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NEW BEST QUADRATURE FORMULAE

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The class W_q^r of all real functions with continuous derivatives up to order $r-1$ and bounded in L_q by 1 absolutely continuous r-th derivative is considered. Quadrature formulas involving values of the integrand and its successive derivatives in two and three points are studied. Detailed results are given for the case $q=2$.

1. Introduction. Let $W_q^r(M; a, b)$, $1 \leq q \leq \infty$, $r=1, 2, \ldots$, be the class of functions f defined on the finite interval $[a, b]$ for which the $(r-1)$-th derivative is absolutely continuous on $[a, b]$ and the r-th satisfies the condition

$$\left\{ \int_a^b |f^{(r)}(t)|^q dt \right\}^{1/q} \leq M.$$

At times we shall abbreviate $W_q^r(M; a, b)$ to W_q^r.

In [1] S. M. Nikolski initiated investigations connected with the construction of a best quadrature formula. It is a quadrature of the form

$$I(f) = \int_a^b f(x)dx \approx \sum_{i=1}^{m} \sum_{k=0}^{\varrho_i} a_{ki} f^{(k)}(x_i) \tag{1}$$

which has a minimal estimation of the error in a given subset of the class $W_q^r(M; a, b)$ among all quadratures of the same type that are precise for polynomials of degree not greater than r. Afterwards the problem of the "best" formula was considered by many other authors. For futher references see [2]. Another group of authors use the characteristic "best" in a sense of A. Sard [3] where the expresion to be minimized is the L_2 norm of a corresponding kernel function related with the quadrature (1). S. A. Smoljak [4] was the first who posed the problem of the best method of integration in a manner that seems to be most natural. In this paper we shall follow his definition.

Suppose that $L_1(f), L_2(f), \ldots, L_N(f)$ are linear functionals defined on W_q^r. Denote

$$T(f) = \{L_1(f), L_2(f), \ldots, L_N(f)\}.$$

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Let $\sigma(T)$ be the set of all admissible methods S of approximate evaluating of the integral $I(f)$ that use only the information $T(f)$ and that apply to every function f from W_q^r. Note that there is not any restriction on $\sigma(T)$ about linearity of the quadratures or exactness for some polynomial class. Denote by $S(f)$ the approximate value of $I(f)$ calculated by the method S. The quantity

$$R(S, T) = \sup_{f \in W_q^r} |I(f) - S(f)|$$

is said to be an error of the method S in the class W_q^r. Let

$$R(T) = \inf_{S \in \sigma(T)} R(S, T).$$

The method S^* for which $R(S^*, T) = R(T)$ is said to be best for the class W_q^r.

In the case

$$T(f) = \{f^{(k)}(x_i), (k=0, 1, \ldots, r-1; \; i=1, 2, \ldots, n)\}, \; a \leq x_1 < \cdots < x_n \leq b,$$

a best formula was constructed in [5]. In [6] the same problem was solved for the information

$$T(f) = \{f(x_i), f'(x_i), i=1, 2, \ldots, n\}, \; -1 < x_1 < \cdots < x_n < 1,$$

and the class F of all real on $[-1, 1]$ functions, which have a bounded by 1 analytic continuation in the unit circle.

In this paper we shall study formulas of integration which are best for the class W_q^r, relative to the information

(2) $$T(f) = \{f(a), \ldots, f^{(r-1)}(a), f(b), \ldots, f^{(m)}(b)\}, \quad m \leq r-1.$$

2. Main results. We shall make use of the following result proved by S. A. Smoljak [4] (the proof can be seen in [7] also).

Lemma 1. *Let Ω is a convex centrally symmetrical subset of a linear metric space. Suppose $L(f), L_1(f), \ldots, L_N(f)$ are linear functionals on Ω and such that*

$$\sup_{f \in \Omega_0} L(f) < \infty,$$

where $\Omega_0 = \{f : f \in \Omega, L_k(f) = 0, k=1, 2, \ldots, N\}$. Then, there exist numbers D_k, $k=1, 2, \ldots, N$, for which

$$\sup_{f \in \Omega} \left| L(f) - \sum_{k=1}^{N} D_k L_k(f) \right| = \inf_S \sup_{f \in \Omega} |L(f) - S(f)|.$$

Here inf is extended over all admissible methods S of approximation of the functional $L(f)$ with information $\{L_1(f), L_2(f), \ldots, L_N(f)\}$.

