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FOUNDATIONS OF OPERATIONAL CALCULI FOR THE BESSEL-TYPE DIFFERENTIAL OPERATORS

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A survey of some results of the author for constructing of operational calculi for Bessel-type linear differential operators

$$B=t^{\alpha_0}\frac{d}{dt}t^{\alpha_1}\ldots t^{\alpha_{m-1}}\frac{d}{dt}t^{\alpha_m}$$

with $\beta=m-\alpha_0-\alpha_1-\ldots-\alpha_m>0$ is given. As a basis of the corresponding operational calculi is taken the right inverse linear operator L of B, i. e. $BL=I$, defined with zero initial conditions. It is considered in a space C_α of continuous in $(0, \infty)$ functions with a power singularity in $t=0$. The basic role in these operational calculi is played by the notion convolution $a * b$ of L in C_α, understanded as a bilinear, commutative and associative operation with $L(a * b)=(La) * b$. (1) A family of convolutions for L in C_α is given, and separately a simpler convolution. (2) A transform approach is given on an integral transformation, introduced by N. Obreshkoff in 1958. An explicit expression of the convolution of the modified Obrechkoff transformation is given. (3) An explicit expression of a Volterra transformation of the first kind is found, serving as a similatude operator of L and the operator of multiple integration l^m. By means of this transformation, an effective way for application of the usual Laplace transformation for a functional construction of the operational calculi for the operators of the Bessel type. It is proved the isomorphism of all the quotient fields, generated by continuous convolutions for the different Bessel-type linear differential operators.

1. Introduction. The class of the linear differential operators of Bessel type can be defined as consisting of all the differential operators of the form

$$B=t^{\alpha_0}\frac{d}{dt}t^{\alpha_1}\frac{d}{dt}\ldots t^{\alpha_{m-1}}\frac{d}{dt}t^{\alpha_m}. \tag{1}$$

Here we restrict ourselves to the case of real α_k, $k=0, 1, \ldots, m$, satisfying the condition

$$\alpha_0+\alpha_1+\ldots+\alpha_m<m. \tag{2}$$

The simplest example of such an operator is the differentiation operator $\frac{d}{dt}$. An important example is the usual Bessel differential operator

$$t^{-\nu-1}\frac{d}{dt}t^{2\nu+1}\frac{d}{dt}t^{-\nu}=\frac{d^2}{dt^2}+\frac{1}{t}\frac{d}{dt}-\frac{\nu^2}{t^2}.$$

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Let us make a brief survey of the-approaches to opperational calculi for Bessel-type differential operators, proposed by different authors. The direct algebraic approach of J. Mikusiński [1] to the operational calculus for the differentiation operator has been extended to the differential operator $\frac{dt}{dt}t\frac{d}{dt}$ by V. A. Ditkin (1957) [2]. In [3] this algebraic approach was supplemented by an integral transformation — a variant of the Meijer transformation of order 0. Another example of such an operational calculus was proposed by N. A. Meller (1961) [4]. She has given algebraic foundations of an operational calculus for the operator $t^{-\alpha}\frac{d}{dt}t^{\alpha+1}\frac{d}{dt}$ with $-1<\alpha<1$. A transform approach to this operational calculus was proposed by E. Krätzel (1965) [5]. A. P. Prudnikov (1962) [6] proposed algebraic and transform approaches to operational calculus for the operator $\frac{d}{dt}t\frac{d}{dt}t\frac{d}{dt}$ In the same year V. A. Ditkin and A. P. Prudnikov [7] have developed an operational calculus for the more general Bessel-type differential operator $\frac{1}{t}\left(t\frac{d}{dt}\right)^n$. The same was done simultaneously by A. I. Botashev [8]. A more involved example of an operational calculus for the Bessel-type differential operator

$$\frac{d}{dt}t^{\frac{1}{n}-\nu}\left(t^{1-\frac{1}{n}}\frac{d}{dt}\right)^{n-1}t^{\nu+1-\frac{2}{n}}$$

was considered by E. Krätzel [9]. In [10]—[12] the author has proposed an algebraic approach to an operational calculus for the general Bessel-type differential operator (1). The latest paper on an operational approach to Bessel-type differential operator was [13]. Its authors, V. A. Ditkin and A. P. Prudnikov, have considered the operator $t^{1-n}\left(\frac{d}{dt}\right)^n t^n\left(\frac{d}{dt}\right)^n t^{n-1}$.

It happens that, all the integral transformations, proposed by different authors for operational calculi for Bessel-type differential operators are only special cases of an integral transformation, proposed by the Bulgarian mathematician N. Obrechkoff (1958) [14]. We shall show that, a modification of this integral transformation of N. Obrechkoff can be used as a transform approach to the operational calculus for the general Bessel-type differential operator (1).

