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# ON A NONSTANDARD BOUNDARY VALUE PROBLEM FOR THE LAPLACE OPERATORIN THE PLANE* 

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#### Abstract

This paper deals with a nonstandard boundary value problem(BVP) for the Laplace operator in the plane. On the boundary of a bounded simply connected domain, say the unit disk, $|\nabla u|=m$ is prescribed and it is shown that in general the corresponding bvp possesses infinitely many solutions which can be classical or generalized depending on the function $m>0$, $m \in C^{0}$, respectively $m>0$ a.e., $\log m \in L_{1}, m \in L_{p}, p \geq 1$. We shortly discuss the same problem in doubly connected domain.


## 1. Introduction

This paper deals with the following nonstandard non-linear boundary value problem (bvp):

$$
\begin{align*}
& \Delta u=0 \text { in } B_{1}=\left\{x^{2}+y^{2}<1\right\}=\left\{z \in \mathbf{C}^{1}:|z|<1\right\} \\
& |\nabla u|=m(\varphi) \geq 0 \text { on } S_{1}=\left\{x^{2}+y^{2}=1\right\}=\left\{z \in \mathbf{C}^{1}:|z|=1\right\} \tag{1}
\end{align*}
$$

$x=r \cos \varphi, y=r \sin \varphi,-\pi \leq \varphi \leq \pi, m(\varphi+2 \pi)=m(\varphi), \forall \varphi$. Later on we shall assume that $\varphi \in[-\pi, \pi], m(-\pi)=m(\pi)$.

We have used the notion "nonstandard bvp" as in general for each fixed $m \geq 0$ (1) possesses infinitely many solutions. Concerning the approach in investigating (1) we are looking for a solution having the form $u=\Re F(z)$, where $F(z)$ is

[^0]some regular (holomorphic) function in the unit disc $B_{1}$. In order to satisfy the boundary condition $|\nabla u|=m(\varphi)$ on the unit circle $S_{1}$ we shall use several well-known properties of the regular functions belonging to the classical Hardy functional classes $H^{p}, 0<p \leq \infty$ and the Nevanlinna class $N$. Certainly, in many cases is very important $F \in C^{0}\left(\overline{B_{1}}\right)$ as it implies that $u \in C^{0}\left(\overline{B_{1}}\right)$. Assume that $\Delta$ is the 3 -dimensional Laplace operator and instead of the interior bvp (1) we study the exterior bvp (1) with $u(\infty)=0$. The 3 -dimensional problem (1) ( $m=m(\theta, \varphi)$ ) is called Backus problem and it has significant applications to gravity and geomagnetic intensity surveys (see [2]). Some papers are devoted to (1) $([2],[6],[10])$ but we try here to propose a further development on the subject dealing with boundary functions $m$ vanishing on some closed subset $E$ of $S_{1}$ having zero Lebesgue measure. The simplest case is when $E$ contains finitely many points but much more interesting is the case when $E \subset S_{1}$ is closed, contains infinitely many points and is of measure zero. Unfortunately, in the latter case we can solve (1) only for specially constructed smooth functions $m,\left.m\right|_{E}=0, m>0$ on $S_{1} \backslash E$. We point out that then we find either boundary functions $m$ flat at $E$ and belonging to some Gevrey class $G_{s}\left(S_{1}\right), s>1$, or $C^{\varepsilon_{n}}$ smooth functions having zeroes of finite order $\varepsilon_{n}>0$, where $\left\{\varepsilon_{n}\right\}$ is some appropriately chosen sequence. As $H^{p}$ spaces are involved in our investigations, we shall have not only classical solutions $u \in C^{2}\left(B_{1}\right) \cap C^{1}\left(\overline{B_{1}}\right)$ but also harmonic solutions $u=\Re F$ admitting the boundary values $m(\varphi)$ almost everywhere (a. e.) on $S_{1}:|\nabla u|=m(\varphi)$ a. e. on $[-\pi, \pi] \ni \varphi, m(\varphi)$ being measurable, $m(\varphi)>0$ a.e.

We also discuss (1) in the case when the corresponding domain is an annulus and find explicit formulas for some solutions. Because of the lack of space we omit the details and the corresponding proofs.