The proof of this elementary lemma contains another useful result that we shall formulate separately as

Corollary 1.

$$\sup_{f \in \Omega_0} L(f) = \inf_S \sup_{f \in \Omega} |L(f) - S(f)|.$$

Now return to our problem of best integration subject to the information (2).

Denote by π_n the set of all real polynomials of degree not greater than n, and let π_n^* be the set of polynomials $Q(t) \in \pi_n$ with coefficient 1 before t^n. Define

$$E_{n,p}^{\alpha,\beta} = \inf_{Q \in \pi_n^*} \left\{ \int_{-1}^{1} (1-t)^\alpha (1+t)^\beta \, |Q(t)|^p dt \right\}^{1/p}.$$

Here and everywhere in this paper $1/p + 1/q = 1$. Denote by $I_{n,p}^{(\alpha,\beta)}(t)$ the extremal polynomial for which

$$\left\{ \int_{-1}^{1} (1-t)^\alpha (1+t)^\beta \, |I_{n,p}^{(\alpha,\beta)}(t)|^p dt \right\}^{1/p} = E_{n,p}^{(\alpha,\beta)}.$$

The basic result of this paper is

Theorem 1. *The quadrature formula*

$$I(f) \approx \frac{(-1)^m}{r!} \sum_{k=0}^{m} (-1)^{r-k-1} U^{(r-k-1)}(b) f^{(k)}(b)$$

$$+ \frac{(-1)^{m+1}}{r!} \sum_{k=0}^{r-1} (-1)^{r-k-1} U^{(r-k-1)}(a) f^{(k)}(a),$$

where

$$U(t) = \left(\frac{b-a}{2}\right)^{m+1} (b-t)^{r-m-1} I_{m+1,p}^{(p(r-m-1),0)} \left(\frac{2}{b-a} t - \frac{a+b}{b-a}\right)$$

is a best method of integration for the class $W_q^r(M; a, b)$ among all quadrature formulas that use only the information T from (2). *Here*

$$R(T) = \frac{M}{r!} \left(\frac{b-a}{2}\right)^{r+1/p} E_{m+1,p}^{(p(r-m-1),0)}.$$

Proof. Let us introduce the following subsets of W_q^r

$$W_a = \{f : f \in W_q^r, \ f^{(k)}(a) = 0 \ (k = 0, 1, \ldots, r-1)\},$$

$$W_{ab} = \{f : f \in W_q^r, \ f^{(k)}(a) = 0, \ f^{(i)}(b) = 0 \ (k = 0, 1, \ldots, r-1; \ i = 0, 1, \ldots, m)\}.$$

Consider the informations

$$T_1(f) = \{f^{(k)}(a) \ (k = 0, 1, \ldots, r-1)\},$$

$$T_2(f) = \{f^{(k)}(b) \ (k = 0, 1, \ldots, m)\}.$$

It is clear that W_q^r and $T = T_1 \cup T_2$ satisfy the conditions of lemma 1. Thus we can apply corollary 1. It gives

$$R(T) = \sup_{f \in W_{ab}} I(f).$$

Now applying the same corollary to the set W_a and information $T_2(f)$ we get

$$R(T)=\sup_{f\in W_{ab}} I(f)=\inf_{S\in\sigma(T_c)}\sup_{f\in W_a}|I(f)-S(f)|.$$

According to the lemma 1 the above minimum is attained for a linear method. So

$$R(T)=\inf_{\{B_k\}_0^m}\sup_{f\in W_a}\left|I(f)-\sum_{k=0}^{m}B_k f^{(k)}(b)\right|.$$