The objective of this investigation is to collect into a coherent whole all the results about operational calculi for Bessel-type differential operators. A new outlook on the problem of developing of an effective operational approach to the general Bessel-type differential operator (1) gives a new theorem we are to prove, stating the isomorphism of all these operational calculi. A special interest presents the isomorphism of a quotient field, generated by (1), with the field of Mikusiński. This isomorphism is given in an explicit and comparatively simple form to allow an immediate transfer of almost all results from the classical Heaviside-Mikusiński operationae calculus to an operational calculus for the general Bessel-type differential operator (1). This isomorphism opens also a way for applying of the Laplace transform to the operator (1).

2. Formulating of the problem of constructing of an operational calculus for the general Bessel-type differential operator. What we mean when we say "operational calculus for a Bessel-type differential operator (1)"? The problem of finding of a Mikusiński-type operational calculus for (1) can be stated in the following way (see [10]). We consider the operator B not directly, but by means of a right inverse operator L of it. This is the linear integral operator

$$La(t)=\frac{t^{\beta}}{\beta^{m}}\int_0^1\ldots\int_0^1 a(t(t_1\ldots t_m)^{\frac{1}{\beta}})\prod_{k=1}^{m} t_k^{\gamma'_k}dt_1\ldots dt_m$$

with $\beta=m-\alpha_0-\alpha_1-\ldots-\alpha_m$ and $\gamma'_k=(\alpha_k+\ldots+\alpha_m-m+k)/\beta$, $k=1, 2, \ldots, m$. We consider the operator L in the space C_a of the functions of the form $a(t)=t^p\widetilde{a}(t)$ in $(0, \infty)$ with an arbitrary $p>\alpha$, where

$$\alpha=\max_{0\leq k\leq m-1}(\alpha_0+\alpha_1+\ldots+\alpha_k-k-1),$$

and with continuous functions $\widetilde{a}(t)$ in the interval $[0, \infty)$. For the sake of simplification, let us rearange the sequence γ'_k, $k=1, 2, \ldots, m$, in increasing order: $\gamma_1\leq\gamma_2\leq\ldots\leq\gamma_m$. Let us also denote the unit m-dimensional cube

$$\{(t_1, \ldots, t_m): 0\leq t_k\leq 1,\ k=1, \ldots, m\}$$

by e_m, writing $de_m=dt_1\ldots dt_m$. Then our operator L takes the form

$$La(t)=\frac{t^{\beta}}{\beta^{m}}\int_{e_m} a[t(t_1\ldots t_m)^{\frac{1}{\beta}}]\prod_{k=1}^{m} t_k^{\gamma_k}de_m, \tag{3}$$

where $a(t)\in C_\alpha$ with $\alpha=-\beta(\gamma_1+1)$. The condition $\beta>0$, equivalent to (2) implies $L: C_\alpha\rightarrow C_\alpha$.

In the basis of our algebraic approach to an operational calculus for the generai Bessel-type differential operator (1), or more precisely, for the operator L, we put the notion convolution of linear operator.

Definition 1. *A convolution of a linear operator U in linear space C with $U: C\rightarrow C$ is called each inner operation $a*b$ in C, which is bilinear, commutative and associative, and satisfies the condition*

$$U(a*b)=(Ua)*b \tag{4}$$

for all $a, b\in C$.

Each convolution of U in C introduces a structure of commutative and associative algebra in the linear space C. The operator U is a multiplier in this algebra.

Given a linear operator U in linear space C, with $U: C\rightarrow C$, the problem of finding of at least one nonzero convolution of U in C is a nontrivial one.

Any convolution of U in C can be used as a basis for a Mikusiński-type operational calculus for the linear operator U provided there exist nondivisors of zero. If such a convolution is free of divisors of zero, the development of such an operational calculus should proceed in many respects as in the Mikusiński operational calculus [1]. In this case we should embed the algebra C into its quotient field. The element Ua/a, as it follows from (4), does not depend on the element $a\in C$ and can be identified with the operator U itself. Then we could consider rational expression in U, fractional powers of U and other elements of the quotient field of C, and in some cases to interpret them as elements of C.

The problem of finding a nontrivial convolution of a linear operator U in a linear space C, $U:C\to C$ could be successfully solved provided we know a convolution of a linear operator $\widetilde{U}$, similar to U (see [15]). Let $\varphi:C\to\widetilde{C}$ be an invertible linear transformation of C in $\widetilde{C}$. Then, if $f\tilde{*}g$ is a convolution of the linear operator $\widetilde{U}=\varphi U\varphi^{-1}$ in $\widetilde{C}$, the operation

$$(5)\qquad a*b=\varphi^{-1}[(\varphi a)\tilde{*}(\varphi b)]$$

is a convolution of U in C. This very simple theorem of N. A. Meller [15] explains the uses of integral transformations for developing of operational calculi for different linear operators.