There is an exuberance of classical papers and monographs dealing with different aspects of the boundary behaviour of regular functions in $B_{1}$ and the corresponding functional spaces. We quote a small part of them here, namely the monographs containing the results to be directly applied in our paper: [4], [5], [7], [8], [9], [12], [13], [14].

## 2. Formulation of the main results

To begin with, we shall propose three definitions useful in studying the bvp (1).
Definition 1. The function $u \in C^{2}\left(B_{1}\right) \cap C^{1}\left(\bar{B}_{1}\right)$ is called a classical solution of (1).

Definition 2. The function $u$ satisfies the bvp (1) if $u \in C^{2}\left(B_{1}\right) \cap C^{0}\left(\bar{B}_{1}\right)$ and $|\nabla u| \in C^{0}\left(\bar{B}_{1}\right)$.

Definition 3. The harmonic in $B_{1}$ function $u$ satisfies (1) if $u \in C^{0}\left(\bar{B}_{1}\right)$ and a. e. on $S_{1}$ there exist the non-tangential (angle) boundary values $\lim _{B_{1} \ni(x, y) \rightarrow\left(x_{0}, y_{0}\right) \in S_{1}} u_{x}(x, y)=\left(u_{x}\right)^{*}\left(x_{0}, y_{0}\right)$ and $\lim _{B_{1} \ni(x, y) \rightarrow\left(x_{0}, y_{0}\right) \in S_{1}} u_{y}(x, y)=$ $\left(u_{y}\right)^{*}\left(x_{0}, y_{0}\right),\left(u_{x}\right)^{*},\left(u_{y}\right)^{*}$ being measurable on $S_{1}$ and $\left|(\nabla u)^{*}\right|\left(x_{0}, y_{0}\right)=m\left(x_{0}, y_{0}\right)$ a. e. on $S_{1}$.

Theorem 1. (Existence results)
(i) The bvp (1) possesses a solution $u$ in the sense of Definition 3 if $\log m \in L_{1}$ for $m \in L_{p}, 1 \leq p<\infty$ and $m>0$ a. e.
(ii) The bvp (1) has a classical solution $u$ if $0<m(\varphi) \in C^{1}\left(S_{1}\right)$.
(iii) The bup (1) has a solution in the sense of Definition 2 if $m \in C^{0}\left(S_{1}\right)$, $0<m(\varphi)$ for $\varphi \neq \varphi_{k}, k=1, \ldots, n, m\left(\varphi_{k}\right)=0$ for $k=1, \ldots, n$ and $\log m(\varphi) \in$ $L_{1}$ (equivalently, $\int_{-\pi}^{\pi} \log ^{-} m(\varphi) d \varphi>-\infty$ ).
(iv) The bvp (1) possesses a classical solution if $m \in C^{1}\left(S_{1}\right), 0<m(\varphi)$ for $\varphi \neq \varphi_{1}, \ldots, \varphi_{k}, m\left(\varphi_{k}\right)=0$ for $k=1, \ldots, n$ and $\int_{-\pi}^{\pi} \log ^{-} m(\varphi) d \varphi>-\infty$.
$(v)$ Let $E \subset S_{1}$ be an arbitrary closed subset of $S_{1}$ of measure zero containing infinitely many points. Then one can find a function $m \in C^{1}\left(S_{1}\right),\left.m\right|_{E}=0$, $m>0$ on $S_{1} \backslash E$, $\log m \in L_{1}(-\pi, \pi)$, and such that (1) possesses a classical solution for this boundary data $m$.

As usual, $\log ^{+} x=\max (\log x, 0), \log ^{-} x=\min (0, \log x, 0)$ for $x>0$.
Remark. Consider bvp (1) and suppose that $0<m(\varphi+2 \pi)=m(\varphi), \forall \varphi$ and $\log m(\varphi)$ can be prolonged analytically in the strip $\varphi+i h,|h| \leq \varepsilon_{0}$. Then (1) has a harmonic solution in the disc $B_{1+\frac{\varepsilon_{0}}{2}}, \varepsilon_{0}>0$.

