbert space and $T$ be a maximal monotone operator on $X$. Then, for all positive $\lambda$, $T$ is inclusion well-posed iff $J^T$ is fixed-point well-posed. Of course, $S = T^{-1}(0) = \text{Fix } J^T$.

**Proof.** (i) Let $\{x_n\}$ be an asymptotically regular sequence for $J^T$. Therefore, $e_n := x_n - J^T(x_n) \rightarrow 0$. But $e_n/\lambda \in T(x_n - e_n)$. Thanks to inclusion well-posedness we get $d(x_n - e_n, S) \rightarrow 0$ and therefore $d(x_n, S) \rightarrow 0$.

(ii) Let $\{x_n\}$ be a stationary sequence for $T$. So there exits $\{y_n\} \subset X$ such that $\|y_n\| \rightarrow 0$ and $y_n \in T(x_n)$, which is equivalent to $x_n = J^T(x_n + \lambda y_n)$. Moreover, $\|x_n + \lambda y_n - J^T(x_n + \lambda y_n)\| = \lambda \|y_n\| \rightarrow 0$. Thanks to fixed-point well posedness we get $d(x_n + \lambda y_n, S) \rightarrow 0$ and therefore $d(x_n, S) \rightarrow 0$. □
3 Conditioning

Recall ([14, 5, 28]) that a function \( f : X \to \mathbb{R} \) with \( S := \text{Argmin} f \neq \emptyset \) is said \( \psi \)-conditioned iff there exists a function \( \psi : \mathbb{R}_+ \to \mathbb{R}_+ \cup \{+\infty\} \) with \( \psi(0) = 0 \) such that

\[
\forall x \in X, \quad f(x) \geq \min f + \psi(d(x, S)).
\]

Let \( T : X \to 2^Y \) with \( S := T^{-1}(0) \neq \emptyset \). We say that \( T \) is \( \psi \)-conditioned iff there exists a function \( \psi : \mathbb{R}_+ \to \mathbb{R}_+ \cup \{+\infty\} \) with \( \psi(0) = 0 \) such that

\[
\forall x \in X, \quad d_Y(0, T(x)) \geq \psi(d(x, S)).
\]

or, equivalently,

\[
\forall (x, y) \in T, \quad \|y\| \geq \psi(d(x, S)).
\]

Let \( P : X \to X \) with \( S := \text{Fix} P \neq \emptyset \). We say that \( P \) is \( \psi \)-conditioned iff there exists a function \( \psi : \mathbb{R}_+ \to \mathbb{R}_+ \cup \{+\infty\} \) with \( \psi(0) = 0 \) such that

\[
\forall x \in X, \quad \|x - P(x)\| \geq \psi(d(P(x), S)).
\]

The two last definitions are motivated by the following two propositions.

**Proposition 3.1** Let \( \psi : \mathbb{R}_+ \to \mathbb{R}_+ \cup \{+\infty\} \) with \( \psi(0) = 0 \). Let \( X \) be a Banach space and \( f \) be a closed proper convex function on \( X \). Then \( f \) is \( \psi \)-conditioned iff \( \partial f \) is \( \psi \)-conditioned, where \( \psi \) denotes the function equal to \( \psi(t)/t \) for \( t > 0 \) and equal to zero for \( t = 0 \).

**Proof.** First we note that \( S := \text{Argmin} f = \partial f^{-1}(0) \). Let \( f \) be \( \psi \)-conditioned and \( (x, y) \in \partial f \). We have

\[
\forall x \in S, \quad \min f \geq f(x) + \langle y, x - x \rangle \geq \min f + \psi(d(x, S)) + \langle y, x - x \rangle.
\]

So, we get

\[
\forall (x, y) \in \partial f, \quad x \notin S, \quad \|y\| \geq \psi(d(x, S))/d(x, S),
\]

that is, \( \partial f \) is \( \psi \)-conditioned.

Reciprocally, noting that \( f \) is \( \psi \)-conditioned iff, for all positive real \( \theta < 1 \), \( f \) is \( \psi \)-conditioned, let \( \partial f \) be \( \psi \)-conditioned and assume that \( f \) is not \( \psi \)-conditioned. So, there exist a positive real \( \theta < 1 \) and \( x_\psi \in X \) such that

\[
f(x_\psi) < \min f + \theta \psi(d(x_\psi, S)).
\]

This implies \( x_\psi \notin S \) and \( 0 \in \partial_f f(x_\psi) \), where \( 0 < f(x_\psi) - \min f \leq \epsilon < \theta \psi(d(x_\psi, S)) \).