In the case of our right inverse operator L of the general Bessel-type differential operator (1) the first step to the solving of the problem of constructing of an operational calculus is to find an explicit expression of a convolution of L in C_α. In the following section such expressions are given for infinetely many convolutions of L in C_α.

In a transform approach to an operational calculus for the operator L, the first step should be to find a linear invertible transformation $\mathfrak{T}:C_\alpha\to\widetilde{C}$ from C_α to a linear space of analytic functions $\widetilde{C}$, such that $\mathfrak{T}(La)=\lambda.(\mathfrak{T}a)$, where λ is a fixed function. In fact it is hard to find such a transformation of the whole space. As in the case of the Laplace transformation, this transformation can be defined on a subspace of C_α. We shall use a transformation, which is a modification of a known integral transformation of N. Obrechkoff (see [14]).

3. Algebraic foundations of operational calculi for the general Bessel-type differential operator. Now we give an explicit expression for a convolution of L in C_α. This convolution shall play a basic role in the following considerations. In this expression takes part an integer s, which characterizes the sequence $\gamma_1\leq\gamma_2\leq\ldots\leq\gamma_m$ of exponents in (3). If among the numbers γ_k, $k=1,\ldots,m$, there are different, we define s as an integer, such that

$$\gamma_1\leq\gamma_2\leq\ldots\leq\gamma_s<\gamma_{s+1}=\ldots=\gamma_m,$$

and as $s=0$, when $\gamma_1=\gamma_2=\ldots=\gamma_m$.

Definition 2. *Let a, $b\in C_\alpha$ with $\alpha=-\beta(\gamma_1+1)$. Then*

$$(6)\qquad a*b=T(a\circ b),$$

where

$$(a\circ b)(t)=t^\beta\int\limits_{e_m}a[t(t_1\ldots t_m)^{1/\beta}]b[t((1-t_1)\ldots(1-t_m))^{1/\beta}]\prod_{k=1}^{m}[t_k(1-t_k)]^{\gamma_k}\,de_m$$

and

$$(7)\qquad Ta(t)=[t^{\gamma_m\beta}/\prod_{k=1}^{s}\Gamma(\gamma_m-\gamma_k)]\int_{e_s}a[t(t_1\ldots t_s)^{1/\beta}]\prod_{k=1}^{s}t_k^{2\gamma_k}(1-t_k)^{\gamma_m-\gamma_k-1}\,de_s$$

for $s>0$, *and* $Ta(t)=t^{\gamma_m\beta}a(t)$ for $s=0$.

Theorem 1. *The operation* $a*b$ *is defined for all* $a, b\in C_\alpha$ *and is a convolution of* L *in* C_α, *free of divisors of zero.*

For a proof, see [11]. Let us note that $a\circ b\in C_{2\alpha+\beta}$, and the linear operator T is defined exactly in $C_{2\alpha+\beta}$ and $T: C_{2\alpha+\beta}\to C_{\alpha+\beta(\gamma_m-\gamma_1)}\subset C_\alpha$. Hence $a*b$ is an inner operation in C_α. By a simple calculation, we receive

$$(8)\quad \{t^p\}*\{t^q\}=t^{p+q+(\gamma_m+1)\beta}\prod_{k=1}^{m}\Gamma\Big(\frac{p}{\beta}+\gamma_k+1\Big)\Gamma\Big(\frac{q}{\beta}+\gamma_k+1\Big)/\Gamma\Big(\frac{p+q}{\beta}+\gamma_k+\gamma_m+2\Big)$$

for $p, q>\alpha$.

Now, using the convolution (6) of the operator L in C_α, is possible to develop a Mikusiński-type operational calculus. In the corresponding quotient field of C_α we can consider rational and other expressions of the element $S=1/L$.

We shall restrict ourselves with the definition of fractional powers of L (or S). If p is an arbitrary real number, then

$$L^p=S^{-p}=L^{[p]-l}\Big\{t^{(\{p\}+l-\gamma_m-1)\beta}/\prod_{k=1}^{m}\Gamma(\{p\}+l-\gamma_m+\gamma_k)\Big\},$$

where $l=[\alpha/\beta+\gamma_m+2]$. Here we denote by $[p]$ and $\{p\}$ the integer and the fractional part of p, respectively. Using (8), it is easy to verify that $L^p.L^q=L^{p+q}$ for real p, q.

It is seen at once that for the cosidered by V. A. Ditkin and A. P. Prudnikov [7] operator $\frac{1}{t}\Big(t\frac{d}{dt}\Big)^n$, the convolution (6) coincides with theirs. In this case we have simply $a*b=a\circ b$. In the case of the operator $t^{-\alpha}\frac{d}{dt}t^{\alpha+1}\frac{d}{dt}$, considered by N. A. Meller and E. Krätzel [4] and [5], the convolution (6) seems to be a simpler one. It deserves to mention that (6) is defined for each real α, without the restriction $-1<\alpha<1$.

