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ON A THEOREM OF BOURGIN — YANG

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The mappings (single and multi-valued) of an n-cohomological sphere with involution in the k-dimensional Euclidean space are considered. The estimates of the dimension of the set of points which belong to the same orbit of the involution and are identified with respect to the mapping are obtained.

Introduction. In [1] K. Borsuk has proved that for a single-valued continuous mapping $f: S^n \to R^k$ of an n-dimensional sphere in the k-dimensional Euclidean space R^k $(k \leq n)$ the set $A(f) = \{x \in S^n \mid f(x) = f(-x)\}$ is not empty.

The question of how big the set $A(f)$ is has been raised in [2, 3]. In these papers it has been proved that $\dim A(f) \geq n-k$, i. e. the covering dimension of the set $A(f)$, is not less than $n-k$. Especially, the following theorem has been established in [2, 3]:

Let $T: X \to X$ be a free involution of a compact n-homological sphere X, $\varphi: X \to R^k$ a single-valued continuous mapping. Then

$$\dim \{x \in X \mid \varphi(x) = \varphi(Tx)\} \geq n-k.$$

In order to prove this Theorem D. Bourgin and G. Yang used in [2, 3] Smith's index and an inductive procedure.

In [4] a multi-valued acyclic mapping $\Phi: X \to R^k$ of a compact space n-homological sphere X with a free involution T has been investigated, and the following results has been obtained: the set $\{x \in X \mid \Phi(x) \cap \Phi(Tx) \neq \emptyset\}$ is not empty.

In the present paper with the aid of a sheaf-theoretic treatment of Smith's theory the result of Bourgin—Yang of [2, 3] is generalized. Also the answer of the question how big is the set $\{x \in X \mid \psi(x) \cap \psi(Tx) \neq \emptyset\}$ is given. Here $\psi: X \to R^k$ is a multi-valued acyclic mapping of an n-cohomological sphere X into the k-dimensional Euclidean space and $\theta: X \to X$ is a multi-valued acyclic involution. The following inequality is established

$$\dim \{x \in X \mid \psi(x) \cap \psi(\theta x) \neq \emptyset\} \geq n-k-\dim \theta,$$

where $\dim \theta = \max \{\dim \theta(x) \mid x \in X\}$. Therefore, in the case of single-valued involution $T = \theta$ we obtain the following result

$$\dim \{x \in X \mid \psi(x) = \psi(Tx)\} \geq n-k.$$

The mappings (single-valued and multi-valued acyclic) of a locally compact n-cohomological sphere X into the k-dimensional Euclidean space R^k are considered.

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It is well-known that for a locally compact Hausdorff n-cohomological sphere and an involution $T: X \to X$, the fixed point set $F = \{x \in X \mid Tx = x\}$ is an r-cohomological sphere, for some $-1 \leq r \leq n$. Suppose that $0 \leq r \leq n-1$. The following theorem is proved:

If $f: X \to R^k$ is a single-valued mapping and $k \leq n-r-1$, then

$$\dim_{Z_2} \{x \in X \setminus F \mid f(x) = f(Tx)\} \geq n-k.$$

For multi-valued acyclic mappings $\psi: X \to R^k$ of the locally compact Hausdorff n-cohomological sphere X with an involution T and a fixed point set F, which is an r-cohomological sphere, $0 \leq r \leq n-1$, the dimension with Z_2 coefficients of the set $\{x \in X \mid \psi(x) \cap \psi(Tx) \neq \emptyset\}$ is not less than $n-k$, under the conditions $k \leq n-r-1$, and the mapping $\psi/F: F \to R^k$ is a single-valued one.

1. Z_2-spaces. As usually, by Z_2 we shall denote the cyclic group $(1, T)$ of order 2.

Definition 1. *The Hausdorff topological space X is called a Z_2-space if the group Z_2 acts on X, i. e., a homeomorphism $T: X \to X$ is given, such that $T^2x = x$ for every $x \in X$.*

By $F(T)$ we shall denote the fixed point set of the action of Z_2 on the space X, i. e., $F(T) = \{x \in X \mid Tx = x\}$. This set is a closed subset of X.

Definition 2. The Z_2-*space X is called a free Z_2-space if $F(T) = \emptyset$, i. e., $Tx \neq x$ for every $x \in X$.*

Given a Z_2-space X, by $\tilde{X}$ we shall denote the orbit space of the action of the group Z_2 on the space X. The space $\tilde{X}$ is the identification space of the space X with respect to the following relation: x is equal to y if and only if $Tx = y$ for $x, y \in X$. The topology of the space $\tilde{X}$ is the identification topology and the projection $\pi: X \to \tilde{X}$ is an open and closed mapping.

The restriction of the mapping π on the set $F(T)$ is a homeomorphism. We shall often consider the set $F(T)$ as a subset of the space $\tilde{X}$, having in mind this homeomorphism.

2. Cohomology. We shall use the cohomology with the coefficients in sheafs of Abelian groups ([5], Ch. 2). For a given sheaf $\mathcal{L}$ of Abelian groups over the space Y and open set U in Y by $\Gamma(U, \mathcal{L})$ we shall denote the group of all continuous sections of the sheaf $\mathcal{L}$ on the set U ([5], Ch. 1, § 1).

Suppose that B is a closed subset of the space Y. There is only one sheaf $\mathcal{L}_B$ such that its restriction on the set B coincides with the sheaf $\mathcal{L}$ and is zero on $Y \setminus B$. There exists also an epimorphism $j: \mathcal{L} \to \mathcal{L}_B$.

By $C^*(Y, \mathcal{L})$ we shall denote the canonical resolution of the sheaf $\mathcal{L}$, and by $H^*(Y, \mathcal{L})$ the cohomology of the space Y with coefficients in $\mathcal{L}$ ([5], Ch. 2). Hereafter we shall use the family of all closed sets as supports for the cohomology.

By Z_2 we shall denote the constant sheaf over the spase Y, i. e., $Z_2 = Y \times Z_2$ and by $H^*(Y)$ the cohomology of the space Y with coefficients Z_2.

Let us consider the mapping $\pi: X \to \tilde{X}$, given in 1. The following lemma follows from Theorem 11.1, Ch. 2, § 11 in [5]:

Lemma 1. *The induced homomorphism*

$$\widehat{\pi}^*: H^*(\tilde{X}, \pi_* Z_2) \to H^*(X)$$

is an isomorphism.

The homomorphism $\widehat{\pi}^*$ is defined as follows. The continuous mapping $\pi: X \to \widetilde{X}$ induces a π-cohomomorphism $\overline{\pi}: \mathsf{A} = \pi_* Z_2 \to Z_2$ ([5, Ch. 1, § 4]. The π-cohomomorphism $\overline{\pi}$ induces a π-cohomomorphism $\overline{\pi}_*: \mathsf{C}^*(\widetilde{X}, \mathsf{A}) \to \mathsf{C}_*(X, Z_2)$ The last π-cohomomorphism $\overline{\pi}^*$ gives us the homomorphism $\widehat{\pi}^*$.