Thanks to Brøndsted-Rockafellar’s theorem, there exists \((\tilde{x}, \tilde{y}) \in \partial f \) such that

\[
\|\tilde{x} - x_\psi\| \leq \theta d(x_\psi, S), \quad \|\tilde{y}\| \leq \epsilon / (\theta d(x_\psi, S)).
\]
Hence, \((\tilde{x}, y)\) satisfies:
\[
(\tilde{x}, y) \in \partial f, \quad \tilde{x} \not\in S, \quad \|y\| < \psi(d(x_\psi, S))/d(x_\psi, S),
\]
a contradiction with \(\psi\)-conditioning of \(\partial f\). \(\square\)

**Proposition 3.2** Let \(X\) be a real Hilbert space and \(T\) be a maximal monotone operator on \(X\). Then, for all \(\lambda > 0\), \(T\) is \(\psi\)-conditioned iff \(J_T^\lambda\) is \(\lambda \psi\)-conditioned.

**Proof.** First we note that, for all positive \(\lambda\), \(S := T^{-1}(0) = \text{Fix } J_T^\lambda\).

Let \(T\) be \(\psi\)-conditioned. As \(x - J_T^\lambda(x) \in \lambda \ T(J_T^\lambda(x))\), we have \(|x - J_T^\lambda(x)| \geq \lambda \psi(d(J_T^\lambda(x), S))\).

Reciprocally, let \((x, y)\) \(\in T\). We have \(x = J_T^\lambda(x + \lambda y)\). As \(J_T^\lambda\) is \(\lambda \psi\)-conditioned we have:
\[
\lambda \|y\| = \|x + \lambda y - J_T^\lambda(x + \lambda y)\| \geq \lambda \psi(d(J_T^\lambda(x + \lambda y), S)) = \lambda \psi(d(x, S)).
\] \(\square\)

The two last propositions lead immediately to the following result.

**Corollary 3.1** Let \(X\) be a real Hilbert space and \(f\) a closed proper convex function on \(X\). Then, for all \(\lambda > 0\), \(f\) is \(\psi\)-conditioned iff \(\partial f\) is \(\psi\)-conditioned, iff \(\text{prox}_{\lambda f}\) is \(\lambda \psi\)-conditioned.

4 Well-posedness and conditioning

Let \(X\) be a Banach space and \(f\) a closed proper convex function on \(X\). It is known ([14]) that \(f\) is minimisation well-posed iff \(f\) is strongly firmly conditioned, that is, \(f\) is \(\psi\)-conditioned where \(\psi\) is strongly firm, i.e. \(\psi\) is firm:
\[
\forall \{t_n\} \subset \mathbb{R}_+ \setminus \{0\}, \quad \psi(t_n)/t_n \to 0 \Rightarrow t_n \to 0.
\]
So, putting together Propositions 2.1 and 3.1 leads to: \(\partial f\) is inclusion well-posed iff \(\partial f\) is firmly conditioned. Actually this can also be obtained as an immediate consequence of the following proposition.

**Proposition 4.1** Let \(X\) and \(Y\) be two real normed spaces and \(T : X \rightarrow 2^Y\) such that \(S := T^{-1}(0)\) is nonempty and closed. Then \(T\) is inclusion well-posed iff \(T\) is firmly conditioned.

**Proof.** That firm conditioning implies inclusion well-posedness is easy to prove. Reciprocally, let us consider the radial regularized of \(T\), i.e. the function \(\psi_T : \mathbb{R}_+ \rightarrow \mathbb{R}_+ \cup \{+\infty\}\) defined by
\[
\psi_T(t) := \inf\{\|y\|; (x, y) \in T, \ d(x, S) \geq t\}.
\]
It is clear that $\psi_{T}(0) = 0$ and that $T$ is $\psi_{T}$-conditioned. Let $\{t_{n}\}$ be a sequence of nonnegative reals such that $\psi(t_{n}) \to 0$. From the definition of the infimum, for all $n \in \mathbb{N}$, there exists $(x_{n}, y_{n}) \in T$, such that $d(x_{n}, S) \geq t_{n}$ and $\|y_{n}\|_{Y} \leq \psi(t_{n}) + 1/n$. Therefore, $\{x_{n}\}$ is an asymptotically solving sequence and, thanks to well-posedness, $d(x_{n}, S) \to 0$ and hence $t_{n} \to 0$. So $\psi_{T}$ is firm. □

Now, let $X$ be a real Hilbert space and $T$ a maximal monotone operator on $X$ such that $S := T^{-1}(0) \neq \emptyset$. Putting together Propositions 2.2 and 3.2 leads to: for all $\lambda > 0$, $J_{\lambda}$ is fixed-point well-posed iff $J_{\lambda}$ is firmly conditioned. Actually, this can also be obtained as an immediate consequence of the following proposition.