Assume that $f\in W_a$. The classical Taylor formula provides the following representation of f

$$f(x)=\frac{1}{(r-1)!}\int_a^x (x-t)^{r-1}f^{(r)}(t)dt.$$

It is clear that

$$\int_a^b f(x)dx=\frac{1}{r!}\int_a^b (b-t)^r f^{(r)}(t)dt,$$

$$f^{(k)}(b)=\frac{1}{(r-k-1)!}\int_a^b (b-t)^{r-k-1}f^{(r)}(t)dt,$$

for $k=0, 1, \ldots, m$. Consequently,

$$R(T)=\inf_{\{B_k\}_0^m}\sup_{f\in W_a}\frac{1}{r!}\int_a^b\left((b-t)^r-\sum_{k=0}^{m}\frac{r!B_k}{(r-k-1)!}\cdot(b-t)^{r-k-1}\right)f^{(r)}(t)dt.$$

By Hölder's inequality

$$R(T)=\frac{M}{r!}\inf_{\{B_k\}_0^m}\left\{\int_a^b(b-t)^{(r-m-1)p}\left|(b-t)^{m+1}-\sum_{k=0}^{m}\frac{r!\,B_k}{(r-k-1)!}(b-t)^{m-k}\right|^p dt\right\}^{1/p}$$

$$=\frac{M}{r!}\left(\frac{b-a}{a}\right)^{r+\frac{1}{p}}E^{(p(r-m-1),0)}_{m+1,p}.$$

On the other hand from the definition of the error

$$R(T)=\inf_{S\in\sigma(T)}\sup_{f\in W_q^r}|I(f)-S(f)|.$$

Integrating the Taylor's formula we get

$$I(f)=S_0(f)+\frac{1}{r!}\int_a^b(b-t)^r f^{(r)}(t)dt,$$

where

$$S_0(f)=\sum_{k=0}^{r-1}\frac{(b-a)^{k+1}}{(k+1)!}f^{(k)}(a).$$

Thus

$$R(T)=\inf_{S\in\sigma(T)}\sup_{f\in W_q^r}\left|\frac{1}{r!}\int_a^b(b-t)^r f^{(r)}(t)dt-(S(f)-S_0(f))\right|.$$

But the method $S_1=S-S_0$ belongs to the class $\sigma(T)$ when $S\in\sigma(T)$. Hence

$$R(T)=\inf_{S_1\in\sigma(T)}\sup_{f\in W_q^r}\left|\frac{1}{r!}\int_a^b(b-t)^r f^{(r)}(t)dt-S_1(f)\right|.$$

An multiple integration by parts shows immediately that the quadrature formula

$$I(f)\approx\frac{1}{r!}\int_a^b(b-t)^{r-m-1}Q(t)f^{(r)}(t)dt, \tag{3}$$

belongs to the class $\sigma(T)$ for every polynomial $Q(t)\in\pi_m$. Therefore

$$R(T)\leq\inf_{Q\in\pi_m}\sup_{f\in W_q^r}\frac{1}{r!}\left|\int_a^b(b-t)^{r-m-1}((b-t)^{m+1}-Q(t))f^{(r)}(t)dt\right|$$

$$=\frac{M}{r!}\left\{\int_a^b(b-t)^{(r-m-1)}\left|\left(\frac{b-a}{2}\right)^{m+1}J_{m+1,p}^{(p(r-m-1),0)}\left(\frac{2}{b-a}t-\frac{a+b}{b-a}\right)\right|^p dt\right\}^{1/p}$$

$$=\frac{M}{r!}\left(\frac{b-a}{2}\right)^{r+\frac{1}{p}}E_{m+1\,p}^{(p(r-m-1),0)}=R(T).$$