The bvp does not possess unique solution.
Example 1. Let $m \equiv 1$. Then (1) possesses infinitely many classical solutions. To verify this we look for a regular function $F(z)$ in $B_{1}, F \not \equiv$ const, $F \in C^{0}\left(\bar{B}_{1}\right)$ and such that $|F|_{S_{1}}=1$. Certainly, there exists then $z_{0} \in B_{1}$ such that $F\left(z_{0}\right)=0$. Let $F$ be a rational function. It is well known then that $F(z)=e^{i \gamma} z^{m} \prod_{k=1}^{n} \frac{z-\alpha_{k}}{1-\bar{\alpha}_{k} z}$, where $\gamma \in \mathbf{R}$ is a constant, $m, n$ are non-negative integers, $0 \neq \alpha_{k} \in B_{1}, k=1, \ldots, n . F(z)$ is a finite Blascke product and $\left|\frac{z-\alpha_{k}}{1-\bar{\alpha}_{k} z}\right|=1$ for $|z|=1$. Consider the holomorphic function $g(z)=\int_{0}^{z} F(\lambda) d \lambda,|z|<1+\varepsilon_{0}$, $\varepsilon_{0}>0$ and define $u(x, y)=\Re g \in C^{1}\left(B_{1+\varepsilon_{0}}\right) \cap C^{2}\left(B_{1}\right)$. Evidently, $\Delta u=0$ in $B_{1}, g^{\prime}(z)=u_{x}+i v_{x}=u_{x}-i u_{y}$ if $g(z)=u+i v$. Therefore, $|\nabla u|=\left|g^{\prime}(z)\right|$ in $B_{1+\varepsilon_{0}} \Rightarrow|\nabla u|_{S_{S_{1}}}=|F|_{\left.\right|_{S_{1}}}=1$.

Let the regular functions $F_{1}^{\prime}, F_{2}^{\prime}$ in $B_{1}$ be defined by the formulas

$$
\begin{align*}
F_{1}^{\prime}(z) & =P_{1}(z) P_{2}(z) h_{1}(z)  \tag{2}\\
F_{2}^{\prime}(z) & =P_{1}(z) P_{2}(z) h_{2}(z) \tag{3}
\end{align*}
$$

where $h_{1,2}$ are regular in $|z|<1, h_{1,2} \in C^{0}\left(\bar{B}_{1}\right), h_{1,2}(z) \neq 0$ in $\bar{B}_{1}, P_{1}(z)=$ $\prod_{j=1}^{m}\left(z-a_{j}\right),\left|a_{j}\right|<1$, multiple roots of the polynomial $P_{1}$ being admissible and $P_{2}(z)=\prod_{j=1}^{n}\left(z-b_{j}\right)^{\alpha_{j}}, \alpha_{j}>0,\left|b_{j}\right|=1, j=1, \ldots, n$, i.e. $P_{2} \in C^{0}\left(\bar{B}_{1}\right)$ (Definition of the branches of $\left(z-b_{j}\right)^{a_{j}}$ in $B_{1}$ and their link with the conformal mappings $B_{1} \rightarrow$ Polygons $\subset \mathbf{C}^{1}$ can be found in [5]). Below we discuss the uniqueness problem for solutions with prescribed zeroes in $\bar{B}_{1}$.

Theorem 2. (Uniqueness) Assume that $u_{1}$ and $u_{2}$ satisfy (1) in the sense of Definition 1 and can be found in the form $u_{i}=\Re F_{i}$, where $F_{i}^{\prime}$ are given by (2) and (3). Then $F_{1}(z)=F_{2}(z) e^{i \gamma}+C$ for some constants $\gamma \in \mathbf{R}^{1}$ and $C \in \mathbf{C}^{1}$.

Combining Theorem 1 (ii) and Theorem 2, we can construct solution $v$ of the bvp (1) having boundary data $n(\varphi)=\left|P_{1}(\varphi) \| P_{2}(\varphi)\right| m(\varphi), 0<m(\varphi) \in C^{1}\left(S_{1}\right)$. Consequently $0<m \in C^{0}\left(S_{1}\right)$ gives rise to solutions with gradient vanishing at finitely many points of the unit circle $S_{1}$ and on finitely many points of the unit disc.