3. Z_2-structure of the sheaf $\mathsf{A} = \pi_* Z_2$ and the Smith theory. Suppose that X is a Z_2-space. The action of Z_2 on the space X induces an action of the group Z_2 on the sheaf A:

Lemma 2 ([6], Ch. 3, § 1). *The sheaf $\mathsf{A} = \pi_* Z_2$ is a module over the sheaf $Z_2 = \widetilde{X} \times Z_2$.*

The action of the group Z_2 on A is defined as follows. The sheaf A is induced by the presheaf $A: A(U) = \Gamma(\pi^{-1}(U), Z_2)$ for every open subset U of the space $\widetilde{X}$ (the restriction operators are the restrictions of the sections). Let $s \in \Gamma(\pi^{-1}(U), Z_2)$, the section s is a continuous mapping of the set U in Z_2. We define $Ts \in \Gamma(U, \mathsf{A})$ by $Ts(x) = s(Tx)$ for every point $x \in \pi^{-1}(U)$. Now we have homomorphisms $T: A(U) \to A(U)$ compatible with restrictions operators. These homomorphisms induce a Z_2-structure in the sheaf A.

Let us consider the action of the group Z_2 on the stalks of the sheaf A. For a given point $z \in \widetilde{X}$ the stalk A_z of the sheaf A over the point z coincides with $Z_2 \oplus Z_2$ if $z \notin F(T)$ and with Z_2 if $z \in F(T)$. Then $T(l_1, l_2) = (l_2, l_1)$ for $(l_1, l_2) \in Z_2 \oplus Z_2$ and $T(l) = l$ for $l \in Z_2$, respectively.

The element $1 + T = \sigma$ of the group ring of the group Z_2 acts on A and let $\sigma\mathsf{A}$ is the image of the homomorphism $\sigma: \mathsf{A} \to \mathsf{A}$.

The following lemma is obvious.

Lemma 3. *The sheaf $\sigma\mathsf{A}$ is isomorphic to the sheaf $Z_{2\widetilde{X} \setminus F(T)}$.*

The following lemma is basic for the Smith's theory of Z_2-spaces.

Lemma 4 ([6], Ch. 3, § 4). *The sequence*

$$0 \to \sigma\mathsf{A} \xrightarrow{i} \mathsf{A} \xrightarrow{\sigma \oplus j} \sigma\mathsf{A} \oplus \mathsf{A}_{F(T)} \to 0 \tag{1}$$

is exact.

Here the homomorphism j is the standard epimorphism of the sheaf A on the sheaf $\mathsf{A}_{F(T)}$, and the homomorphism i is the identity inclusion.

Corollary 1. *For every free Z_2-space the following sequence is exact*

$$0 \to \sigma\mathsf{A} \xrightarrow{i} \mathsf{A} \xrightarrow{\sigma} \sigma\mathsf{A} \to 0. \tag{2}$$

Corollary 2. *For every Z_2-space X the following sequence is exact*

$$\cdots \to H^i(\widetilde{X}, \sigma\mathsf{A}) \to H^i(X) \to H^i(\widetilde{X}, \sigma\mathsf{A}) \oplus H^i(F(T)) \xrightarrow{\partial^i} H^{i+1}(\widetilde{X}, \sigma\mathsf{A}) \to \cdots \tag{3}$$

Under the condition that the space X is a free Z_2-space (*i.e.*, $F(T) = \varnothing$), it follows from (2) that the sequence

$$\cdots \to H^i(\widetilde{X}) \to H^i(X) \to H^i(\widetilde{X}) \xrightarrow{\partial^i} H^{i+1}(\widetilde{X}) \to \cdots \tag{4}$$

is exact.

The homomorphism $H^i(\widetilde{X}) \to H^i(X)$ in (4) is the transfer homomorphism.

Corollary 3. *For every locally compact Hausdorff Z_2-space X the sequence*

(5) $$\cdots \to H_c^i(\tilde{X} \setminus F(T)) \to H_c^i(X) \to H_c^i(\tilde{X} \setminus F(T)) \oplus H_c^i(F(T)) \xrightarrow{\partial^i} H_c^{i+1}(\tilde{X} \setminus F(T)) \to \cdots$$

is exact.

Here $H_c^*(Y)$ are the cohomologies of the space Y with compact supports.

The sequences (3), (4), (5), are Smith's exact sequences of the space with the involution.

We put $H_c^i(X)$ instead of $H_c^i(\tilde{X}, \Delta)$ in the corollaries 2 and 3 having in mind Lemma 1.

4. Smith's homomorphism. Let X be a free Z_2-space.

Definition 3 (Smith). *The Smiths' homomorphism of the free Z_2-space is the homomorphism $s_{i,k}(X): H^i(\tilde{X}) \to H^{i+k}(\tilde{X})$ given by $s_{i,k}(X) = \partial^{i+k-1} \ldots \partial^{i+1}\partial^i$. The homomorphisms ∂^i are the same as ∂^i in the exact sequence* (4).

If $i=0$, we shall abreviate our notations and shall write $s_k(X)$ instead of $s_{0,k}(X)$. Sometimes we shall omit X and shall write s_k or $s_{i,k}$ instead of $s_k(X)$ or $s_{i,k}(X)$, respectively (if this does not lead to a confusion).

Lemma 5. *Let X_1 and X_2 be free Z_2-spaces and $f: X_1 \to X_2$ an equivariant mapping, i. e., $fT = Tf$. The followiug diagram is commutative*

$$\begin{array}{ccc} H^i(\tilde{X}_2) & \xrightarrow{s_{i,k}(X_2)} & H^{i+k}(\tilde{X}_2) \\ \tilde{f}^i \downarrow & & \downarrow \tilde{f}^{i+k} \\ H^i(\tilde{X}_1) & \xrightarrow{s_{i,k}(X_1)} & H^{i+k}(\tilde{X}_1). \end{array}$$

Here $\tilde{X}_i$ are the orbit spaces of X_i, $i=1, 2$. The mapping $f: X_1 \to X_2$ is an equivariant one, therefore this mapping induces a mapping $\tilde{f}: \tilde{X}_1 \to \tilde{X}_2$ of the orbit spaces. By $\tilde{f}^*$ we denote the homomorphism of the cohomology induced by $\tilde{f}$.