The above chain of relations implies that the quadrature

$$I(f)\approx S^*(f)=\frac{1}{r!}\int_a^b(b-t)^{r-m-1}Q(t)f^{(r)}(t)dt$$

with

$$Q(t)=(b-t)^{m+1}-(-1)^{m+1}\left(\frac{b-a}{2}\right)^{m+1}J_{m+1,p}^{(p(r-m-1),0)}\left(\frac{2}{b-a}t-\frac{a+b}{b-a}\right)$$

is the best method of integration. Futher we calculate

$$S^*(f)=I(f)-\frac{(-1)^{m+1}}{r!}\int_a^b U(t)f^{(r)}(t)dt, \tag{4}$$

where $U(t)$ is defined as in the assertion of the theorem. Evaluating the above integral using only integration by parts we obtain

$$S^*(f)=\frac{(-1)^m}{r!}\sum_{k=0}^{r-1}(-1)^{r-k-1}U^{(r-k-1)}(b)f^{(k)}(b)$$

$$+\frac{(-1)^{m+1}}{r!}\sum_{k=0}^{r-1}(-1)^{r-k-1}U^{(r-k-1)}(a)f^{(k)}(a).$$

It remains to observe that $U^{(i)}(b)=0$ for $i=0,1,\ldots,r-m-2$. This completes the proof.

Note that the constructed best formula is precise for polynomials of degree not greater than r. This follows immediately from (4).

For the sake of completeness we shall formulate as a separate theorem the analogous result treating the case

$$T(f)\equiv\{f(a),\ldots,f^{(m)}(a),f(b),\ldots,f^{(r-1)}(b)\}. \tag{5}$$

Theorem 1′. *The quadrature formula*

$$I(f)\approx\frac{(-1)^{r+m-1}}{r!}\sum_{k=0}^{m}U^{(r-k-1)}(b)f^{(k)}(a)+\frac{(-1)^{r+m}}{r!}\sum_{k=0}^{r-1}U^{(r-k-1)}(a)f^{(k)}(b)$$

is the best method of integration for the class W_q^r with information (5). The error $R(T)$ is the same as in theorem 1.

The assertion follows from theorem 1 and the equality $I(f)=I(g)$, where $g(t)=f(a+b-t)$.

Now consider a more general case when the information consists of values of the integrand in the points a, b and c, $a<c<b$. Precisely, let

$$T'\equiv\{f(a),\ldots,f^{(m)}(a),f(c),\ldots,f^{(r-1)}(c)\},$$

$$T''\equiv\{f(c),\ldots,f^{(r-1)}(c),f(b),\ldots,f^{(m)}(b)\}.$$

Theorem 2. *Let S_1 be the best method of approximation of the integral $I_1(f)=\int_a^c f(x)dx$ for the class $W_q^r(M;a,c)$ with information T' and S_2 be the best one for the integral $I_2(f)=\int_c^b f(x)dx$, the class $W_q^r(M;c,b)$ and information T''. Then the method $I(f)\approx S_1(f)+S_2(f)$ is the best for the class $W_q^r(M;a,b)$ with information $T=T'\cup T''$.* Here

$$R(T)=\frac{M}{r!}\left\{\left(\frac{c-a}{2}\right)^{rp+1}+\left(\frac{b-c}{2}\right)^{rp+1}\right\}^{1/p}E_{m+1,p}^{(p(r-m-1),0)}.$$

Proof. Denote, for convenience

$$e_1=\left(\frac{c-a}{2}\right)^{r+\frac{1}{p}}E_{m+1,p}^{(p(r-m-1),0)},$$

$$e_2=\left(\frac{b-c}{2}\right)^{r+\frac{1}{p}}E^{(p(r-m-1),0)}_{m+1,p}.$$

Let us define the numbers M_1 and M_2 as follows $M_i=Me_i^{p-1}/(e_1^p+e_2^p)^{1/q}$ $(i=1, 2)$. Suppose $\varepsilon\in(0, 1)$. It follows by corollary 1 that there exists functions $g_1(t)\in W^r_q(M_1; a, c)$ and $g_2(t)\in W^r_q(M_2; c, b)$ such that

$$I_1(g_1)=\varepsilon\frac{M_1}{r!}e_1,\quad I_2(g_2)=\varepsilon\frac{M_2}{r!}e_2$$

and

$$g_1^{(k)}(a)=g_2^{(k)}(b)=0\quad (k=0, 1, \ldots, m),$$

$$g_1^{(i)}(c)=g_2^{(i)}(c)=0\quad (i=0, 1, \ldots, r-1).$$

Define the function $g(t)$ in the following way

$$g(t)=\begin{cases}g_1(t) & \text{for}\quad t\in[a, c],\\ g_2(t) & \text{for}\quad t\in[c, b].\end{cases}$$

