9. Series representation for analytic functions

9.1. Power series.

Definition: A power series is the formal expression

$$S(z) := \sum_{n=0}^{\infty} c_n (z-a)^n, a,\ c_i, i = 0, 1, \cdots, \text{–fixed},\ z \in \mathbb{C}. \qquad (1)$$

The n.th partial sum $S_n(z)$ is the sum of the first $n+1$ terms. We say that the power series does converge for a fixed $z \in \mathbb{C}$, if the sequence $\{S_n(z)\}$ converges, as $n \to \infty$. If $S_n(z)$ diverges, then we say that the power series diverges at the point z. ℵ

We will write $S(z) < \infty$ and $S(z) = \infty$, respectively.

We draw the reader's attention to the fact that the partial sums S_n are polynomials of degree n. A convergent power series is the limes of polynomial sequences, e.g.

$$S(z) := \lim_{k \to \infty} \sum_{n=0}^{k} c_n (z-a)^n, \qquad (2)$$

so that all results from Chpr. 2 are applicable.

Definition: The power series

$$S(z) := \sum_{n=0}^{\infty} c_n (z-a)^n$$

is *absolutely convergent*, if the series with nonnegative terms

$$\sum_{n=0}^{\infty} |c_n||z-a|^n < \infty$$

and *uniformly convergent*, if $S_n \to S$ uniformly in the metric of Chebyshev on compact sets. ℵ

We recall some well known facts.

Theorem 9.1. *Suppose that $S(z) := \sum_{n=0}^{\infty} c_n(z-a)^n$ is absolutely convergent at z_0. Then the power series converges in the regular sense at the same point.*

Proof: Fix an arbitrary ε. Taking into account the absolute convergence, we may write

$$|\sum_{0}^{k+m} |c_n||z_0-a|^n - \sum_{0}^{k} |c_n||z_0-a|^n| \le \varepsilon;$$

this inequality is valid for every k great enough (say $k \ge k_0$) and for every $m \in \mathbb{N}$. It turns out that

$$|S_{k+m}(z_0) - S_k(z_0)| \le \varepsilon, k \ge k_0, m \in \mathbb{N}.$$

Our statement follows immediately from Cauchy's fundamental theorem[1] **Q.E.D.**

Theorem 9.2. *Let $S(z) := \sum_{n=0}^{\infty} c_n(z-a)^n$ be convergent at z_0. Then $c_n(z_0-a)^n \to 0, n \to \infty$.*

Proof: The statement follows from the fact that

$$S_k(z_0) - S_{k-1}(z_0) = c_k(z-a)^k \to 0,\ k \to \infty.$$

Q.E.D.

Theorem 9.3., Abel's theorem *Suppose that$S(z) := \sum_{n=0}^{\infty} c_n(z-a)^n$ converges at z_0. Then S is absolutely convergent at every z such that $|z-a| < |z_0-a|$.*

Proof: Take z with $|z-a| < |z_0-a|$. Fix $\varepsilon > 0$. By the previous theorem,

$$|c_n(z-a)^n| \le \varepsilon$$

for all $n \ge n_0$. We have further,

$$\sum_{0}^{\infty} |c_n||z-a|^n = \sum_{0}^{\infty} |c_n| \frac{|z-a|^n}{|z_0-a|^n} |z_0-a|^n \le$$

$$\sum_{0}^{n_0} |c_n| \frac{|z-a|^n}{|z_0-a|^n} |z_0-a|^n + \sum_{n_{0+1}}^{\infty} |c_n||z_0-a|^n \frac{|z-a|^n}{|z_0-a|^n}$$

[1] *Cauchy's fundamental theorem: the infinite sequence $\{a_n\}$ converges iff for every $\varepsilon > 0$ there exists a number k_0 great enough such that $|a_{k+m} - a_k| < \varepsilon$ whenever $k \ge k_0$ and $m \in \mathbb{N}$.*

$$\leq \sum_{0}^{n_0} |c_n| \frac{|z-a|^n}{|z_0-a|^n} |z_0-a|^n + \sum_{n_{0+1}}^{\infty} \varepsilon \frac{|z-a|^n}{|z_0-a|^n}.$$

Using common notations, we may write

$$\sum_{0}^{\infty} |c_n||z-a|^n \ll \sum_{0}^{\infty} \frac{|z-a|^n}{|z_0-a|^n}.$$

To complete the proof, we need to remember that $|z-a| < |z_0-a|$. **Q.E.D.**
Corollary 9.4. : *Suppose that* $S(z) := \sum_{n=0}^{\infty} c_n(z-a)^n$ *diverges at* z_0. *Then it diverges at every point* z *with* $|z-a| > |z_0-a|$.