Now, we shall introduce infinitely many convolutions of L in C_α, depending on a real parameter.

Definition 3. *Let* λ *be an arbitrary real number with* $\lambda>\gamma_m$, *and* T_λ, $T_\lambda: C_{2\alpha+\beta}\to C_\alpha$, *be the linear operator*

$$(9)\qquad T_\lambda a(t)=\frac{t^{\lambda\beta}}{\prod_{k=1}^{m}\Gamma(\lambda-\gamma_k)}\int_{e_m}a[t(t_1\ldots t_m)^{\frac{1}{\beta}}]\prod_{k=1}^{m}t_k^{2\gamma_k}(1-t_k)^{\lambda-\gamma_k-1}\,de_m.$$

Then

$$(10)\qquad [a*b]_\lambda=T_\lambda(a\circ b).$$

T h e o r e m 2. *The operation* $[a*b]_\lambda$ *is a convolution of* L *in* C_α, *free of divisors of zero.*

For a proof, see [10]. Let us note that the convolutions (10) are in some respects more involved than the convolution (6).

The convolutions (6) and (10) were both found without any reference to integral transformations. However, it happens that for both can be proposed, a posteriori, integral transformations, such that, (6) and (10) to be their convolutions. These transformations are modifications of the integral transformation of N. O b r e c h k o f f [14].

4. Transform approach to the operational calculus for the general Bessel-type differential operator, by means of a modification of the integral transformation of N. Obrechkoff. In 1958 the Bulgarian mathematician N. Obrechkoff had proposed and investigated (see [14]) the integral transformation

$$F(z)=\int_0^\infty \Phi(zt)f(t)dt \tag{11}$$

with the kernel

$$\Phi(z)=\int_0^\infty\dots\int_0^\infty u_1^{\alpha_1}\dots u_m^{\alpha_m}\exp(-u_1-\dots-u_m-z/u_1\dots u_m)du_1\dots du_m,$$

where $\alpha_1,\dots,\alpha_m$ are arbitrary real numbers.

The transformation (11) is defined in the linear subspace of C_α, $\alpha=\max(-1,\alpha_1+1,\dots,\alpha_m+1)$ for functions $f(t)=O(\exp\lambda t^{1/(m+1)})$ for $t\to\infty$, with arbitrary $\lambda>0$. For such a function $f(t)$, its Obrechkoff transform $F(z)$ is an analytic function in the halfplane $\operatorname{Re} z>\lambda$.

The Obrechkoff integral transformation (11) can be used as a transform basis of an operational calculus for the Bessel-type differential operator

$$t^{-\alpha_1-1}\frac{d}{dt}t^{\alpha_1-\alpha_2+1}\frac{d}{dt}t^{\alpha_2-\alpha_3+1}\frac{d}{dt}\dots t^{\alpha_{m-1}-\alpha_m+1}\frac{d}{dt}t^{\alpha_m}\frac{d}{dt}.$$

It is near to suppose that a slight modification of (11) could serve as a transform basis for an operational calculus for the general differential operator of Bessel type (1).

D e f i n i t i o n 4. *The modified Obrechkoff transformation, corresponding to the Bessel-type differential operator* (1), *we call the transformation*

$$F(z)=\Re\{f\}=\beta\int_0^\infty \Phi[(zt)^\beta]t^{(\gamma_m+1)\beta-1}f(t)dt, \tag{12}$$

with the kernel-function

$$\Phi(z)=\int_0^\infty\dots\int_0^\infty \exp(-u_1-\dots-u_{m-1}-z/u_1\dots u_{m-1})\prod_{k=1}^{m-1}u_k^{\gamma_k-\gamma_m-1}du_1\dots du_{m-1}$$

for the functions of C_α, $\alpha=-\beta(\gamma_1+1)$, *which are* $O(\exp \lambda t^{\beta/m})$ for $t\to\infty$, *with real* $\lambda>0$.

We consider $F(z)$ as an analytic function. It exists in the domain $D_f=\{z:\operatorname{Re} z>\lambda\}\cap\{z:|\arg z|<\pi\beta/2m\}$.