**Proposition 4.2** Let $X$ be a normed space and $P : X \to X$ such that $S := \text{Fix } P$ is nonempty and closed. Then, $P$ is fixed-point well-posed iff $P$ is firmly conditioned.

**Proof.** The proof is analogue to the one of Proposition 4.1 considering the radial regularized of $P$ defined by

$$\psi_{P}(t) := \inf\{\|x - P(x)\|; x \in X, d(P(x), S) \geq t\},$$

and noticing that, for an asymptotically regular sequence $\{x_{n}\}$, $d(x_{n}, S) \to 0$ and $d(P(x_{n}), S) \to 0$ are equivalent. □

5 Iteration and well-posedness

Let $X$ be a real Banach space and $P$ a self mapping on $X$. We consider the approximate iterative scheme

$$\|x_{n} - P(x_{n})\| \leq \epsilon_{n}, \ n = 1, 2, \ldots$$

**Proposition 5.1** Let us assume that $P$ is $\theta$-firmly nonexpansive, i.e.

$$\exists \theta > 0, \forall x, y \in X, \|P(x) - P(y)\|^{2} \leq \|x - y\|^{2} - \theta\|I - P\|^{2} + \theta\|I - P\|^{2},$$

that $P$ is fixed-point well-posed (which implies $S := \text{Fix } P \neq \emptyset$), and that $\sum_{n=1}^{+\infty} \epsilon_{n} < +\infty$. Then $x_{n}$ converges in norm to some $x_{\infty}$ in $S$.

**Proof.** Thanks to nonexpansiveness we have

$$\forall n \in \mathbb{N}, \forall \bar{x} \in X, \|x_{n} - \bar{x}\| \leq \|x_{n-1} - \bar{x}\| + \epsilon_{n}.$$ 

Therefore,

$$\forall m > n, \|x_{m} - x_{n}\| \leq 2d(x_{n}, S) + \sum_{k=n+1}^{m} \epsilon_{k}.$$ 

Let $\epsilon_{n} := x_{n} - P(x_{n-1})$. Thanks to $\theta$-firm nonexpansiveness we have

$$\forall n \in \mathbb{N}, \forall \bar{x} \in S, \|x_{n} - \epsilon_{n} - \bar{x}\|^{2} \leq \|x_{n-1} - \bar{x}\|^{2} - \theta\|I - P\|^{2} + \theta\|I - P\|^{2}.$$
Therefore, \( \{x_n\} \) is asymptotically regular for \( P \). Thanks to well-posedness, \( d(x_n, S) \to 0 \). Finally, \( \{x_n\} \) is a Cauchy sequence so converges to some \( x_\infty \) and, as \( S \) is closed and \( d(., S) \) is continuous, \( x_\infty \in S \). \( \square \)

As a direct consequence of Propositions 2.1, 2.2 and 5.1 we get

**Corollary 5.1** ([15]).

Let \( T \) be a maximal monotone operator on the real Hilbert space \( X \), inclusion well-posed (for instance \( T := \partial f \) with \( f \) closed proper convex, minimization well-posed). Then, for all positive \( \lambda \), any sequence \( \{x_n\} \) generated by the approximate proximal iterative scheme

\[
\|x_n - J_T^\lambda x_{n-1}\| \leq \epsilon_n,
\]

with \( \sum_{n=1}^{+\infty} \epsilon_n < +\infty \), converges in norm to some zero of \( T \).

### 6 Regularization

As it is well known for minimization ([27]), the Tikhonov regularization method consists in replacing an ill-posed problem by a family (in practice a sequence) of Tikhonov well-posed ones of same type. Let \( X \) be a real Hilbert space. For a special subclass of each of the three classes above we define below the regularized problem and the regularized solution, that is, the unique solution of the regularized problem.