Below is the proof of the Theorem 2.
Proof. Having in mind that $F_{1,2}^{\prime} \in C^{0}\left(\bar{B}_{1}\right) \subset H^{1} \Rightarrow F_{1,2} \in C^{0}\left(\bar{B}_{1}\right)$ we can construct classical solutions $u_{1,2} \in C^{1}\left(\bar{B}_{1}\right)$ of (1) via the formula $u_{1,2}=\Re F_{1,2}$ as $u_{1,2 x}-i u_{1,2 y}=F_{1,2}^{\prime}$ in $B_{1}$. Evidently, $\left|\nabla u_{1,2}\right|_{\bar{B}_{1}}=\left|F_{1,2}^{\prime}\right|_{\bar{B}_{1}},\left|\nabla u_{1,2}\right|_{\left.\right|_{S_{1}}}=m \geq 0$, $m(\varphi)=0 \Leftrightarrow b_{j}=e^{i \varphi_{j}}, j=1, \ldots, n$.

Consider now the regular function in $B_{1}: \beta(z)=\frac{F_{1}^{\prime}(z)}{F_{2}^{\prime}(z)}=\frac{h_{1}(z)}{h_{2}(z)}, \beta \in C^{0}\left(\bar{B}_{1}\right)$, $|\beta|>0$ in $\bar{B}_{1},\left.|\beta|\right|_{S_{1}}=1$. Then $\log \beta(z)=\log |\beta(z)|+i \arg \beta(z)=\tau+i \theta$ in $B_{1}$, $\tau \in C^{0}\left(\bar{B}_{1}\right) \Rightarrow \left\lvert\, \begin{aligned} & \Delta \tau=0 \text { in } B_{1} \\ & \left.\tau\right|_{S_{1}}=0\end{aligned} \Rightarrow \tau \equiv 0 \Rightarrow \theta=\gamma=\right.$ const $\in \mathbf{R}_{1}$ according to the Cauchy-Riemann equations in $B_{1}: \theta_{x}=\tau_{y}, \theta_{y}=-\tau_{x}$, i.e. $\beta(z)=e^{i \gamma} \Rightarrow F_{1}(z)=$ $F_{2}(z) e^{i \gamma}+C, C=$ const.

In other words the classical solutions of (1) constructed via the formula $u=$ $\Re F$ and having finitely many prescribed zeroes by $P_{1,2}$ in $\bar{B}_{1}$ are determined up to a rotation and translation.

## 3. Appendix

Fig. 1 is illustrating the geometrical configuration linearly perturbed saddle. Thus, $u_{1}=2 y(x+1)$ is harmonic function in $\mathbf{R}^{2}$, while $\left|\nabla u_{1}\right|$ vanishes at $S_{1}$ only. Other example is the bvp $\left\lvert\, \begin{aligned} & \Delta u=0 \text { in } B_{1} \\ & \left.|\nabla u|^{2}\right|_{S_{1}}=n^{2}+m^{2} A^{2}+2 n m A \cos (n-m) \varphi\end{aligned}\right.$, where $n>m>1$ are integers, $A>0$. We look for $u(r, \varphi)=\operatorname{Re}\left(z^{n}+A z^{m}\right)$, $z=r e^{i \varphi}$.

Concluding remarks. To complete this paper we consider a bvp similar to (1) but in the annulus $A:\{1>|z|>r>0\}$, i.e. in doubly connected domain in $\mathbf{R}^{2}$.

$$
\begin{align*}
& \Delta u=0 \text { in } A=\left\{0<r^{2}<x^{2}+y^{2}<1\right\} \\
& |\nabla u|_{S_{1}}=m_{1}(\varphi)>0  \tag{4}\\
& |\nabla u|_{S_{r}}=m_{2}(\varphi)>0
\end{align*}
$$

$S_{r}=\left\{x^{2}+y^{2}=r^{2}\right\}$. Because of the lack of space we shall work rather formally. We shall deal with analytic functions in the sense of Weierstrass. They could be multivalued and are produced via analytical continuation of an element along a curve. Below we remind a well known fact ([5], Chapter VIII).

