9. Series representation for analytic functions


Definition: A power series is the formal expression

\[ S(z) := \sum_{n=0}^{\infty} c_n(z - a)^n, \quad a, c_i, i = 0, 1, \cdots, \text{fixed}, \quad z \in \mathbb{C}. \]  

(1)

The \( n \)th partial sum \( S_n(z) \) is the sum of the first \( n + 1 \) terms. We say that the power series does converge for a fixed \( z \in \mathbb{C} \), if the sequence \( \{S_n(z)\} \) converges, as \( n \to \infty \). If \( S_n(z) \) diverges, then we say that the power series diverges at the point \( z \).

We will write \( S(z) < \infty \) and \( S(z) = \infty \), respectively.

We draw the reader’s attention to the fact that the partial sums \( S_n \) are polynomials of degree \( n \). A convergent power series is the limes of polynomial sequences, e.g.

\[ S(z) := \lim_{k \to \infty} \sum_{n=0}^{k} c_n(z - a)^n, \]  

(2)

so that all results from Chpr. 2 are applicable.

Definition: The power series

\[ S(z) := \sum_{n=0}^{\infty} c_n(z - a)^n \]

is absolutely convergent, if the series with nonnegative terms

\[ \sum_{n=0}^{\infty} |c_n||z - a|^n < \infty \]

and uniformly convergent, if \( S_n \to S \) uniformly in the metric of Chebyshev on compact sets.

We recall some well known facts.
Theorem 9.1. Suppose that \( S(z) := \sum_{n=0}^{\infty} c_n(z - a)^n \) is absolutely convergent at \( z_0 \). Then the power series converges in the regular sense at the same point.

**Proof:** Fix an arbitrary \( \varepsilon \). Taking into account the absolute convergence, we may write
\[
|\sum_{n=0}^{k+m} c_n||z_0 - a|^n - \sum_{n=0}^{k} c_n||z_0 - a|^n| \leq \varepsilon;
\]
this inequality is valid for every \( k \) great enough (say \( k \geq k_0 \)) and for every \( m \in \mathbb{N} \). It turns out that
\[
|S_{k+m}(z_0) - S_k(z_0)| \leq \varepsilon, k \geq k_0, m \in \mathbb{N}.
\]
Our statement follows immediately from Cauchy’s fundamental theorem\(^1\) Q.E.D.

Theorem 9.2. Let \( S(z) := \sum_{n=0}^{\infty} c_n(z - a)^n \) be convergent at \( z_0 \). Then \( c_n(z_0 - a)^n \to 0, n \to \infty \).

**Proof:** The statement follows from the fact that \( S_k(z_0) - S_{k-1}(z_0) = c_k(z - a)^k \to 0, k \to \infty \). Q.E.D.

Theorem 9.3., Abel’s theorem Suppose that \( S(z) := \sum_{n=0}^{\infty} c_n(z - a)^n \) converges at \( z_0 \). Then \( S \) is absolutely convergent at every \( z \) such that \( |z - a| < |z_0 - a| \).

**Proof:** Take \( z \) with \( |z - a| < |z_0 - a| \). Fix \( \varepsilon > 0 \). By the previous theorem,
\[
|c_n(z - a)^n| \leq \varepsilon
\]
for all \( n \geq n_0 \). We have further,
\[
\sum_{0}^{\infty} |c_n||z - a|^n = \sum_{0}^{\infty} |c_n||z - a|^n|z_0 - a|^n \leq \\
\sum_{0}^{n_0} |c_n||z - a|^n |z_0 - a|^n + \sum_{n_0+1}^{\infty} |c_n||z_0 - a|^n \frac{|z - a|^n}{|z_0 - a|^n}
\]
\(^1\)Cauchy’s fundamental theorem: the infinite sequence \( \{a_n\} \) converges iff for every \( \varepsilon > 0 \) there exists a number \( k_0 \) great enough such that \( |a_{k+m} - a_k| < \varepsilon \) whenever \( k \geq k_0 \) and \( m \in \mathbb{N} \).
\[
\leq \sum_{n=0}^{n_0} |c_n| \frac{|z - a|^n}{|z_0 - a|^n} |z_0 - a|^n + \sum_{n=n_0+1}^{\infty} \varepsilon |z - a|^n |z_0 - a|^n.
\]

Using common notations, we may write
\[
\sum_{n=0}^{\infty} |c_n| |z - a|^n \ll \sum_{n=0}^{\infty} \varepsilon |z - a|^n.
\]

To complete the proof, we need to remember that \(|z - a| < |z_0 - a|\). \textbf{Q.E.D.}

\textbf{Corollary 9.4}: Suppose that \(S(z) := \sum_{n=0}^{\infty} c_n (z - a)^n\) diverges at \(z_0\). Then it diverges at every point \(z\) with \(|z - a| > |z_0 - a|\).