Evidently, $g\in W^r_q(M; a, b)$. Indeed,

$$\left\{\int_a^b|g^{(r)}(t)|^q dt\right\}^{1/q}=\left\{\int_a^c|g^{(r)}(t)|^q dt+\int_c^b|g^{(r)}(t)|^q dt\right\}^{1/q}$$

$$\leqq\{M_1^q+M_2^q\}^{1/q}=\frac{M}{(e_1^p+e_2^p)^{1/q}}\{e_1{}^{(p-1)q}+e_2{}^{(p-1)q}\}^{1/q}=M.$$

By corollary 1

$$R(T)\geqq I(g)=\frac{\varepsilon}{r!}(M_1e_1+M_2e_2)=\varepsilon\frac{M}{r!}(e_1^p+e_2^p)^{1/p}$$

In so far as ε was an arbitrary number less than 1, we get

$$R(T)\geqq\frac{M}{r!}(e_1^p+e_2^p)^{1/p}. \tag{6}$$

Now denote by $R^*(T)$ the error of the method described in the theorem. It follows from theorem 1 and theorem 1′ that

$$R^*(T)\leqq\sup_{f\in W^r_q}\left\{\frac{e_1}{r!}\left(\int_a^c|f^{(r)}(t)|^q dt\right)^{1/q}+\frac{e_2}{r!}\left(\int_c^b|f^{(r)}(t)|^q dt\right)^{1/q}\right\}.$$

By Hölder's inequality

$$R^*(T)\leqq\frac{M}{r!}\{e_1^p+e_2^p\}^{1/p}.$$

This and (6) complete the proof of the theorem.

3. Special cases. Below we shall formulate some immediate consequences of the results proved in the preceding section.

Theorem 3. *The quadrature formula*

(7) $$I(f)\approx\sum_{k=0}^{m}(-1)^k\frac{\binom{m+1}{k+1}\binom{2r-m-1}{r}}{\binom{2r}{k+1}\binom{2r-m-k-2}{r-k-1}}\frac{(b-a)^{k+1}}{(k+1)!}f^{(k)}(b)$$

$$+\sum_{k=0}^{r-1}\left(\sum_{\substack{\nu=\max\\(0,m-k)}}^{\substack{\min(r-k\\-1,m+1)}}\frac{\binom{m+1}{\nu}\binom{m+1}{\nu}\binom{r-m-1}{k+\nu-m}}{\binom{2r}{m+1+\nu}\binom{r}{k+1}}\right)\frac{(b-a)^{k+1}}{(k+1)!}f^{(k)}(a)$$

is the best method of integration for the class $W_2^r(M;\, a,\, b)$ among all quadrature formulas that use only the information

$$T(f)\equiv\{f(a),\ldots,f^{(r-1)}(a),f(b),\ldots,f^{(m)}(b)\}.$$

Here

$$R(T)=M(b-a)^{r+\frac{1}{2}}\Big/\left[\binom{2r}{m+1}r!\sqrt{2r+1}\right].$$

Proof. According to theorem 1 we have to calculate $U^{(r-k-1)}(b)$, $(k=0,1,\ldots,m)$ and $U^{(r-k-1)}(a)$ $(k=0,1,\ldots,r-1)$, where

$$U(t)=\left(\frac{b-a}{2}\right)^{m+1}(b-t)^{r-m-1}J_{m+1,2}^{(2(r-m-1),0)}\left(\frac{2}{b-a}t-\frac{a+b}{b-a}\right).$$