Lemma 5 follows immediately from the following

Lemma 6. *The Smith's exact sequence of Z_2-space is canonical, i. e. if X_1 and X_2 are Z_2-spaces and $f: X_1 \to X_2$ is an equivariant mapping, then the following diagram is commutative*

$$\begin{array}{ccccccccc} \cdots \to & H^i(\tilde{X}_2) & \to & H^i(X_2) & \to & H^i(\tilde{X}_2) & \xrightarrow{\partial^i} & H^{i+1}(\tilde{X}_2) & \to \cdots \\ & \tilde{f}^i \downarrow & & f^i \downarrow & & \tilde{f}^i \downarrow & & \tilde{f}^{i+1} \downarrow & \\ \cdots \to & H^i(\tilde{X}_1) & \to & H^i(X_1) & \to & H^i(\tilde{X}_1) & \xrightarrow{\partial^i} & H^{i+1}(\tilde{X}_1) & \to \cdots \end{array}$$

Lemma 6 follows directly from the definition of Smith's exact sequence of a free Z_2-space.

Corollary 4. *Let X_i, $i=1, 2$, be free Z_2-spaces, $f: X_1 \to X_2$ an equivariant mapping.*

a) *if $s_k(X_1) \neq 0$, then $s_k(X_2) \neq 0$;*

b) *if $\tilde{f}^k$ is a monomorphism and $s_k(X_2) \neq 0$, then $s_k(X_1) \neq 0$.*

Suppose that Y is a locally compact Hausdorff Z_2-space. By $F(T)$ we denote, as above, the fixed point set of the involution T.

Definition 4 (Smith). *Smith's homomorphism of the Z_2-space X $s_{i,k}(X): H_c^i(F(T)) \to H_c^{i+k}(\widetilde{X}\setminus F(T))$ is the homomorphism $s_{i,k}(X)=\partial_2^{i+k-1}\ldots\partial_2^{i+1}\partial_1^i$, where $\partial_1^i=\partial^i \mid H_c^i(F(T)): H_c^i(F(T)) \to H_c^{i+1}(\widetilde{X}\setminus F(T))$ and*

$$\partial_2^s=\partial^s \mid H_c^s(\widetilde{X}\setminus F(T)): H_c^s(\widetilde{X}\setminus F(T)) \to H_c^{i+1}(\widetilde{X}\setminus F(T)).$$

Lemma 7. *Let the locally compact Hausdorff spaces X_1 and X_2 be Z_2-spaces, $F_i(T)$ — the fixed point set of X_i, $i=1, 2$. If the mapping $f: X_1 \to X_2$ is a proper and equivariant one and $f^{-1}F_2(T)=F_1(T)$, then the following diagram is commutative*

$$\begin{array}{ccc} H_c^l(F_2(T)) & \xrightarrow{s_{l,k}(X_2)} & H_c^{k+l}(U_2) \\ f_F^l\downarrow & & \downarrow \widetilde{f}_U^{k+l} \\ H_c^l(F_1(T)) & \xrightarrow{s_{l,k}(X_1)} & H_c^{k+l}(\widetilde{U}_1). \end{array}$$

This diagram needs some clarifications. From the assumptions of Lemma 7 the sets $U_i=X_i\setminus F_i(T)$ are free Z_2-spaces and $f(U_1)\subset U_2$. By $\widetilde{U}_i$ we denote the orbit space of the Z_2-space U_i and by $f_U=f\mid_{U_1}: U_1 \to U_2$. The mapping f_U is an equivariant one, hence this mapping induces a mapping $\widetilde{f_U}: \widetilde{U}_1 \to \widetilde{U}_2$. By f_F we denote the mapping $f_F=f \mid F_1(T)$. The homomorphisms f_F^* and $\widetilde{f}_U^*$ are induced by f_F and $\widetilde{f}_U$, respectively.

Lemma 7 follows from

Lemma 8. *Under the assumptions of Lemma 7, the following diagram is commutative*

$$\begin{array}{ccccccccc} \to H_c^s(\widetilde{U}_2) & \to & H_c^s(X_2) & \to & H_c^s(\widetilde{U}_2)\oplus H_c^s(F_2(T)) & \xrightarrow{\partial^s} & H_c^{s+1}(U_2) & \to \\ \downarrow \widetilde{f}_U^s & & \downarrow f^s & & \downarrow \widetilde{f}_U^s\oplus f_F^s & & \downarrow \widetilde{f}_U^{s+1} & \\ \to H_c^s(\widetilde{U}_1) & \to & H_c^s(X_1) & \to & H_c^s(\widetilde{U}_1)\oplus H_c^s(F_1(T)) & \xrightarrow{\partial^s} & H_c^{s+1}(\widetilde{U}_1) & \to \end{array}$$

From Lemma 7 we obtain

Corollary 5. *From the assumptions of Lemma 7 it follows* that

a) *if f_F^l is an isomorphism and $s_{l,k}(X_1)\neq 0$, then $s_{l,k}(X_2)\neq 0$;*

b) *if $\widetilde{f}_U^{k+l}$ is an monomorphism and $s_{l,k}(X_2)\neq 0$, then $s_{l,k}(X_1)\neq 0$.*

5. Another way to obtain the exact sequence (4). In this section we shall assume that the space X is a free Z_2-space.

Lemma 9. *The sheaf $\mathcal{C}^*(X, Z_2)$ is a Z_2-sheaf and the differentials are Z_2-homomorphisms.*

To prove this Lemma one can use an induction with respect to the dimension i for $\mathcal{C}^i(X, Z_2)$.

First of all it follows directly that the sheaf $Z_2=X\times Z_2$ is a Z_2-sheaf. The action of T, $T: Z_{2x} \to Z_{2T(x)}$ is the identical isomorphism of Z_2 on Z_2 (here Z_{2y} is the stalk of the sheaf Z_2 over the point $y\in X$). The sheaf $\mathcal{C}^0(X, Z_2)$ is generated by the presheaf $U\to \text{Map}\ (U, Z_2)$, here Map (U, Z_2) is the set of all mappings of the set U in Z_2. For $f\in \text{Map}(U, Z_2)$ we define $(Tf)(x)=f(Tx)$ for every $x\in U$. This in an action of Z_2 in the set Map(U, Z_2) which is compatible with the restriction operators. This Z_2-structure of the set Map(U, Z_2) induces the Z_2-structure of the sheaf $\mathbb{C}^\circ(X, Z_2)$. Therefore the sheaf $\mathbb{R}^\circ(X, Z_2) = \mathbb{C}^\circ(X, Z_2)/Z_2$ is a Z_2-sheaf and the projection $\mathbb{C}^\circ(X, Z_2) \to \mathbb{R}(X, Z_2)$ is a Z_2-mapping. To conclude the proof of Lemma 9 it is sufficient to remind that $\mathbb{C}^{i+1}(X, Z_2) = \mathbb{C}^\circ(X, \mathbb{R}^i(X, Z_2))$ and $\mathbb{R}^i(X, Z_2) = \mathbb{C}^i(X, Z_2)/(\mathbb{R}^{i-1}(X, Z_2)$.