Proposition 1. The function $u(z)$ is harmonic in the multiconnected domain $D$ if and only if $u(z)=\operatorname{ReF}(z)$, where $F(z)$ is analytic in $D$ and satisfies the conditions:

1. $F^{\prime}(z)$ is regular in $D$;
2. The integral $\int_{\gamma} F^{\prime}(z) d z$ along any closed path $\gamma \subset D$ is purely imaginary number or 0 , i.e. $\int_{\gamma} F^{\prime}(z) d z \in i \mathbf{R}$.

To solve (4) we are looking for a regular in $A$ function $F^{\prime}(z) \neq 0, \forall z$. Therefore, $F(z), \log F^{\prime}(z)$ are analytic in $A$. Put $u=$ Re F. According to Proposition $1 \Delta u=0$ in $A$ iff: 1) $F^{\prime}(z)$ is regular in $A$, and 2) $\int_{\gamma} F^{\prime}(z) d z \in i \mathbf{R}$ for each closed path $\gamma \subset$ A. Then $F^{\prime}(z)=u_{x}(x, y)-i u_{y}(x, y), z=$ $x+i y,(x, y) \in A \Rightarrow\left|F^{\prime}(z)\right|=\left|\nabla_{x, y} u\right| \Rightarrow$ $\left|F^{\prime}(z)\left\|_{S_{1}}=\left|\nabla u\left\|_{S_{1}}=m_{1}(\varphi), \mid F^{\prime}(z)\right\|_{S_{r}}=\right.\right.\right.$ $m_{2}(\varphi)$ (say if $u \in C^{1}(\bar{A})$ ). Certainly,
(5) $\log F^{\prime}(z)=\log \left|F^{\prime}\right|+i \operatorname{vararg} F^{\prime}$.


Figure 1

Consider now the Dirichlet problem in $A$ :

$$
\begin{gather*}
\Delta \tau=0 \text { in } A \\
\left.\tau\right|_{S_{1}}=\log m_{1}  \tag{6}\\
\left.\tau\right|_{S_{r}}=\log m_{2}
\end{gather*}
$$

where $\tau=\log \left|F^{\prime}\right|=\Re \log F^{\prime}(z), \theta=\operatorname{vararg} F^{\prime}, F^{\prime}=\left|F^{\prime}\right| e^{i \theta}$. As it is well known, there exists a unique classical solution of (6), say if $0<m_{1,2} \in C^{0}\left(S_{1}, S_{r}\right)$, $+\infty>\log \left|F^{\prime}\right|>-\infty$ in $\bar{A}$. It can be expressed explicitly by the Green formula and the Green function of the annulus $A$. The Green function in that special case
can be constructed by using the method of inversions with respect to $S_{1}$ and $S_{r}$ obtaining this way an infinite series. On the other hand let $f$ be a regular function in $A$, i.e. $\bar{\partial} f(z)=0$, and $\Phi(\varphi)=\operatorname{Re} f(z)$ on $S_{1}$, while $\Psi(\varphi)=\operatorname{Re} f(z)$ on $S_{r}$, $\Phi, \Psi$ being prescribed in $C^{0}[-\pi, \pi]$. Moreover, let $\int_{-\pi}^{\pi} \Phi(\varphi) d \varphi=\int_{-\pi}^{\pi} \Psi(\varphi) d \varphi$. Then according to Villa (see [1], Chapter XI)

$$
\begin{align*}
f(z) & =\frac{i \omega}{\pi^{2}} \int_{-\pi}^{\pi} \Phi(\varphi) \zeta\left(\frac{\omega}{\pi i} \log z-\frac{\omega}{\pi} \varphi\right) d \varphi-\frac{i \omega}{\pi^{2}} \times \\
& \times \int_{-\pi}^{\pi} \Psi(\varphi)\left[\zeta\left(\frac{\omega}{\pi i} \log z-\frac{\omega}{\pi} \varphi-\omega^{\prime}\right)+\eta^{\prime}\right] d \varphi+i C, \tag{7}
\end{align*}
$$