It is easily verified that

$$U^{(r-k-1)}(b)=(-1)^{r-m-1}(r-m-1)!\binom{r-k-1}{r-m-1}\left(\frac{b-a}{2}\right)^{k+1}\frac{d^{m-k}}{dt^{m-k}}J_{m+1,2}^{(2(r-m-1),0)}(1)$$

for $k=0,1,\ldots,m$ and

$$U^{(r-k-1)}(a)=\sum_{\substack{\nu=\max\\(0,m-k)}}^{\substack{\min(r-k\\-1,m+1)}}(-1)^{r-k-\nu-1}\binom{r-k-1}{\nu}\frac{(r-m-1)!}{(k+\nu-m)!}\frac{(b-a)^{k+1}}{2^{m+1-\nu}}$$

$$\times\frac{d^{\nu}}{dt^{\nu}}J_{m+1,2}^{(2(r-m-1),0)}(-1)$$

for $k=0,1,\ldots,r-1$. It remains to evaluate the derivatives of the polynomial $J_{m+1,2}^{(2(r-m-1),0)}(t)$. By the known [8] relation

$$\frac{d}{dt}J_{n,2}^{(\alpha,\beta)}(t)=\frac{1}{2}(n+\alpha+\beta+1)J_{n-1,2}^{(\alpha+1,\beta+1)}(t)$$

we find

$$\frac{d^{m-k}}{dt^{m-k}}J_{m+1,2}^{(2(r-m-1),0)}(1)=2^{k+1}\frac{(2r-k-1)!(m+1)!}{(2r)!}\binom{2r-m-1}{k+1},$$

$$\frac{d^{\nu}}{dt^{\nu}}J_{m+1,2}^{(2(r-m-1),0)}(-1)=(-1)^{m+1-\nu}2^{m+1-\nu}\frac{(2r-m+\nu-1)!(m+1)!}{(2r)!}\binom{m+1}{\nu}.$$

So,

$$U^{(r-k-1)}(b)=(-1)^{r-m-1}\binom{2r-m-1}{k+1}\frac{(r-m-1)!\,(2r-k-1)!\,(m+1)!}{(2r)!}(b-a)^{k+1}$$

$$=(-1)^{r-m-1}r!\,\frac{\binom{m+1}{k+1}\binom{2r-m-1}{r}}{\binom{2r}{k+1}\binom{2r-m-k-2}{r-k-1}}\,\frac{(b-a)^{k+1}}{(k+1)!},$$

$$U^{(r-k-1)}(a)=(-1)^{r-k+m}\sum_{\substack{\nu=\max\\(0,m-k)}}^{\substack{\min(r-k\\-1,m+1)}}\frac{(2r-m+\nu-1)!\,(m+1)!\,(r-m-1)!}{(2r)!\,(k+\nu-m)!}\times$$

$$\binom{r-k-1}{\nu}\binom{m+1}{\nu}(b-a)^{k+1}=(-1)^{r-k+m}r!\,\frac{(b-a)^{k+1}}{(k+1)!}\sum_{\substack{\nu=\max\\(0,m-k)}}^{\substack{\min(r-k\\-1,m+1)}}\frac{\binom{m+1}{\nu}\binom{m+1}{\nu}\binom{r-m-1}{k+\nu-m}}{\binom{2r}{m+1-\nu}\binom{r}{k+1}}.$$

From these equalities and theorem 1 we conclude that the quadrature (7) is the best one. Next, by the quoted theorem and the known formula [8]

$$E_{m+1,2}^{(2(r-m-1),0)}=\frac{2^{m+1}}{\binom{2r}{m+1}}\left\{\frac{2^{2(r-m-1)+1}}{2r+1}\right\}^{1/2}$$

we get

$$R(T)=\frac{M}{r!}\,\frac{(b-a)^{r+\frac{1}{2}}}{\binom{2r}{m+1}\sqrt{2r+1}}.$$

The theorem is proved.

Putting $m=r-1$ in (7) we find after an easy computation that the quadrature

$$I(f)\approx\sum_{k=0}^{r-1}\frac{\binom{r}{k+1}}{\binom{2r}{k+1}}\,\frac{(b-a)^{k+1}}{(k+1)!}\,[f^{(k)}(a)+(-1)^k f^{(k)}(b)] \tag{8}$$

is the best one for the class $W_2^r(M;a,b)$. The above formula can be obtained from the appropriate Hermite interpolation polynomial, by integration.