**Convex minimization.**

Let \( f \) be a closed proper convex function on \( X \), \( x \in X \) and \( t > 0 \). The regularized problem of the minimization of \( f \) is the minimization of

\[
f_{x,t} := f + \frac{t}{2} \| \cdot - x \|^2.
\]

As \( f_{x,t} \) is closed proper, strongly convex, it has a unique minimizer, namely the \( f \)-proximal point to \( x \) with parameter \( \frac{1}{t} : \text{prox}_x \).

**Maximal monotone inclusion.**

Let \( T \) be a maximal monotone operator on \( X \), \( x \in X \) and \( s > 0 \). The regularized problem of the inclusion for \( T \) is the inclusion for

\[
T_{x,s} := T + s(I - x).
\]

As \( T_{x,s} \) is maximal monotone and strongly monotone, it has a unique zero, namely the \( T \)-proximal point to \( x \) with parameter \( \frac{1}{s} : J_T^s x \).
When $T$ is the subdifferential of a closed proper saddle function ([24]) $L$ on the product $X := X_1 \times X_2$ then the inclusion problem for $T_{x,s}$ with $x := (x_1, x_2)$ is equivalent to the saddle-point problem for

$$L_{x,s} := L + \frac{s}{2}(\|x_1\|^2_1 - \|x_2\|^2_2).$$

So, convergence for saddle-point regularization can be deduced from convergence for inclusion regularization (see Proposition 6.1 (iii) below).

**Nonexpansive mapping fixed-point.**

Let $P$ be a nonexpansive self mapping on $X$, $x$ in $X$ and $0 < r \leq 1$. The regularized problem of fixed-point for $P$ is the fixed-point problem for

$$P_{x,r} := P((1-r)x + rx).$$

As $P_{x,r}$ is strongly nonexpansive it has a unique fixed-point we call $R^P_{x,r}$.

This new proximal mapping $R^P_{x,r}$ has the following easy to prove properties:

(i) $\text{Fix } R^P_{x,r} = \text{Fix } P$,

(ii) $R^P_{x,r}$ is nonexpansive, 1-firmly nonexpansive if $P$ is 1-firmly nonexpansive,

(iii) For $T$ maximal monotone, $R^J_{x,r} = J^{T}_{1/r}$.

Now let $\{r_n\}$ be a sequence of positive reals that tends to 0 and $x$ be fixed.

**Proposition 6.1**

(i) $f(\text{prox}_{1/r}^f x) \to \inf f$,

(ii) If $S := \text{Argmin } f \neq \emptyset$, then $\text{prox}_{1/r} f x$ converges in norm to $\text{proj}_S x$,

(iii) If $S := T^{-1}(0) \neq \emptyset$, then $J^T_{1/r} x$ converges in norm to $\text{proj}_S x$.

**Proof.** (i), (ii) and (iii) are well known ([27, 12, 1, 26]). In fact (ii) is a consequence of (iii) which in turns is a consequence of (iv) the proof of which is analogue to the one of (iii) ([26]) using Lemma 6.1 below and the fact that $I - P$ is maximal monotone. □

**Lemma 6.1**

Let $P$ be a nonexpansive self mapping on the real Hilbert space $X$ such that $S := \text{Fix } P \neq \emptyset$. Then

(i) $\forall x \in X, \forall 0 < r (\leq 1), \|x - R^P_{x,r} x\| \leq \frac{2}{2-r} d(x, S),$

(ii) If $P$ is 1-firmly nonexpansive then $\forall x \in X, \forall r > 0, \|x - R^P_{x,r} x\| \leq d(x, S)$.

**Proof.** (i) Let $\overline{x} \in S$ and $x_r := R^P_{x,r} x$. Thanks to the nonexpansiveness of $P$ we get

$$\|x_r - \overline{x}\|^2 \leq \|x_r - \overline{x} + r(x - x_r)\|^2$$

$$= \|x_r - \overline{x}\|^2 + 2r(x_r - \overline{x}, x - x_r) + r^2 \|x - x_r\|^2,$$

from which we deduce easily the result.

(ii) In the righthandside of the first inequality of (i), thanks to firmness, we can substract $\|r(x - x_r)\|^2$. □
Remark 6.1 Of course, we can define fixed-point Tikhonov regularization of $P$ as inclusion Tikhonov regularization of $I - P$, leading to the regularized fixed-point problem

$$y_s = \frac{1}{1 + s} P(y_s) + \frac{s}{1 + s} x.$$

A simple calculation shows that, with the correspondence of parameters $1 - r = \frac{1}{1 + s}$, then $x_r = (1 + s)y_s - sx$. So, as $r \to 0$ iff $s \to 0$, $x_r \to \bar{x}$ iff $y_s \to \bar{x}$.