There exists a simple relation between the modified Obrechkoff transformation (12) and the m-dimensional Laplace transformation

$$\mathcal{L}_m\{f(t_1,\ldots,t_m)\}=\int_0^\infty\ldots\int_0^\infty \exp(-z_1t_1-\ldots-z_mt_m)f(t_1,\ldots,t_m)dt_1\ldots dt_m$$

T h e o r e m 3. *Let* $F(z)$ *be the modified Obrechkoff transform* (12) *of a function* $f(t)\in C_\alpha$, $f(t)=O(\exp\lambda t^{\beta/m})$ for $t\to\infty$ *with* $\lambda>0$. *Then*

$$F[(z_1\ldots z_m)^{1/\beta}]=\left(\prod_{k=1}^{s} z_k^{\gamma_k-\gamma_m}\right)\mathcal{L}_m\left\{f[(t_1\ldots t_m)^{1/\beta}]\prod_{k=1}^{m}t_k^{\gamma_k}\right\}. \tag{13}$$

P r o o f. The relation (13) can be written so:

$$\beta\int_0^\infty\left[\int_0^\infty\ldots\int_0^\infty \exp(-u_1-\ldots-u_m-(z_1\ldots z_{m-1}t^\beta)/(u_1\ldots u_{m-1})\right.$$

$$\left.\times\prod_{k=1}^{m-1}u_k^{\gamma_k-\gamma_m-1}du_1\ldots du_{m-1}\right]t^{\beta(\gamma_m+1)-1}f(t)dt$$

$$=\prod_{k=1}^{s}z_k^{\gamma_k-\gamma_m}\int_0^\infty\ldots\int_0^\infty\exp(-z_1t_1-\ldots-z_mt_m)f[t(t_1\ldots t_m)^{1/\beta}]\prod_{k=1}^{m}t_k^{\gamma_k}dt_1\ldots dt_m.$$

According to the principle of the analytical continuation, it is sufficient to prove this identity only for real $z_k>0$, $k=1,\ldots,m$. If we consider the left part of the above identity as an m-dimensional integral, we can do the following change of the variables $u_1,\ldots,u_{m-1},t$:

$$u_k=z_kt_k,\quad k=1,2,\ldots,m-1,\quad t=(t_1\ldots t_m)^{1/\beta}.$$

Though the integral is improper, it is readily seen that, for a function $f(t)$, satisfying the condition of the theorem, we can use the standard theorem for the change of variables in multiple integrals. After routine calculations, the left part of the identity, we are to prove, can be transformed into the right part.

What allows to use the modified Obrechkoff transformation for developing of an operational calculus for the general Bessel-type differential operator? The answer is given by the following theorem.

T h e o r e m 4. *Let* $f(t)\in C_\alpha$ *be* $O(\exp\lambda t^{\beta/m})$ *for* $t\to\infty$. *Then*

$$\mathfrak{R}\{L\,f(t)\}=\frac{1}{z^\beta}\,\mathfrak{R}\{f(t)\}, \tag{14}$$

where L is the operator (3) *and* $\Re$ *is the modified Obrechkoff transformation* (12).

Proof. We shall use the relation (13) and the well known property of the m-dimensional Laplace transformation:

$$(z_1 \dots z_m)^{-1}\mathfrak{L}_m\{f(t_1, \dots, t_m)\} = \mathfrak{L}_m\left\{\int_0^{t_1} \dots \int_0^{t_m} f(\tau_1, \dots, \tau_m)d\tau_1 \dots d\tau_m\right\}.$$

The identity (14), we are to prove, is equivalent to

$$\mathfrak{L}_m\left\{\prod_{k=1}^m t_k^{\gamma_k+1} \int_{e_m} f[(t_1\tau_1 \dots t_m\tau_m)^{1/\beta}] \prod_{k=1}^m \tau_k^{\gamma_k} de_m\right\}$$

$$= (z_1 \dots z_m)^{-1}\mathfrak{L}_m\left\{f[(t_1 \dots t_m)^{1/\beta}] \prod_{k=1}^m t_k^{\gamma_k}\right\}.$$

Now, by introducing the multiplier $(z_1 \dots z_m)^{-1}$ in the right part into the sign $\mathfrak{L}_m$, we receive an evident identity. Hence, (14) is proved.

A basic problem connected with the modified Obrechkoff transformation is that of finding an explicit expression for the convolution of this transformation. To this end, let us use the relation (13).

Let $F(z) = \Re\{f(t)\}$ and $G(z) = \Re\{g(t)\}$ be the modified Obrechkoff transforms of the functions $f(t)$, $g(t) \in C_\alpha$, which are $O(\exp \lambda t^{\beta/m})$ with the same $\lambda > 0$. It is not a priori clear whether the function $H(z) = F(z).G(z)$ is an image by (12) of a function $h(t)$, or not.