Corollary 6. *$\Gamma^*(X, Z_2) = \Gamma(X, \mathbb{C}^*(X, Z_2))$ is a differential Z_2-group, i. e., $\Gamma^q(X, Z_2) = \Gamma(X, \mathbb{C}^q(X, Z_2))$ is a vector space over Z_2 and differentials in $\Gamma^*(X, Z_2)$ are Z_2-linear mappings.*

To check the assertion of Corollary 6 it is sufficient to remind that $\Gamma(X, \mathbb{C}^*(X, Z_2))$ consists of all continuous sections of the sheaf $\mathbb{C}^*(X, Z_2)$ over the space X. If $f \in \Gamma(X, \mathbb{C}^*(X, Z_2))$, then $(Tf)(x) = Tf(x)$ for every $x \in X$.

Lemma 10 ([5], Ch. 2, § 11). *The differential sheaf $\pi_*\mathbb{C}^*(X, Z_2)$ is a flabby resolution of the sheaf $\mathbb{A} = \pi_* Z_2$. Every sheaf $\pi_*\mathbb{C}^q(X, Z_2)$ is Z_2-sheaf and the differentials in $\pi_*\mathbb{C}^*(X, Z_2)$ are Z_2-homomorphisms.*

The Z_2-structure of the sheafs $\pi_*\mathbb{C}^*(X, Z_2)$ is inherited by the Z_2-structure of $\mathbb{C}^*(X, Z_2)$. Here and later we shall consider $\widetilde{X}$ as a Z_2-space in which Z_2 acts trivially, i. e., $Tz = z$ for every $z \in \widetilde{X}$.

Now we need some definitions.

Let Y and Z be Z_2-spaces and $f: Y \to Z$ be an equivariant mapping. Suppose also that L and M are sheafs over Y and Z respectively.

Definition 5. *The Z_2,f-cohomomorphism $k: M \to L$ is a collection of Z_2-homomorphisms $k_U: \Gamma(U, M) \to \Gamma(f^{-1}(U), L)$ for every open U in Z, compatible with restrictions.*

The standard f-cohomomorphism ([5], Ch. 2, § 4) $f_*: f_* L \to L$ is an Z_2, f-cohomomorphism (the Z_2-structure of the sheaf f_*L is inherited by the Z_2-structure of the sheaf L).

For each Z_2, f-cohomomorphism $k: M \to L$ there corresponds unique Z_2-homomorphism $h: M \to f^*L$ such that $k = f_*h$.

By $\widetilde{\pi}: \mathbb{A} \to Z_2$ we shall denote now the standard Z_2, π-cohomomorphism ([5], Ch. 2, § 8). This cohomomorphism induces a Z_2, π-cohomomorphism $\mathbb{C}^*(\widetilde{\pi}): \mathbb{C}^*(\mathbb{A}) \to \mathbb{C}^*(X, Z_2)$. The above representation of Z_2, π-cohomomorphism $\mathbb{C}^*(\widetilde{\pi})$ gives us a Z_2-homomorphism $\widehat{\pi}^*: \mathbb{C}^*(\mathbb{A}) \to \pi_*\mathbb{C}^*(X, Z_2)$.

Lemma 11. *The Z_2-homomorphism $\widehat{\pi}^*$ is an isomorphism.*

The sheaf $\mathbb{A}$ is a subsheaf of the Z_2-sheaf $\mathbb{C}^0(\mathbb{A})$. Let us apply the functor direct image π_* to the sequence of sheafs $0 \to Z_2 \to \mathbb{C}^*(X, Z_2)$. We obtain the exact sequence $0 \to \mathbb{A} \to \pi_*\mathbb{C}^0(X, Z_2)$, because the functor π_* is left exact. So $\mathbb{A}$ is a subsheaf of the Z_2-sheaf $\pi_*\mathbb{C}^0(X, Z_2)$. It follows straightforward that $\hat{\pi}^0 \mathbb{A}$ maps $\mathbb{A}$ into $\mathbb{A}$ identical. Let us consider $\hat{\pi}^0: \mathbb{C}^0(\mathbb{A}) \to \pi_*\mathbb{C}^0(X, Z_2)$.

For a given open set U in the space $\widetilde{X}$ we consider the Z_2-homomorphism $\hat{\pi}^0(U): \Gamma(U, \mathbb{C}^0(\mathbb{A}) \to \Gamma(U, \pi_*\mathbb{C}^0(X, Z_2))$. It is sufficient to check that $\hat{\pi}^0(U)$ is an isomorphism.

According to the definition of $\mathbb{C}^0(\mathbb{A})$, $\Gamma(U, \mathbb{A}) = \Pi \mathbb{A}_{y, y \in U}$ (here $\mathbb{A}_y$ is he stalk of the sheaf $\mathbb{A}$ over the point $y \in U$). The mapping π is closed and $\pi^{-1}(y)$ consists only of two points, hence $\mathbb{A}_y = \text{Map}(\pi^{-1}(y), Z_2)$, i. e., it coincides with the set of mappings of $\pi^{-1}(y)$ into Z_2. Therefore $\Gamma(U, \mathbb{C}^0(\mathbb{A})) = \Pi\,\text{Map}(\pi^{-1}(y), Z_2)$, $y \in U$.

For the group $\Gamma(U, \pi_*\mathbb{C}^0(X, Z_2))$ we have

$$\Gamma(U, \pi_* \mathrm{C}^0(X, Z_2)) = \Gamma(\pi^{-1}(U), \mathrm{C}^0(X, Z_2)) = \mathrm{Map}(\pi^{-1}(U), Z_2).$$

If $s \in \Gamma(U, \mathrm{C}^0(\mathrm{A}))$, then $s = \{s_y\}$, $y \in U$, where $s_y \in \mathrm{Map}(\pi^{-1}(y), Z_2)$. According to the definition of the homomorphism $\hat{\pi}^0(U)$, $\hat{\pi}^0(U)(s)(x) = s_{\pi(x)}(x)$ for every $x \in \pi^{-1}(U)$. It is obvious that the homomorphism $\hat{\pi}^0(U)$ is a monomorphism. Let us prove that $\hat{\pi}^0(U)$ is an epimorphism. Given $t \in \mathrm{Map}(\pi^{-1}(U), Z_2)$, let $s_y \in \Gamma(\pi^{-1}(y), Z_2)$ be $s_y(x) = t(x)$, where $y = \pi(x)$. Now if $s = \{s_y\}$, $y \in U$, then $\hat{\pi}^0(U)(s) = t$, i. e., $\hat{\pi}^0(U)$ is an epimorphism. It is proved that $\hat{\pi}^0$ is a Z_2-isomorphism.