7 Exact regularization

In the framework of the previous section we prove that, under a specific kind of conditioning, exact regularization holds true, that is, the regularized solution is a solution to the original problem for all $r$ small enough. In fact we obtain more, namely that the selected solution (the projection of $x$ onto the solution set $S$) is achieved if $x$ is close enough to $S$ with given $r$ or, equivalently, if $r$ is small enough with given $x$.

Let $f$ be a closed proper convex function on $X$ such that $S := \text{Argmin } f \neq \emptyset$. Let $\gamma > 0$. Recall that $f$ is said $\gamma$-linear conditioned if

$$\forall x \in X, f(x) \geq \min f + \gamma d(x, S).$$

We note that linear conditioning is a particular strongly firm conditioning. In fact, in this case, $f$ is $\psi$-conditioned with $\psi(t) := \gamma t$ and hence $\psi(t) = \gamma$ if $t > 0$ and $\psi(0) = 0$. More precisely, if $\widehat{\psi}(t_n) \to 0$ then $t_n = 0$ for all $n$ large enough.

It has been proved ([15]) that, under $\gamma$-linear conditioning, if $d(x, S) < \gamma/r$ then $\text{prox}_f x = \text{proj}_S x$, and consequently that the proximal point algorithm has finite termination, more precisely, denoting $\{x_n\}$ the generated sequence, $\exists N, \forall n > N, x_n = \text{proj}_S x_N$.

This exact regularization result for convex minimization can be extended to maximal monotone inclusion and nonexpansive mapping fixed-point as follows.

First we introduce constant conditioning for this two classes.

Let $\gamma > 0$. An operator $T : X \to 2^Y$ such that $S := T^{-1}(0) \neq \emptyset$ is said $\gamma$-constant conditioned if $T$ is $\psi$-conditioned with $\psi(t) = \gamma$ if $t > 0$ and $\psi(0) = 0$. We note that this is equivalent to

$$\forall (x, y) \in T, \text{ if } \|y\| < \gamma, \text{ then } x \in S.$$

Let $\delta > 0$. We say that a self mapping $P$ of the real normed vector space $X$ such that $S := \text{Fix } P \neq \emptyset$ is $\delta$-constant conditioned if $P$ is $\psi$-conditioned with $\psi(t) = \delta$ if $t > 0$ and $\psi(0) = 0$. We note that this is equivalent to

$$\forall x \in X, \text{ if } \|x - P(x)\| < \delta, \text{ then } P(x) \in S.$$

As corollaries of Propositions 3.1 and 3.2 we get immediately the following two propositions.
Proposition 7.1 ([23]). Let $X$ be a Banach space and $f$ be a closed proper convex function on $X$. Then $f$ is $\gamma$-linear conditioned iff $\partial f$ is $\gamma$-constant conditioned.

Proposition 7.2 Let $X$ be a real Hilbert space and $T$ be a maximal monotone operator on $X$. Then, for all $\lambda > 0$, $T$ is $\gamma$-constant conditioned iff $J^T_\lambda$ is $\lambda \gamma$-constant conditioned.

We can now present the general exact regularization results.

Proposition 7.3 Let $P$ be a nonexpansive self mapping of the real Hilbert space $X$ with $\delta$-constant conditioning. Let $S := \text{Fix } P$.

(i) If $d(x, S) < \frac{2 - r}{2r} \delta$ or, equivalently, $0 < r < \min\{1, \frac{\delta}{2d(x, S)}\}$, then $R^P_r x \in S$.
(ii) If $P$ is 1-firmly nonexpansive and $d(x, S) < \frac{\delta}{r}$, then $R^P_r x = \text{proj}_S x$.

Proof. By definition of $x_r := R^P_r x$ we have $\|((1 - r)x_r + rx - P((1 - r)x_r + rx))\| = r\|x - x_r\|$.

(i) From lemma 6.1 (i) we have $r\|x - x_r\| < \delta$. Therefore, thanks to constant conditioning, we get $x_r = P((1 - r)x_r + rx) \in S$.