From (13) we have

$$H[(z_1 \dots z_m)^{1/\beta}] = F[(z_1 \dots z_m)^{1/\beta}]G[(z_1 \dots z_m)^{1/\beta}]$$

$$= \left(\prod_{k=1}^s z_k^{2(\gamma_k-\gamma_m)}\right)\mathfrak{L}_m\left\{f(t_1 \dots t_m)^{1/\beta}] \prod_{k=1}^m t_k^{\gamma_k}\right\}\mathfrak{L}_m\left\{g[(t_1 \dots t_m)^{1/\beta}] \prod_{k=1}^m t_k^{\gamma_k}\right\}.$$

Now, let us use the Faltung-theorem of the m-dimensional Laplace transformation:

$$\mathfrak{L}_m\{a(t_1, \dots, t_m)\}\mathfrak{L}_m\{b(t_1, \dots, t_m)\}$$

$$= \mathfrak{L}_m\left\{\int_0^{t_1} \dots \int_0^{t_m} a(t_1-\tau_1, \dots, t_m-\tau_m)b(\tau_1, \dots, \tau_m)d\tau_1 \dots d\tau_m\right\}.$$

If we denote $\tilde{h}(t) = (f \circ g)(t)$, we get

$$H[(z_1 \dots z_1)^{1/\beta}] = \prod_{k=1}^s z_k^{2(\gamma_k-\gamma_m)} \mathfrak{L}_m\left\{\left(\prod_{k=1}^m t_k^{2\gamma_k}\right)\tilde{h}[(t_1 \dots t_m)^{1/\beta}]\right\}.$$

It remains only to introduce the multiplier $\prod_{k=1}^{s} z_k^{\gamma_k-\gamma_m}$ into the sign $\mathfrak{L}_m$. Thus we get

$$H[(z_1 \ldots z_m)]^{1/\beta}] = \left(\prod_{k=1}^{s} z_k^{\gamma_k-\gamma_m}\right)\mathfrak{L}_m\{\int_0^{t_1}\ldots\int_0^{t_s}\prod_{k=1}^{s}\frac{(t_k-\tau_k)^{\gamma_m-\gamma_k-1}}{\Gamma(\gamma_m-\gamma_k)}\prod_{k=1}^{s}\tau_k^{2\gamma_k}$$

$$\times(t_{s+1}\ldots t_m)^{2\gamma_m}\tilde{h}[(\tau_1\ldots\tau_s t_{s+1}\ldots t_m)^{1/\beta}]d\tau_1\ldots d\tau_s\}.$$

Now it is seen that there exists a function $h(t)\in C_\alpha$, $h(t)=O(\exp\lambda t^{\beta/m})$ for $t\to\infty$, such that $H(z)=\mathfrak{R}\{h(t)\}$. This is the function

$$h(t)=T(f\circ g)=f*g,$$

where the operator T was defined by (7), and $f*g$ is the convolution (6). Thus we have proved

Theorem 5. (The Faltung-theorem of the modified Obrechkoff transformation (12).) *If $f(t)$ and $g(t)$ are Obrechkoff-transformable functions of C_α, then*

$$\mathfrak{R}\{f*g\}=\mathfrak{R}\{f\}\,.\,\mathfrak{R}\{g\},$$

where $(f*g)(t)$ is the convolution (6) *of the functions $f(t)$ and $g(t)$.*

Now it is not difficult to point out an integral transformation, the convolution of which is to be the operation $[f*g]_\lambda$ from (9) with $\lambda>\gamma_{\max}$. This is the "modified" modified Obrechkoff transformation

$$\mathfrak{R}_\lambda\{f(t)\}=z^{-(\lambda-\gamma_m)\beta}\mathfrak{R}\{f(t)\},\quad \lambda>\gamma_m. \tag{15}$$

We have $\mathfrak{R}_\lambda\{[f(t)*g(t)]_\lambda\}=\mathfrak{R}_\lambda\{f(t)\}\,.\,\mathfrak{R}_\lambda\{g(t)\}$, i. e. the Faltung-theorem of the transformation (15).

The convolutions $f*g$ and $[f*g]_\lambda$ of the operator L in C_α, had been found without any reference to integral transformation. But by means of the modified Obrechkoff transformation, now is seen the connection between these convolutions.

5. An alternative approach to operational calculi for the general Bessel-type differential operator. The two approaches above proposed to the Bessel-type differential operator (1) solve the problem of foundations of the operational calculus for this operator. But there still remains a great mass of technicalities to be done, in order to have an operational calculus, developed at such extent as the classical Heaviside-Mikusiński operational calculus. There arises a need for new special functions. Happily enough, there is no need of such efforts. There exists another approach to the problem of developing of operational calculus for (1). This approach opens the way for using the classical operational calculus to do this. It is based on a similarity relation between the operator L and the operator of multiple integration l^m. This relation can be expressed explicitly by a new linear operator φ, connected with the operator L (or B).