Let us consider the diagram

$$\begin{array}{ccccccccc} 0 \to \mathrm{A} & \longrightarrow & \mathrm{C}^0(\mathrm{A}) & \longrightarrow & \mathrm{R}^0(\mathrm{A}) \to 0 \\ \hat{\pi}^0 \downarrow & & \hat{\pi}^0 \downarrow & & \downarrow \hat{\pi}^0_1 \\ 0 \to \mathrm{A} & \longrightarrow & \pi_* \mathrm{C}^0(X, Z_2) & \longrightarrow & \pi_* \mathrm{R}^0(X, Z_2) \to 0. \end{array} \tag{6}$$

The Z_2-homomorphism $\hat{\pi}^0_1$ is induced by the Z_2-homomorphism $\hat{\pi}^0$. The sheaf $\mathrm{R}^0(X, Z_2)$ is the sheaf $\mathrm{C}^0(X, Z_2)/Z_2$.

It follows from lemma "five" that the homomorphism $\hat{\pi}^0_1$ is an isomorphism. Lemma 11 follows by induction.

Definition 6. *Let $\mathfrak{L}$ be a Z_2-sheaf over the space $\tilde{X}$. By $\mathfrak{L}^\sigma$ we shall denote the subsheaf of all Z_2-invariant elements of the sheaf $\mathfrak{L}$, i. e., $\mathfrak{L}^\sigma = \{l \in \mathfrak{L} \mid Tl = l\}$.*

The inclusion $i: \sigma\mathrm{A} \to \mathrm{A}$ induces an inclusion $\mathrm{C}^*(i)$: $\mathrm{C}^*(\sigma\mathrm{A}) \to \mathrm{C}^*(\mathrm{A})$ (here $\sigma\mathrm{A}$ is the image of homomorphism $\sigma = 1 + T: \mathrm{A} \to \mathrm{A}$).

Lemma 12. *The image of the homomorphism $\mathrm{C}^*(i)$ is $\mathrm{C}^*(\mathrm{A})^\sigma$.*

The Z_2-structure of the sheaf $\mathrm{C}^0(\mathrm{A})$ is given by the Z_2-structure of $\Gamma(U, \mathrm{C}^0(\mathrm{A}))$ for every open set in U in $\tilde{X}$. Let us remind how the group Z_2 acts on the set $\Gamma(U, \mathrm{C}^0(\mathrm{A}))$. We know that $\Gamma(U, \mathrm{C}^0(\mathrm{A})) = \Pi\, \mathrm{Map}(\pi^{-1}(y), Z_2)$. For every $y \in U$ the set $\pi^{-1}(y)$ is a Z_2-set, because $\pi^{-1}(y)$ coincides with some orbit of the action of Z_2 on the space X, $\pi^{-1}(y) = \{x, Tx\}$. Therefore the set $\mathrm{Map}(\pi^{-1}(y), Z_2)$ is a Z_2-set, indeed if $s_y \in \mathrm{Map}(\pi^{-1}(y), Z_2)$, then $(Ts_y)(x) = s_y(Tx)$ for $x \in \pi^{-1}(y)$. The group $\Gamma(U, \mathrm{C}^0(\mathrm{A}))$ is a product of Z_2-sets $\mathrm{Map}(\pi^{-1}(y), Z_2)$ and inherited its Z_2-structure from the Z_2-structure of the sets $\mathrm{Map}(\pi^{-1}(y), Z_2)$. If $\Gamma(U, \mathrm{C}^0(\mathrm{A}))^\sigma = \{s \in \Gamma(U, \mathrm{C}^0(\mathrm{A})) \mid Ts = s\}$, then $s = \{s_y\} \in \Gamma(U, \mathrm{C}^0(\mathrm{A}))^\sigma$ if and only if $s_y(x) = s_y(Tx)$ for every $x \in \pi^{-1}(y)$ and $y \in U$.

The group of continuous sections over the set U of the sheaf $\mathrm{C}^0(\sigma\mathrm{A})$ coincides with $\Gamma(U, \mathrm{C}^0(\tilde{X}, Z_2))$ because $\sigma\mathrm{A}$ is isomorphic to the sheaf $Z_2 = \tilde{X} \times Z_2$. If we denote by $\Gamma(U, \mathrm{C}^0(\mathrm{A}))^\sigma$ the set of all Z_2-invariant sections in $\Gamma(U, \mathrm{C}^0(\mathrm{A}))^\sigma$, then $\Gamma(U, \mathrm{C}^0(\tilde{X}, Z_2)) = \Gamma(U, \mathrm{C}^0(\mathrm{A}))^\sigma$.

The sheaf $\mathrm{C}^0(\mathrm{A})^\sigma$ is defined by the presheaf $U \to \Gamma(U, \mathrm{C}^0(\mathrm{A}))^\sigma$, therefore the image of the homomorphism $\mathrm{C}^0(i)$ is $\mathrm{C}^0(\mathrm{A})^\sigma$.

We have the following commutative diagram

$$\begin{array}{ccccccccc} 0 \to \sigma\mathrm{A} & \longrightarrow & \mathrm{C}^0(\sigma\mathrm{A}) & \longrightarrow & \mathrm{R}^0(\sigma\mathrm{A}) \to 0 \\ i \downarrow & & \mathrm{C}^0(i) \downarrow & & \downarrow \mathrm{C}^0_1(i) \\ 0 \to \mathrm{A} & \longrightarrow & \mathrm{C}^0(\mathrm{A}) & \longrightarrow & \mathrm{R}^0(\mathrm{A}) \to 0. \end{array} \tag{7}$$

The homomorphisms i and $\mathfrak{C}^0(i)$ are monomorphisms. The Z_2-structure of the sheafs $\mathfrak{C}^0(\sigma\mathfrak{A})$ and $\mathfrak{C}^0(\mathfrak{A})$ induces the Z_2-structure of the sheafs $\mathfrak{R}^0(\sigma\mathfrak{A})$ and $\mathfrak{R}^0(\mathfrak{A})$, respectively, and the homomorphism $\mathfrak{C}_1^0(i)$ is a Z_2-homomorphism.

From the diagram (7) we obtain that the image of the homomorphism $\mathfrak{C}_1^0(i)$ coincides with the sheaf $\mathfrak{R}^0(\mathfrak{A})^\sigma$, and that the homomorphism $\mathfrak{C}_1^0(i)$ is a monomorphism. By induction with respect to the dimension i we prove that the image of the homomorphism $\mathfrak{C}^i(i):\mathfrak{C}^i(\sigma\mathfrak{A})\to\mathfrak{C}^i(\mathfrak{A})$ is the sheaf $\mathfrak{C}^i(\mathfrak{A})^\sigma$.

The following corollary follows from Lemma 11:

C o r o l l a r y 7. *The image of the sheaf* $\mathfrak{C}^*(\mathfrak{A})^\sigma$ *with respect to the homomorphism* $\hat{\pi}^*$ *coincides with the sheaf* $(\pi_*\mathfrak{C}^*(X, Z_2))^\sigma$.

Let us denote by $\Gamma^q(L)$ the group $\Gamma(\tilde{X}, \mathfrak{C}^q(L))$. If $\alpha: L\to M$ is a homomorphism of the sheaf L in the sheaf M then by $\Gamma^*(\alpha):\Gamma^*(M)\to\Gamma^*(M)$ we shall denote the homomorphism induced by the homomorphism α.