(ii) From lemma 6.1 (ii) we have $r\|x - x_r\| < \delta$ and therefore $x_r \in S$. Thanks again to lemma 6.1 (ii) we get $\|x - x_r\| = d(x, S)$.

Corollary 7.1 Let $T$ be a maximal monotone operator on the real Hilbert space $X$, with $\gamma$-constant conditioning. Let $S := T^{-1}(0)$. If $d(x, S) < \frac{\gamma}{r}$, then $J^T_\lambda x = \text{proj}_S x$.

Proof. Apply (ii) of Proposition 7.3 with $P := J^T_1$ (so $R^T_r = J^T_\frac{r}{2}$) and, thanks to Proposition 7.2, $\delta := \gamma$.

Linear conditioning can be defined for a saddle function which also implies constant conditioning of its subdifferential and hence, exact regularization for saddle-point problems ([10]).

8 Iteration and regularization

Let $X$ be a real Hilbert space and $P$ a nonexpansive self mapping on $X$ with set of fixed-points $S$. As shown in the previous section, the Tikhonov regularization method allows to approximate a particular fixed-point, namely the projection onto $S$ of a given $x$. Now the iteration method applied to the regularized mapping $P_x, r$ for fixed $r$ allows to approximate the regularized solution $x_r := R^P_r x$, since classically the generated sequence converges in norm to $x_r$. So this give a two stages approximation. We prove in the following that if in the iteration method we use variable $r_n$ tending to 0 not too fast, then the sequence generated by this diagonal iterative scheme converges also to the projection of $x$ onto $S$. 
More precisely we consider a sequence \( \{x_n\} \) generated by the following approximate iterative scheme
\[
\|x_n - P((1 - r_n)x_{n-1} + r_n x)\| \leq \epsilon_n, \ n = 1, 2, \ldots
\]
where \( 0 < r_n \leq 1 \) and \( \epsilon_n \geq 0 \).

**Proposition 8.1** Let us assume:

- \( P \) is nonexpansive,
- \( r_n \to 0 \), \( \sum_{n=1}^{+\infty} r_n = +\infty \), \( \epsilon_n/r_n \to 0 \),
- \( |1 - r_n/r_{n-1}| \to 0 \), \( S \neq \emptyset \). Then \( x_n \) converges in norm to \( \text{proj}_S x \).

**Proof.** Though the result can be deduced from [18] (Proposition 5.1), we prefer to give here a self-contained proof. Noting \( x(n) := R_{r_n} x \), we get easily the estimate
\[
\|x_n - x(n)\| \leq (\|x_{n-1} - x(n-1)\| + \|x(n-1) - x(n)\| + (1 + r_n)\epsilon_n)/(1 + r_n).
\]

We invoke the following direct consequence of [6] (Corollary 5.4):

- let sequences of nonnegative reals \( a_n, r_n, \gamma_n \) be such that \( \sum_{n=1}^{+\infty} r_n = +\infty \), \( \gamma_n \to 0 \) and \( a_n \leq (a_{n-1} + \gamma_n r_n)/(1 + r_n) \); then \( a_n \to 0 \).

For that we take \( \gamma_n := \epsilon_n + \epsilon_n/r_n + \|x(n) - x(n-1)\|/r_n \) showing that \( \|x(n) - x(n-1)\|/r_n \to 0 \). In fact, thanks to nonexpansiveness we can obtain the estimate
\[
\|x(n) - x(n-1)\| \leq |1 - r_n/r_{n-1}| \|x - x(n)\|,
\]
and, from Lemma 6.1 (i),
\[
\|x(n) - x(n-1)\| \leq |1 - r_n/r_{n-1}| \|x - x(n)\|.
\]

So we get \( \|x_n - x(n)\| \to 0 \) and the result since \( x(n) \to \text{proj}_S x \). \( \Box \)

**Corollary 8.1** Let \( T \) be a maximal monotone operator on \( X \) such that \( S := T^{-1}(0) \neq \emptyset \). Let \( \lambda > 0 \). Under the same assumptions on \( r_n \) and \( \epsilon_n \) than in Proposition 8.1, any sequence \( \{x_n\} \) generated by the approximate iterative scheme
\[
\|x_n - J_T^\lambda (1 - \lambda r_n)x_{n-1} + \lambda r_n x)\| \leq \epsilon_n, \ n = 1, 2, \ldots,
\]
converges in norm to \( \text{proj}_S x \).

**References**


Well-posedness and regularization of fixed-point problems


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