Definition 5. *Let* $\lambda_k=\gamma_m-\gamma_k+\frac{k}{m}$, $k=1, 2, \ldots, m-1$. *Then* φ: $C_\alpha\to C_{-1}$ *is the linear operator*

$$\varphi a(t)=[t^{m(\gamma_m+1)-1}/\prod_{k=1}^{m-1}\Gamma(\lambda_k)]\int\limits_{e_{m-1}}\prod_{k=1}^{m-1}\left[(1-t_k)^{\lambda_k-1}t_k^{\gamma_k}\right]a[t^{m/\beta}(t_1\ldots t_{m-1})^{1/\beta}]de_{m-1}.$$

Theorem 6. *The operator* φ *is invertible. If* $\varphi f(t)=F(t)$, *then the following inversion formula is true:*

$$(16)\qquad f[(x_1\ldots x_{m-1})^{1/\beta}]$$

$$=\left(\prod_{k=1}^{m-1}x_k^{-\gamma_k}\right)\frac{\partial^{\lambda_1+\cdots+\lambda_{m-1}}}{\partial x_1^{\lambda_1}\ldots\partial x_{m-1}^{\lambda_{m-1}}}\left\{F[(x_1\ldots x_{m-1})^{1/m}]\prod_{k=1}^{m-1}x_k^{\frac{k+1}{m}-1}\right\},$$

where $\partial/\partial x_k^{\lambda_k}$ *denotes the operator of fractional differentiation, when* λ_k *is not an integer.*

Proof. Let us consider the equality $\varphi f(t)=F(t)$ as an equation for $f(t)\in C_\alpha$. We transform it by the substitution $t=(x_1\ldots x_{m-1})^{1/\beta}$ with independent variables $x_1,\ldots,x_{m-1}$, each varying in $(0,\infty)$. If we denote by $l_1^{\lambda_1},\ldots,l_{m-1}^{\lambda_{m-1}}$ the operators of (eventually) fractional integration with respect to the corresponding variable x_k, from 0 to x_k, of the corresponding order, we can write our equation in the form

$$l_1^{\lambda_1}\ldots l_{m-1}^{\lambda_{m-1}}\left\{f[(x_1\ldots x_{m-1})^{1/\beta}]\prod_{k=1}^{m-1}x_k^{\gamma_k}\right\}=F[(x_1\ldots x_{m-1})^{1/m}]\prod_{k=1}^{m-1}x_k^{\frac{k+1}{m}-1}.$$

From here it follows that φ is an invertible operator, and the inversion formula (16).

We shall have a need to describe exactly the image-space $\varphi(C_\alpha)$ of the operator φ. It is easily seen that $\varphi(C_\alpha)\subset C_{m(\gamma_m-\gamma_1)-1}$. We know that the mapping $\varphi: C_\alpha\to C_{m(\gamma_m-\gamma_1)-1}$ is an injective one, but the image space $\varphi(C_\alpha)$ is a proper part of $C_{m(\gamma_m-\gamma_1)-1}$. By a closer examination of the way of receiving the formula (16) is seen that (16) gives, for $F(t)\in C_{m(\gamma_m-\gamma_1)-1}$, a function of C_α, if and only if $F(t)$ is of the form

$$(17)\qquad F(t)=t^p\widetilde{F}(t^m),$$

where $p>m(\gamma_m-\gamma_1)-1$, and $F(t)$ is at least $q=[\lambda_1]+\ldots+[\lambda_{m-1}]+m-k+1$ times continuously differentiable in $[0,\infty)$, with k equal to the number of the integer members of the sequence $\lambda_1,\ldots,\lambda_{m-1}$.

The main result in this section is

Theorem 7. *Let* l, $l: C_{-1}\to C_0$ *be the integration operator*

$$lf(t)=\int_0^t f(\tau)d\tau.$$

Then the following similarity relation is satisfied:

(18) $$L=\varphi^{-1}\left(\frac{m}{\beta}\, l\right)^m \varphi.$$

Proof. The relation (18) is equivalent to

(19) $$\varphi L\{f(t)\}=\frac{m^m}{\beta^m}\, l^m \varphi\{f(t)\},\quad f(t)\in C_\alpha .$$

It is easy to verify (19) for a power $f(t)=t^p$ with $p>-\beta(\gamma_1+1)$, by simple calculations with gamma- and beta-functions.

Hence, we have

$$\varphi L\{t^p\}=\frac{m^m}{\beta^m}\, l^m \varphi\{t^p\}.$$

Now, let $f(t)=t^q\tilde{f}(t)$, $q>\alpha=-\beta(\gamma_1+1)$ be any function of C_α with continuous in $[0, \infty)$ function $\tilde{f}(t)$. Then, by the Weierstrass' approximation theorem, there exists a polynomial sequence $\{\tilde{f}_n(t)\}_{n=1}^{\infty}$, which converges uniformly to $\tilde{f}(t)$ in each finite interval $[0, A]$. Having proved (19) for a power ot t, we can assert that

$$\varphi L\{t^q\tilde{f}_n(t)\}=(m^m/\beta^m)l^m\varphi\{t^q\tilde{f}_n(t)\}.$$

It remains to let $t\to\infty$ to get (19).