C o r o l l a r y 8. *The homomorphism* $\Gamma^*(\hat{\pi})$ *is a* Z_2*-isomorphism.*

This Corollary follows from Lemma 11.

It follows from Lemma 12 the following

C o r o l l a r y 9. *The homomorphism* $\Gamma^*(\hat{\pi})\Gamma^*(i):\Gamma^*(\sigma\mathfrak{A})\to\Gamma^*(\mathfrak{A})^\sigma$ *is an isomorphism.*

The group $\Gamma^*(X, Z_2)=\Gamma(X, \mathfrak{C}^*(X, Z_2))$ inherited a structure of Z_2-module from the Z_2-sheaf $\mathfrak{C}^*(X, Z_2)$. Let $\Gamma^*(X, Z_2)^\sigma$ be the set of all Z_2-invariant elements of $\Gamma^*(X, Z_2)$.

C o r o l l a r y 10. *The following diagram is commutative*

$$\begin{array}{ccccccccc} 0\to & \Gamma^*(X, Z_2)^\sigma & \longrightarrow & \Gamma^*(X, Z_2) & \xrightarrow{\sigma} & \Gamma^*(X, Z_2)^\sigma & \to 0 \\ & \uparrow & & \uparrow & & \uparrow & \\ 0\to & \Gamma^*(\sigma\mathfrak{A}) & \longrightarrow & \Gamma^*(\mathfrak{A}) & \xrightarrow{\sigma} & \Gamma^*(\sigma\mathfrak{A}) & \to 0. \end{array}$$

The homomorphism $\Gamma^*(\mathfrak{A})\to\Gamma^*(X, Z_2)$ is induced by the standard π-cohomomorphism $\tilde{\pi}:\mathfrak{A}\to Z_2$ ([5], Ch. 2, § 11). The other vertical arrows are the homomorphisms defined above.

C o r o l l a r y 11. *The cohomology exact sequence induced by the exact sequence*

$$0\to\Gamma^*(X, Z_2)^\sigma\longrightarrow\Gamma^*(X, Z_2)\xrightarrow{\sigma}\Gamma^*(X, Z_2)^\sigma\to 0$$

coincides with (4).

R e m a r k. If X is a locally compact Hausdorff space, then the exact sequence (5) is induced by the exact sequence

$$0\to\Gamma_c^*(X, Z_2)^\sigma\longrightarrow\Gamma_c^*(X, Z_2)\xrightarrow{\sigma}\Gamma^*(X, Z_2)^\sigma\to 0.$$

Here $\Gamma_c^q(X, Z_2)=\Gamma_c(X, \mathfrak{C}^q(X, Z_2))$ and Γ_c is the functor of sections with compact supports.

6. Yang's homomorphism of a free Z_2-space. In this section the space X shall be a free Z_2-space.

Suppose that F is a closed subset of a space X, with the following property $F\cup T(F)=X$. By B we shall denote the set $F\cap T(F)$. Let

$$r_1 : Z_2 \to Z_{2\,F}, \quad r_2 : \quad Z_2 \to Z_{2\,T(F)},$$
$$s_1 : Z_{2\,F} \to Z_{2\,B}, \quad s_2 : \quad Z_{2\,T(F)} \to Z_{2\,B}$$

are the standard epimorphisms.

Lemma 13 ([5], Ch. 2, § 13). *The following sequence is exact*

$$0 \to Z_2 \xrightarrow{\alpha} Z_{2\,F} \oplus Z_{2\,T(F)} \xrightarrow{\beta} Z_{2\,B} \to 0. \tag{8}$$

Here $\alpha = (r_1, r_2)$, $\beta = s_1 + s_2$.

From the sequence (8) we obtain the exact sequence

$$0 \to \Gamma^*(X, Z_2) \xrightarrow{\alpha} \Gamma^*(F, Z_2) \oplus \Gamma^*(T(F), Z_2) \xrightarrow{\beta} \Gamma^*(B, Z_2) \to 0.$$

The homomorphism $T : \Gamma^*(X, Z_2) \to \Gamma^*(X, Z_2)$ induces an isomorphism $T : \Gamma^*(F, Z_2) \to \Gamma^*(T(F), Z_2)$.

Lemma 14. If $\zeta \in \Gamma^i(X, Z_2)$ *and* $\zeta|_{T(F)} = T\zeta|_F$, *then* $\zeta \in \Gamma^i(X, Z_2)^\sigma$.

This Lemma is proved by an induction with respect to the dimension i.

Now we are in a positlon to define Yang's homomorphism.

The space $B = F \cap T(F)$ is a free Z_2-space. Therefore we can apply the results from the sections 1—4 for the space B.

Let $\{\xi\} \in H^i(\widetilde{B})$ and $\xi \in \Gamma^i(B, Z_2)^\sigma$ be a representative element of the cohomology class $\{\xi\}$. From Corollary 10 it follows: there is an element $\eta \in \Gamma^i(B, Z_2)$ such that $\sigma\eta = \xi$. Let us consider $(\eta|_F, T\eta|_{T(F)})$, this is an element ot $\Gamma^i(F, Z_2) \oplus \Gamma^i(T(F), Z_2)$. The image of $(\eta|_F, T\eta|_{T(F)})$ with respect to the homomorphism β coincides with ξ. The element $\hat{\eta}$ — the differential of the element η, $\hat{\eta} = (\delta\eta|_F, \delta T\eta|_{T(F)})$ belongs to the image of $\Gamma^*(X, Z_2)$ with respect to the homomorphism α, because ξ is a cycle: $\beta\hat{\eta} = \delta(\eta|_T + T\eta|_T) = \delta\sigma\eta = \delta\xi = 0$. Let $\zeta \in \Gamma^{i+1}(X, Z_2)$ be such an element that $\alpha\zeta = \hat{\eta}$, i. e., $\zeta|_F = \delta\eta|_F$, $\zeta|_{T(F)} = T\delta\eta|_{T(F)}$. By Lemma 14, $\zeta \in \Gamma^{i+1}(X, Z_2)^\sigma$. The cohomology class of the element ζ we shall denote by

$$\Delta^i(\{\xi\}) \quad (\Delta^i(\{\xi\}) \in H^{i+1}(\widetilde{X})).$$

The element $\Delta^i(\{\xi\})$ depends only on the cohomology class $\{\xi\}$. Therefore we have the homomorphism $\Delta^i : H^i(\widetilde{B}) \to H^{i+1}(\widetilde{X})$ — this is Yang's homomorphism.