where $\zeta(w)$ is the Weierstrass $\zeta$ function, $C$ is an arbitrary real constant, $\omega>0$ is arbitrary, $\omega^{\prime} \in i \mathbf{R}$ is s.t. $e^{\pi i \frac{\omega^{\prime}}{\omega}}=r, \zeta\left(w+2 \omega^{\prime}\right)=\zeta(w)+2 \eta^{\prime}$. As we know, $\zeta^{\prime}=-\rho, \rho$ being the famous Weierstrass $\rho$ function. Conversely, (7) gives a regular solution of $\bar{\partial} f=0$, Re $\left.f\right|_{S_{1}}=\Phi$, Re $\left.f\right|_{S_{r}}=\Psi$. Taking $v=R e f$ we get an explicit formula for the solution of the Dirichlet bvp

$$
\left\lvert\, \begin{aligned}
& \nabla v=0 \mathrm{in} A \\
& \left.v\right|_{S_{1}}=\Phi(\varphi) \\
& \left.v\right|_{S_{r}}=\Psi(\varphi)
\end{aligned}\right.
$$

Applying again Proposition 1 to (5) with $\tau=\log \left|F^{\prime}(z)\right|=R e \log F^{\prime}(z), \log F^{\prime}(z)$ being possibly multivalued analytic in $A$, we conclude that: 1) $\frac{d}{d z} \log F^{\prime}$ is regular in $A$, and 2) $\int_{\gamma} \frac{d}{d z} \log F^{\prime}(z) d z \in i \mathbf{R}$ for each closed path $\gamma$ in $A$.

Therefore, $\frac{F^{\prime \prime}}{F^{\prime}}$ is regular (obviously) and $\int_{\gamma} \frac{F^{\prime \prime}(z)}{F^{\prime}(z)} d z \in i \mathbf{R}$.
Proposition 2. The bvp (4) possesses a solution $u$ of the form $u=\operatorname{Re} F(z)$, where $F(z)$ is some analytic function in $A$ if the following conditions hold true: $F^{\prime}(z)$ is regular in $A, F^{\prime}(z) \neq 0$ in $A, \int_{S_{r_{1}}} F^{\prime}(z) d z \in i \mathbf{R}, \int_{S_{r_{1}}} \frac{F^{\prime \prime}}{F^{\prime}} d z \in i \mathbf{R}$ for some $r_{1} \in(r, 1), \log \left|F^{\prime}(z)\right|$ verifies (6) and $\int_{-\pi}^{\pi} \log m_{1} d \varphi=\int_{-\pi}^{\pi} \log m_{2} d \varphi$.

In the special case $F^{\prime}, F^{\prime \prime} \in C^{0}(\bar{A}), F^{\prime} \neq 0$ in $\bar{A}$ we can assume that $\int_{S_{1}} F^{\prime}(z) d z \in i \mathbf{R}^{1}, \int_{S_{1}} \frac{F^{\prime \prime}}{F^{\prime}} d z \in i \mathbf{R}^{1}$.

Let at $z_{0} \neq 0, \infty$ be prescribed the value $\log z_{0}$ and the curve $\gamma$ joins $z_{0}, z$. Then the analytic continuation along $\gamma$ gives: $\log z=\log |z|+i\left[\Im\left(\log z_{0}\right)+\Delta_{\gamma} \arg z\right]$, $\Delta_{\gamma}$ is the increment of $\arg z$ along $\gamma$.

## 4. Possible generalizations of the previous results for different bounded simply connected domains

Let $f: D \rightarrow B_{1}$ be a conformal mapping, where $f$ is regular and univalent, while $\partial D \in C^{k, \alpha}, k \in \mathbf{N}, 0<\alpha<1 ; w=f(z), z=x+i y \in D, w=\xi+i \eta \in B_{1}$.

Then according to S.E. Warschawski (see Proc. AMS, vol. 12:4, 1961, pp. 614-620) $f(z)$ and $f^{-1}(w)$ can be prolonged to functions belonging to $C^{k, \alpha}(\bar{D})$, respectively $C^{k, \alpha}\left(\bar{B}_{1}\right)$ with $f^{\prime} \neq 0$ in $\bar{D},\left(f^{-1}\right)^{\prime} \neq 0$, in $\bar{B}_{1}$.