Remark. The quadrature (8) is a particular case of the L. Tschakaloff — N. Obrechkoff [9, 10] formula

$$I(f)\approx\frac{1}{\binom{r+m+1}{r}}\sum_{k=0}^{r-1}\binom{r+m-k}{r-k-1}\frac{(b-a)^{k+1}}{(k+1)!}f^{(k)}(a) \tag{9}$$

$$+\frac{1}{\binom{r+m+1}{r}}\sum_{k=0}^{m}(-1)^k\binom{r+m-k}{r}\frac{(b-a)^{k+1}}{(k+1)!}f^{(k)}(b).$$

L. Tschakaloff has proved in [9] that it is the unique quadrature of this type that is precise for the polynomials of degree not greater than $r+m$. It is interesting to note that (9) coincides with the best formula given in theorem 3 only in the case $m=r-1$.

The next two formulas are very special cases of theorem 3. They use the same information as Simpson formula.

Corollary 2. *The quadrature formula*

$$I(f)\approx\frac{b-a}{16}\left(3f(a)+10f\left(\frac{a+b}{2}\right)+3f(b)\right)$$

is the best method of integration for the class $W_2^2(M;a,b)$ *with information* $T(f)=\left\{f(a),f(b),f\left(\frac{a+b}{2}\right),f'\left(\frac{a+b}{2}\right)\right\}$ *and the corresponding* $R(T)$ *has the value* $M(b-a)^{5/2}/32\sqrt{5}$.

Proof. Indeed, by theorem 2 and theorem 3

$$R(T)=\frac{M}{2!}\left\{2\left(\frac{b-a}{4}\right)^5\right\}^{1/2}E_{1,2}^{(2,0)}=\frac{M}{32\sqrt{5}}(b-a)^{5/2}.$$

Further, as far as $J_{1,2}^{(2,0)}(t)=t+1/2$ we have

$$S(f)=-\frac{1}{2}U_1'\left(\frac{a+b}{2}\right)f(a)+\frac{1}{2}\left\{U_1'(a)f\left(\frac{a+b}{2}\right)+U_1(a)f'\left(\frac{a+b}{2}\right)\right\}$$

$$-\frac{1}{2}U_2'(b)f(b)-\frac{1}{2}\left\{-U_2'\left(\frac{a+b}{2}\right)f\left(\frac{a+b}{2}\right)+U_2\left(\frac{a+b}{2}\right)f'\left(\frac{a+b}{2}\right)\right\},$$

where

$$U_1(t)=\frac{b-a}{4}\left(\frac{a+b}{2}-t\right)\left(\frac{4}{b-a}t-\frac{b+3a}{b-a}+\frac{1}{2}\right),$$

$$U_2(t)=\frac{b-a}{4}(b-t)\left(\frac{4}{b-a}t-\frac{a+3b}{b-a}+\frac{1}{2}\right)$$

and the assertion follows.

Corollary 3. *The quadrature formula*

$$I(f)\approx\frac{b-a}{8}\left(\sqrt{2}f(a)+(8-2\sqrt{2})f\left(\frac{a+b}{2}\right)+\sqrt{2}f(b)\right)$$

is the best method of integration for the class $W_\infty^2(M;a,b)$ *with information*

$$T(f)=\left\{f(a),f(b),f\left(\frac{a+b}{2}\right),f'\left(\frac{a+b}{2}\right)\right\}.$$

Here

$$R(T)=(2-\sqrt{2})M(b-a)^3/48.$$

Proof. It is easily verified that

$$E_{1,1}^{(1,0)}=\inf_c\int_{-1}^{1}(1-t)\,|t-c|\,dt=4(2-\sqrt{2})/3.$$

$$J_{1,1}^{(1,0)}(t) = t - 1 + \sqrt{2}.$$

The statement follows from theorem 2 and theorem 3.

R e m a r k. Corollary 3 was announced in [11] with misprints.

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