Naturally we come to the definition of a \textit{radius of convergence}:

\textbf{Definition}: Given \(S(z) := \sum_{n=0}^{\infty} c_n (z - a)^n\), we set
\[
R := \sup \{ \rho \mid S(z) < \infty \text{ for } |z - a| < \rho \}.
\]

The number \(R\) is called \textit{radius of convergence} of the power series.

Regarding Abel’s theorem, we conclude that the power series converges in the disk \(D_{\rho}(R)\) and diverges outside.

\textbf{Theorem 9.5, H’Adamard ’s formula}:

\[
R = \frac{1}{\limsup_{n \to \infty} |c_n|^{1/n}}.
\]

\textbf{Proof}: If \(R = 0\) then we are done. We assume that \(R\) is positive. Fix \(\rho < R\). We will show that the power series is uniformly convergent on \(D_{\rho}(R)\).

Select a positive number \(\varepsilon\) in such a way that \(\rho + \varepsilon < R\). Viewing the definition of \(R\), we get for every \(n > n_0\) (\(n\) large enough) \(n\) the estimation
\[
|c_n| \leq \frac{1}{(R - \varepsilon)^n}.
\]

Consequently,
\[
\sum_{n=0}^{\infty} |c_n| |z - a|^n = \sum_{n=0}^{n_0-1} |c_n| |z - a|^n + \sum_{n=n_0}^{\infty} |c_n| |z - a|^n <
\]
\[
\sum_{n=0}^{n_0-1} |c_n| |z - a|^n + \sum_{n=n_0}^{\infty} \left( \frac{\rho}{R - \varepsilon} \right)^n,
\]

3
or
\[ \sum_{n=0}^{\infty} |c_n||z - a|^n \ll \sum_{n=0}^{\infty} \left( \frac{\rho}{R - \varepsilon} \right)^n. \]

The right-hand side series is a convergent geometric progression, (recall the choice of \( \varepsilon \).)
Therefore, \( S(z) \) is absolutely convergent, and thus, convergent in the regular sense. \textbf{Q.E.D.}

\textbf{Remark:} There is no statement about the behavior on the circle \( C_a(R) \).

\textbf{Theorem 9.6.} Suppose that the power series \( S(z) := \sum_{n=0}^{\infty} c_n(z - a)^n \) is of positive radius of convergence \( R \). Then it converges uniformly on compact subsets of the disk \( D_a(R) \) and absolutely at every point \( z \in D_a(R) \).

\textbf{Proof:} Fix \( \rho < R \). By the previous theorem,
\[ \sum_{n=0}^{\infty} |c_n|\rho^n < \infty. \]

Rearranging the difference \( S_{n+m} - S_n \) yields
\[ \|S_{n+m} - S_n\|_{D_a(\rho)} = \| \sum_{k=n+1}^{n+m} c_k(z - a)^k \| \leq \sum_{k=n+1}^{n+m} |c_k|\rho^k. \]

Applying again Cauchy’s fundamental theorem, for all \( n \) large enough we get \( \|S_{n+m} - S_n\|_{D_a(\rho)} \), which means a uniform convergence on \( \overline{D_a(\rho)} \). \textbf{Q.E.D.}

\textbf{9.2. Taylor’s theorem and consequences}

\textbf{Theorem 9.7.} (Taylor’s theorem) Let \( f \in A(\overline{D_a(\rho)}), \rho > 0, a \in \mathbb{C} \). Then \( f \) can be represented as a Taylor series\(^2\) \( \sum_{n} (z - a)^n \), which is convergent uniformly inside \( \mathcal{D}_a(\rho) \) and
\[ c_n = \frac{1}{2\pi i} \int_{C_a(\rho)} \frac{f(\zeta)}{(\zeta - a)^{n+1}} d\zeta. \]

\textbf{Proof:} In view of the integral formula,
\[ f(z) = \frac{1}{2\pi i} \int_{C_a(\rho)} \frac{f(\zeta)}{\zeta - z} d\zeta, \quad z \in \mathcal{D}_a(\rho). \]