Lemma 15. *The following diagram is commutative*

$$\begin{array}{ccc} H^i(\widetilde{X}) & \xrightarrow{s_{i,k}(X)} & H^{i+k}(\widetilde{X}) \\ \widetilde{k}^i \downarrow & & \uparrow \Delta^{i+k-1} \\ H^i(\widetilde{B}) & \xrightarrow{s_{i,k-1}(B)} & H^{i+k-1}(\widetilde{B}). \end{array}$$

Here $k : B \to X$ is the inclusion, and $\widetilde{k} : \widetilde{B} \to \widetilde{X}$ is induced by the equivariant mapping k.

It is sufficient to prove that the following diagram is commutative

$$\begin{array}{ccc} H^{i-1}(\widetilde{X}) & \xrightarrow{d^i(X)} & H^i(\widetilde{X}) \\ \widetilde{k}^{i-1} \downarrow & \nearrow \Delta^{i-1} & \\ H^{i-1}(\widetilde{B}) & & \end{array}$$

The homomorphism $\partial^t(X)$ is from Smith's exact sequence of the Z_2-space X.

Let $\{\theta\} \in H^{t-1}(X)$ and $\theta \in \Gamma^{t-1}(X, Z_2)^\sigma$ be a representative element of the cohomology class $\{\theta\}$. The representative element of the cohomological class $\widetilde{k}^{t-1}(\{\theta\})$ is $\theta|_B \in \Gamma^{t-1}(B, Z_2)^\sigma$. Let $\Sigma \in \Gamma^{t-1}(X, Z_2)$ and $\sigma\Sigma = \theta$ and $\eta = \Sigma|_B$. If $\hat{\eta} = (\delta^{t-1}\eta|_F, T\delta^{t-1}\eta|_{T(F)})$ and $\alpha\zeta = \hat{\eta}$, then $\zeta = \delta^{t-1}\Sigma$. By the construction of the homomorphism Δ^{t-1} follows $\Delta^{t-1}(\{\eta\}) = \{\zeta\}$. By the definition of the homomorphism $\partial^t(X)$ we obtain $\partial^t(X)\{\theta\} = \{\delta^{t-1}\Sigma\} = \{\zeta\}$.

L e m m a 16. *If* $s_n(X) \neq 0$, *then* $s_{n-1}(B) \neq 0$.

Indeed, by Lemma 15, we have $s_n(X) = \Delta^{n-1} s_{n-1}(B)\widetilde{k}^0$.

L e m m a 17. *Let X be a free Z_2-space and* $s_n(X) \neq 0$. *If* $f: X \to R^k$ *is a single-valued continuous mapping of the space X in the k-dimensional Euclidean space R^k, $k \leq n$, and $C = \{x \in X \mid f(x) = f(Tx)\}$, then C is a free Z_2-space and* $s_{n-k}(C) \neq 0$.

We follow Yang [3]. Let $f = (f_1, \ldots, f_k)$, where $f_i: X \to R$ (R is the real line), and $X_0 = X$, $X_j = \{x \in X \mid f_i(x) = f_i(T(x)),\ i = 1, \ldots, j\}$, $X_k = C$. If $F_j = \{x \in X_{j-1} \mid f_j(x) \leq f_j(Tx)\}$, then F_j is a closed subset of the space X_{j-1}, $F_j \cup T(F_j) = X_{j-1}$, and $F_j \cap T(F_j) = X_j$. It follows from Lemma 16 that $s_{n-i}(X_i) \neq 0$, hence $s_{n-k}(C) \neq 0$.

7. **Single-valued continuous mappings of a free Z_2-space in R^k — Yang's theorem**. We need the definition and some basic properties of the cohomological dimension ([8], ch. 2, §§ 2,3).

D e f i n i t i o n 7. *Let X be a paracompact space, $\mathfrak{L}$ a sheaf of Abelian groups over the space X. We shall say that* $\dim_{\mathfrak{L}} X \leq n$ *if* $H^{n+1}(X, \mathfrak{L}_U) = 0$ *for every open set U in the space X.*

By $\dim X$ we shall denote the covering dimension of the space X ([7], Ch. 1, § 1.4).

T h e o r e m 1 ([11]). *Let X be paracompact space. If Z is the group of integers and Z is the constant sheaf with the stalk Z, then* $\dim_Z X = \dim X$.

T h e o r e m 2 ([8]). *Let X be a paracompact space and F a closed subset of X. If $\mathfrak{L}$ is a sheaf of Abelian groups over the space X, then* $\dim_{\mathfrak{L}} F \leq \dim_{\mathfrak{L}} X$.

T h e o r e m 3 ([8]). *If X is a paracompact space, then* $\dim_{Z_2} X \leq \dim_Z X$.

Suppose that a paracompact space X is a free Z_2-space. The mapping $\pi: X \to \widetilde{X}$ is a closed one, hence the space $\widetilde{X}$ is also a paracompact space [9]. Suppose also that the space X is a n-cohomological sphere over Z_2, i e., $H^*(S^n, Z_2) = H^*(X)$, and $\dim_{Z_2} X < \infty$. Here S^n is the n-dimensional Euclidean sphere.

It follows from the Smith exact sequence of the space X that $s_n(X) \neq 0$.

If $f: X \to R^k$ is a single-valued continuous mapping of the space X in the k-dimensional Euclidean space R^k, then $s_{n-k}(C) \neq 0$, where $C = \{x \in X \mid f(x) = f(Tx)\}$ (Lemma 6). Therefore $H^{n-k}(\widetilde{C}) \neq 0$, where $\widetilde{C}$ is the orbit space of the Z_2-space C. By the definition of the cohomological dimension $\dim_{Z_2} \widetilde{C} \geq n-k$. The space $\widetilde{C}$ is a closed subset of the space $\widetilde{X}$, hence $\widetilde{C}$ is a paracompact space.

Let us consider the mapping $\pi_1 = \pi|C: C \to \widetilde{C}$. This mapping is a local homeomorphism. Having in mind that C and $\widetilde{C}$ are paracompact spaces, we obtain $\dim_{Z_2} C = \dim_{Z_2} \widetilde{C}$, hence we have $\dim_{Z_2} C \geq n-k$. But $\dim C \geq \dim_{Z_2} C$ because C is a paracompact space. Therefore the following theorem is proved.

Theorem 4. *Let X be a paracompact n-cohomological sphere and $\dim_{Z_2} X < \infty$. If X is a free Z_2-space and $f: X \to R^k$ is a continuous single-valued mapping, then*

$$\dim \{x \in X \mid f(x) = f(T(x))\} \geq n-k.$$

8. Acyclic mappings of a free Z_2-space into R^k. Let X and Y be Hausdorff topological spaces and $\psi: X \to Y$ a multi-valued mapping.