Naturally we come to the definition of a *radius of convergence*:

Definition: Given $S(z) := \sum_{n=0}^{\infty} c_n(z-a)^n$, we set

$$R := \sup\{\rho, S(z) < \infty \text{ for } |z-a| < \rho\}.$$

The number R is called *radius of convergence* of the power series. $\aleph$

Regarding Abel's theorem, we conclude that the power series converges in the disk $D_a(R)$ and diverges outside.

Theorem 9.5., H'Adamard 's formula:

$$R = \frac{1}{\limsup_{n\to\infty} |c_n|^{1/n}}.$$

Proof: If $R =$ then we are done. We assume that R is positive. Fix $\rho < R$. We will show that the power series is uniformly convergent on $D_a(\rho)$.

Select a positive number ε in such a way that $\rho + \varepsilon < R$. Viewing the definition of R, we get for every $n > n_0$ (n large enough) n the estimation

$$|c_n| \leq \frac{1}{(R-\varepsilon)^n}.$$

Consequently,

$$\sum_{n=0}^{\infty} |c_n||z-a|^n = \sum_{n=0}^{n=n_0-1} |c_n||z-a|^n + \sum_{n=n_0}^{\infty} |c_n||z-a|^n <$$

$$\sum_{n=0}^{n=n_0-1} |c_n||z-a|^n + \sum_{n=n_0}^{\infty} (\frac{\rho}{R-\varepsilon})^n,$$

or

$$\sum_{n=0}^{\infty}|c_n||z-a|^n \ll \sum_{n=0}^{\infty}(\frac{\rho}{R-\varepsilon})^n.$$

The right-hand side series is a convergent geometric progression, (recall the choice of ε.)

Therefore, $S(z)$ is absolutely convergent, and thus, convergent in the regular sense. **Q.E.D.**

Remark: There is no statement about the behavior on the circle $C_a(R)$.

Theorem 9.6. *Suppose that the power series $S(z) := \sum_{n=0}^{\infty} c_n(z-a)^n$ is of positive radius of convergence R. Then it converges uniformly on compact subsets of the disk $D_a(R)$ and absolutely at every point $z \in D_a(R)$.*

Proof: Fix $\rho < R$. By the previous theorem,

$$\sum_{n=0}^{\infty}|c_n|\rho^n < \infty.$$

Rearranging the difference $S_{n+m} - S_n$ yields

$$\|S_{n+m} - S_n\|_{\overline{D_a(\rho)}} = \|\sum_{k=n+1}^{n+m} c_k(z-a)^k\| \leq \sum_{k=n+1}^{n+m}|c_k|\rho^k.$$

Applying again Cauchy's fundamental theorem, for all n large enough we get $\|S_{n+m} - S_n\|_{\overline{D_a(\rho)}}$, which means a uniform convergence on $\overline{D_a(\rho)}$. **Q.E.D.**

9.2. Taylor's theorem and consequences

Theorem 9.7. (Taylor's theorem) *Let $f \in \mathcal{A}(\overline{\mathcal{D}}_a(\rho)), \rho > 0, a \in \mathbb{C}$. Then f can be represented as a Taylor series*[2] $\sum_{c_n}(z-a)^n$, *which is convergent uniformly inside* $\mathcal{D}_a(\rho)$ and

$$c_n = \frac{1}{2\pi i}\int_{C_a(\rho)} \frac{f(\zeta)}{(\zeta-a)^{n+1}}d\zeta.$$

Proof: In view of the integral formula,

$$f(z) = \frac{1}{2\pi i}\int_{C_a(\rho)} \frac{f(\zeta)}{\zeta - z}d\zeta,\ z \in \mathcal{D}_a(\rho).$$