Now, we proceed to apply the theorem 7 for an alternative development of an operational calculus for the general Bessel-type differential operator (1). In this case we shall follow an idea of N. A. Meller [15], on which was said in section 2. This idea was used in a more restricted case by V. A. Ditkin and A. P. Prudnikov [13] for operational calculi for the Bessel-type differential operators

$$\frac{1}{t^{n-1}}\left(\frac{d}{dt}\right)^n t^n\left(\frac{d}{dt}\right)^n t^{n-1} \quad\text{and}\quad \frac{1}{t^{n-1}}\frac{d^n}{dt^n}\, t^{n-1}.$$

By a linear transformation φ, these authors reduced the problem to the operational calculus for the operator $\frac{d}{dt}\, t\, \frac{d}{dt}$.

Let us by $f*g$ denote the usual convolution

(20) $$(f*g)(t)=\int_0^t f(t-\tau)g(\tau)d\tau$$

of the functions $f(t), g(t)\in C_{-1}$ (do not confuse with (6)!)

Using the formula (5), we can assert that the operation in C_α:

(21) $$(f\,\tilde{*}\,g)(t)=\varphi^{-1}[(\varphi f)*(\varphi g)]$$

is a convolution of L in C_α. This follows from the above-mentioned theorem of N. A. Meller and from the fact that the operation (20) is a convolution of the operator $\left(\frac{m}{\beta}\, l\right)^m$ in C_{-1}.

The convolution (21) of L in C_α is free of divisors of zero and can be used for developing of a Mikusiński-type operational calculus for the general differential operator of the Bessel type (1). The corresponding quotient field of C_α is isomorphic with a subfield of the field of Mikusiński, namely, with the subfield, received as a quotient field of the image space $\varphi(C_\alpha)$ with respect to the convolution (20). But by means of (17) it is easy to see that this subfield of the field of Mikusiński coincides with the field of Mikusiński itself.

In [12] we have proved that all the quotient fields of C_α with respect to continuous convolutions of L in C_α, are isomorphic. The convolution (21) is continuous in the sense of [12]. Therefore, all the quotient fields of C_α with respect to continuous convolutions are isomorphic to the field of Mikusiński.

Thus we have proved the following

Theorem 8. *The quotient fields of the algebras C_α with continuous convolutions of the right inverse operators L of all the Bessel-type differential operators, as multiplications, are isomorphic.*

In particular, the quotient fields of all C_α with all possible convolutions (6), (10) and (21), as multiplications, are isomorphic.

At least in principle, all the results, known for the field of Mikusiński, can be transferred to an operational calculus for the general Bessel-type differential operator (1).

6. A possibility for using of the Laplace transformation for an operational calculus for the general differential operator of the Bessel type. The theorem 7 have not only a theoretical aspect, but it can be used for an effective application of the Laplace to the operator (1). Let by

$$\mathfrak{L}\{f(t);\ p\}=\int_0^\infty e^{-pt}f(t)dt$$

we denote the usual Laplace transform of a function $f(t)\in C_{-1}$, $f(t)=O(\exp \gamma t)$ for $t\to\infty$, with some real γ. Now we can define the following integral transformation.

Definition. *Let $f(t)\in C_\alpha$ be $O(\exp \gamma t^{\beta/m})$ for $t\to\infty$. Then*

$$\widetilde{\mathfrak{K}}\{f(t);\ p\}=\mathfrak{L}\{\varphi f(t);\ p\}. \tag{22}$$

The possibility for using the transformation $\widetilde{\mathfrak{K}}$ for a transform basis of an operational calculus for the general Bessel-type differential operator (1) is seen from the following theorem.

Theorem 9. *If $f(t)\in C_\alpha$ is $O(\exp \lambda t^{\beta/m})$ for $t\to\infty$, then*

$$\widetilde{\mathfrak{K}}\{Lf(t);\ p\}=(m/\beta p)^m\widetilde{\mathfrak{K}}\{f(t);\ p\}.$$

Proof. We shall use the basic property of the Laplace transformation:

$$\mathfrak{L}\{lf(t);\ p\}=(1/p)\mathfrak{L}\{f(t);\ p\}.$$

We have, using (19),

$$\widetilde{\mathfrak{K}}\{L f(t);\ p\}=\mathfrak{L}\{\varphi Lf(t);p\}=\mathfrak{L}\{(m/\beta)^m l^m(\varphi f(t));\ p\}$$

$$=(m/\beta)^m \frac{1}{p^m} \mathfrak{L}\{\varphi f(t);\ p\} = (m/\beta p)^m \widetilde{\mathfrak{R}}\{f(t);\ p\}.$$

It is easy to see that the operation (21) is the convolution of the integral transformation (22).

Let us note at last that there are real and complex inversion formulas for the modified Obrechkoff transformation and for the transformation (22) resembling the corresponding inversion formulas for the Laplace transformation.

7. Conclusion. It seems that the above considerations form a sufficient basis for an effective development of operational calculi for the different Bessel-type differential operators.

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