In general, such a continuation is impossible for $k=1, \alpha=0$.
Assume now that $D_{1}$ and $D_{2}$ are two bounded multiconnected domains in $\mathbf{C}^{1}, \partial D_{1}, \partial D_{2} \in C^{k, \alpha}, k \in \mathbf{N}, 0<\alpha<1$ which are conformly equivalent via the regular univalent function $f(z)=w$. Then again $f, f^{-1}$ can be prolonged to functions belonging to $C^{k, \alpha}\left(\bar{D}_{1}\right)$, respectively to $C^{k, \alpha}\left(\bar{D}_{2}\right)$ and $f^{\prime} \neq 0$ in $\bar{D}_{1}$, while $\left(f^{-1}\right)^{\prime} \neq 0$ in $\bar{D}_{2}$.

Consider the bvp

$$
\left\lvert\, \begin{align*}
& \Delta_{x, y} u=0 \text { in } D  \tag{8}\\
& \left.\left|\nabla_{x, y} u\right|\right|_{\partial D}=m \text { on } \partial D
\end{align*}\right.
$$

By using the change $z=f^{-1}(w)$ and putting $\left\lvert\, \begin{aligned} & x=x(\xi, \eta),(\xi, \eta) \in \bar{B}_{1} \\ & y=y(\xi, \eta),(\xi, \eta) \in \bar{B}_{1},\end{aligned}\right.$, $\tilde{u}(\xi, \eta)=u(x(\xi, \eta), y(\xi, \eta))$ we get the bvp

$$
\begin{align*}
& \Delta_{\xi, \eta} \tilde{u}=0 \text { in } B_{1} \\
& \left.\left|\nabla_{\xi, \eta} \tilde{u}\right|\right|_{S_{1}}=\left.\tilde{m} \sqrt{\left(\frac{\partial x}{\partial \xi}\right)^{2}+\left(\frac{\partial x}{\partial \eta}\right)^{2}}\right|_{S_{1}} \tag{9}
\end{align*}
$$

where $\tilde{m}(\xi, \eta)=m(x(\xi, \eta), y(\xi, \eta)),(\xi, \eta) \in S_{1}, x, y \in C^{k, \alpha}\left(\bar{B}_{1}\right),\left|\left(f^{-1}\right)^{\prime}(w)\right|^{2}=$ $\left(\frac{\partial x}{\partial \xi}\right)^{2}+\left(\frac{\partial x}{\partial \eta}\right)^{2}>0$ in $\bar{B}_{1}$.

The bvp (9) coincides with (1) that we studied in the first part of our paper.
Suppose now that $D_{1}$ is doubly connected bounded domain having two contours: $\partial D_{1}^{(1)}, \partial D_{1}^{(2)} \in C^{k, \alpha}$ which is conformly equivalent to the annulus $A$ (see Chapter V from [7]). Then in a similar way as in the case (8) we can reduce the bvp

$$
\left\lvert\, \begin{align*}
& \Delta_{x, y} u=0 \text { in } D_{1}  \tag{10}\\
& \left.\left|\nabla_{x, y} u\right|\right|_{\partial D_{1}^{(1)}}=m_{1}(\varphi)>0 \quad \text { on } \partial D_{1}^{(1)} \\
& \left.\left|\nabla_{x, y} u\right|\right|_{\partial D_{1}^{(2)}=m_{2}(\varphi)>0} \text { on } \partial D_{1}^{(2)} \\
& \partial D_{1}^{(1)} \cap \partial D_{1}^{(2)}=\emptyset
\end{align*}\right.
$$

to the solvability of a bvp of the type (4).

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[^0]:    2010 Mathematics Subject Classification: 35J05, 35J65, 35J67, 30D55
    Key words: Laplace operator in $\mathbf{R}^{2}$, nonlinear boundary value problem, classical solution, generalized solution, Hardy $H_{p}$ space
    *This paper is supported by the bilateral project between Serbian Academy of Sciences and BAS.