\(^2\)it if \( a = 0 \), then we speak about MacLaurin series.
We remember that $|z - a| < |\zeta - a|$. Consequently,

$$
\frac{1}{2\pi i} \int_{C_n(\rho)} \frac{f(\zeta)}{(\zeta - a) - (z - a)} d\zeta = \frac{1}{2\pi i} \int_{C_n(\rho)} \frac{f(\zeta)}{\zeta - a} \sum_{n=0}^{\infty} \frac{(z - a)}{(\zeta - a)^n}.
$$

Using known estimates, we obtain

$$
f(z) = \sum_{n=0}^{\infty} (z - a)^n \frac{1}{2\pi i} \int_{C_n(\rho)} \frac{f(\zeta)}{\zeta - a} \frac{1}{(\zeta - a)^{n+1}} d\zeta := \sum_n c_n (z - a)^n. \quad (3)
$$

**Q.E.D.**

**Remark:**

$$
c_n = \frac{f^{(n)}(a)}{n!}. \quad (4)
$$

**Example:** Write down the Taylor series of $\text{Log} z := \ln |z| + i \text{Arg} z$ around $z = 1$.

**Solution:** Since

$$
\frac{d^j}{dz^j} \text{Log} z = (-1)^{j+1} (j - 1)! z^{-j}, \ j = 1, 2, \ldots
$$

we get

$$
\text{Log} z = 0 + (z - 1) - (z - 1)^2/2! + 2!(z - 1)^3/3! - 3!(z - 1)^4/4! + \cdots = \sum_{j=1}^{\infty} (-1)^{j+1} (z - 1)^j / j.
$$

The series converges uniformly on $D_1(r)$ for every $r < 1$.

**Theorem 9.8.** Suppose that $f$ is analytic at the point $z = a$, $f(z) = \sum c_n (z - a)^n$,\(^3\) Then

$$
f'(z) = \sum_{n=1}^{\infty} nc_n (z - a)^{n-1}.
$$

**Proof:** Indeed, from Chpt. 8 we know that $f'(z)$ is also analytic at $z = a$. From Theorem 9.7, we get

$$
f'(z) = \sum_{n=0}^{\infty} \frac{(f'(z))^{(n)}(a)}{n!} (z - a)^n.
$$

\(^3\)analytic in some domain $D_n(r), \ r > 0$. 

Recalling that 
\[(f'(z))^n(a) = (f(z))^{n+1}(a),\]
we obtain the required statement. Q.E.D.

The proof of the following theorems is left to the reader.

**Theorem 9.9.** Let the functions \(f(z)\) and \(g(z)\) be analytic at \(z = 1\),

\[f(z) = \sum f_n(z-a)^n, \quad g(z) = \sum g_n(z-a)^n.\]

Denote by \(R(f), R(g)\) the radii of convergence of both Taylor series. Then \(f \pm g\) and \(fg\) are analytic at \(z = a\) and

\[R(f \pm g) \geq \min(R(f), R(g)), \quad R(fg) \geq \min(R(f), R(g)).\]

and

\[(f \pm g)(z) = \sum (f_n \pm g_n)(z-a)^n,\]
\[f(z)g(z) = \sum c_n(z-a)^n\]

with

\[c_n = \sum_{k=0}^{n} f_k g_{n-k}.\]

**9.3. The point of infinity.**

**Definition:** The function \(f(z)\), defined at infinity, is said to be **analytic at** \(z = \infty\), if the function

\[g(\zeta) := f\left(\frac{1}{\zeta}\right)\]

is analytic at \(\zeta = 0\).

The checking of the validity of the following theorem is left to the reader:

**Theorem 9.10.** Suppose that \(f\) is analytic at infinity. Then it is expandable into Taylor series

\[f(z) = \sum_{n=0}^{\infty} \frac{c_n}{z^n}.\]

The series converges at every point \(z, |z| > \lim sup |c_n|^{1/n}\) and is uniformly convergent in the exterior of every circle \(D_0(R)\) with \(R > \lim sup |c_n|^{1/n} \).
Exercises:
1. Find the Taylor series of
   \[ f(z) := z^2 \cos \frac{1}{3z} \]
   at \( z = 0 \).
2. Find the Taylor series of
   \[ f(z) = \frac{1}{z - 2} \]
   at \( z = \infty \).
3. Find the Taylor series of
   \[ f(z) = \frac{1}{z(z - 2)} \]
   at \( z = 1 \).