[2]it if $a = 0$, then we speak about *MacLaurin series.*

We remember that $|z-a| < |\zeta - a|$. Consequently,

$$\frac{1}{2\pi i}\int_{C_a(\rho)} \frac{f(\zeta)}{(\zeta-a)-(z-a)}d\zeta = \frac{1}{2\pi i}\int_{C_a(\rho)} \frac{f(\zeta)}{\zeta-a}\sum_{n=0}^{\infty}(\frac{z-a}{\zeta-a})^n.$$

Using known estimates, we obtain

$$f(z) = \sum_{n=0}^{\infty}(z-a)^n\frac{1}{2\pi i}\int_{C_a(\rho)} \frac{f(\zeta)}{\zeta-a)^{n+1}}d\zeta := \sum c_n(z-a)^n. \qquad (3)$$

Q.E.D.

Remark:

$$c_n = \frac{f^{(n)}(a)}{n!}. \qquad (4)$$

Example: Write down the Taylor series of $\text{Log}z (:= \ln|z| + i\text{Arg}z, .)$ around $z = 1$.

Solution; Since

$$\frac{d^j\ \text{Log}z}{dz^j} = (-1)^{j+1}(j-1)!z^{-j},\ j = 1, 2, \cdots$$

we get

$$\text{Log}z = 0 + (z-1) - (z-1)^2/2! + 2!(z-1)^3/3! - 3!(z-1)^4/4! + \cdots =$$

$$= \sum_{j=1}^{\infty}(-1)^{j+1}(z-1)^j/j.$$

The series converges uniformly on $D_1(r)$ for every $r < 1$.

Theorem 9.8. *Suppose that f is analytic at the point $z = a$, $f(z) = \sum_{c_n}(z-a)^n$,*[3] *Then*

$$f'(z) = \sum_{n=1}^{\infty} nc_n(z-a)^{n-1}.$$

Proof: Indeed, from Chpt. 8 we know that $f'(z)$ is also analytic at $z = a$. From THeorem 9.7, we get

$$f'(z) = \sum_{n=0}^{\infty}\frac{(f'(z))^{(n)}(a)}{n!}(z-a)^n.$$

[3] analytic in some domain $D_a(r)$, $r > 0$.

Recalling that

$$(f'(z))^{(n)}(a) = (f(z))^{(n+1)}(a),$$

we obtain the required statement. **Q.E.D.**

The proof of the following theorems is left to the reader.

Theorem 9.9. *Let the functions $f(z)$ and $g(z)$ be analytic at $z = 1$,*

$$f(z) = \sum f_n(z-a)^n,\ g(z) = \sum g_n(z-a)^n.$$

Denote by $R(f), R(g)$ the radii of convergence of both Taylor series. Then $f \pm g$ and fg are analytic at $z = a$ and

$$R(f \pm g) \geq \min(R(f), R(g)),$$
$$R(fg) \geq \min(R(f), R(g)).$$

and

$$(f \pm g)(z) = \sum (f_n \pm g_n)(z-a)^n,$$

$$f(z)g(z) = \sum c_n(z-a)^n$$

with

$$c_n = \sum_{k=0}^{n} f_k g_{n-k}.$$

9.3. The point of infinity.

Definition: The function $f(z)$, defined at infinity, is said to be *analytic at* $z = \infty$, if the function

$$g(\zeta) := f(\frac{1}{\zeta})$$

is analytic at $\zeta = 0$. ℵ

The checking of the validity of the following theorem is left to the reader:

Theorem 9.10. *Suppose that f is analytic at infinity. Then it is expandable into Taylor series*

$$f(z) = \sum_{n=0}^{\infty} \frac{c_n}{z^n}.$$

The series converges at every point $z, |z| > \limsup |c_n|^{1/n}$ and is uniformly convergent in the exterior of every circle $D_0(R)$ with $R > \limsup |c_n|^{1/n}$.

Exercises:

1. Find the Taylor series of

$$f(z) := z^2 \cos \frac{1}{3z}$$

at $z = 0$.

2. Find the Taylor series of

$$f(z) = \frac{1}{z-2}$$

at $z = \infty$.

3. Find the Taylor series of

$$f(z) = \frac{1}{z(z-2)}$$

at $z = 1$.