Definition 8. *The mapping ψ is called an acyclic one if a) ψ is an upper-semicontinuous multi-valued mapping* ([10], ch. 1), *b) the set $\psi(x)$ is an acyclic compact set with respect to the Cech cohomology with Z_2 coefficients for every $x \in X$.*

Definition 9. *The compact space K is called acyclic if K is a connected space and $H^i(K)=0$, for $i \geq 1$.*

Denoting by $\Gamma(\psi)$ the graph of the mapping ψ, i. e., $\Gamma(\psi) = \{(x, y) \in X \times Y \mid y \in \psi(x)\}$, there are two projections $p: \Gamma(\psi) \to X$, $q: \Gamma(\psi) \to Y$ given by $p(x, y) = x$, $q(x, y) = y$ for every $(x, y) \in \Gamma(\psi)$.

The projection p has the following properties:

a) p is a closed mapping, b) the set $p^{-1}(x)$ is an acyclic compact set for every $x \in X$.

It follows from the Vietoris-Begle theorem ([5], Ch. 2, § 11) that the homomorphism $p^*: H^*(X) \to H^*(\Gamma(\psi))$ is an isomorphism (here p^* is the homomorphism induced by the mapping p).

Definition 10. *The multi-valued mapping $\Phi: X \to X$ is said to be an involution of the space X if the graph of the mapping Φ, $\Gamma(\Phi) = \{(x, y) \in X \times X \mid y \in \Phi(x)\}$ is symmetrical with respect to the diagonal of the space $X \times X$, i. e., $(x, y) \in \Gamma(\Phi)$ if and only if $(y, x) \in \Gamma(\Phi)$.*

Definition 11. *The multi-valued involution $\Phi: X \to X$ is called a free involution of the space X if $x \notin \Phi(x)$ for every $x \in X$.*

Definition 12. *The multi-valued involution $\Phi: X \to X$ is called an acyclic involution if the mapping Φ is an acyclic one.*

Let $F: X \to R^k$ be an acyclic mapping of the space X in the k-dimensional Euclidean space R^k. Let us consider the mapping $F \times F: X \times X \to R^k \times R^k$ given by $F \times F(x, y) = F(x) \times F(y)$ for $(x, y) \in X \times X$.

Lemma 18. *The mapping $F \times F$ is an acyclic one.*

This lemma follows from the Küneth formula ([5], Ch. 2, § 18) and the definition of an upper-semicontinuous mapping.

Suppose that $\Phi: X \to X$ is an acyclic free involution, the graph $\Gamma(\Phi)$ is a subset of the space $X \times X$, let us put $\theta = F \times F \mid \Gamma(\Phi)$. It follows by Lemma 18 that the mapping θ is an acyclic one. Let $\Gamma(\theta)$ be the graph of the mapping θ, i. e.,

$$\Gamma(\theta) = \{(x, y, u, v) \in X \times X \times R^k \times R^k \mid y \in \Phi(x), u \in F(x), v \in F(y)\}.$$

The space $\Gamma(\theta)$ is a free Z_2-space. Indeed, the mapping $T: \Gamma(\theta) \to \Gamma(\theta)$ given by $T(x, y, u, v) = (y, x, v, u)$ is a free involution of the space $\Gamma(\theta)$. The space $\Gamma(\Phi)$ is also a free Z_2-space — the involution $T: \Gamma(\Phi) \to \Gamma(\Phi)$ is given by $T(x, y) = (y, x)$ for $(x, y) \in \Gamma(\Phi)$.

Let us consider the projection $r: \Gamma(\theta) \to \Gamma(\Phi)$, $r(x, y, u, v) = (x, y)$ for $(x, y, u, v) \in \Gamma(\theta)$. The mapping r has the following properties: a) $r(\Gamma(\theta)) = \Gamma(\Phi)$

b) $r^{-1}(x, y)$ is an acyclic compact set for every $(x, y) \in \Gamma(\Phi)$, c) r is a closed mapping, d) r is an equivariant mapping, i. e., $rT = Tr$.

It follows from property d) of the mapping r that this mapping induces the mapping $\tilde{r}$ of the orbit space $\tilde{\Gamma}(\theta)$ in the orbit space $\tilde{\Gamma}(\Phi)$.

Lemma 19. *The mapping* $\tilde{r}: \tilde{\Gamma}(\theta) \to \tilde{\Gamma}(\Phi)$ *has the following properties:*

a) $\tilde{r}(\tilde{\Gamma}(\theta)) = \tilde{\Gamma}(\Phi)$; b) $\tilde{r}$ *is a closed mapping,* c) *the space* $\tilde{r}^{-1}(z)$ *is an acyclic compact space for* $z \in \tilde{\Gamma}(\Phi)$.

The proof of this lemma is straightforward.

Applying the Vietoris-Begle theorem ([5], Ch. 2, § 11) to the mapping $\tilde{r}$ we obtain that the homomorphism $\tilde{r}^*$ of the group $H^*(\tilde{\Gamma}(\Phi))$ into the group $H^*(\tilde{\Gamma}(\theta))$ is an isomorphism (here $\tilde{r}^*$ is the homomorphism induced by the mapping $\tilde{r}$).

Lemma 20. *If* $s_n(\Gamma(\Phi)) \neq 0$, *then* $s_n(\Gamma(\theta)) \neq 0$.

This lemma follows from Corollary 4, b).

Let us consider the mapping $\varphi: \Gamma(\theta) \to R^k$ given by $\varphi(x, y, u, v) = u$ for $(x, y, u, v) \in \Gamma(\theta)$ and denote by $C = \{z \in \Gamma(\theta) \mid \varphi(z) = \varphi(Tz)\}$. The set C is a Z_2-equivariant subset of $\Gamma(\theta)$. If $B = r(C)$ and $t = r|_C: C \to B$, then t is an equivariant mapping.

Lemma 21. *If* $s_n(\Gamma(\theta)) \neq 0$, *then* $s_{n-k}(B) \neq 0$.

This lemma follows from Lemma 17 and Corollary 4, b).

Corollary 12. *If* $s_n(\Gamma(\Phi)) \neq 0$, *then* $\{x \in X \mid F(x) \cap F(\Phi(x)) \neq \emptyset\} \neq \emptyset$.

Lemma 22. *If X is an n-cohomological sphere and* $\dim_{Z_2} X < \infty$ *then* $s_n(\Gamma(\Phi)) \neq 0$.

We know that $p^*: H^*(X) \to H^*(\Gamma(\Phi))$ is an isomorphism, hence the space $\Gamma(\Phi)$ is an n-cohomological sphere. It follows from the Hurewitz-Sklyarenko theorem [12] that

$$\dim_{Z_2} \Gamma(\Phi) \leq 2 \dim_{Z_2} X < \infty.$$

It follows from the Smith exact sequence that $s_n(\Gamma(\Phi)) \neq 0$.

Corollary 13. *If X is n-cohomological sphere and* $\dim_{Z_2} X < \infty$, *then* $\{x \in X \mid F(x) \cap F(\Phi(x)) \neq \emptyset\} \neq \emptyset$.

In the case of compact space X, this has been proved in [4].

Corollary 14. *Let X be a paracompact n-cohomological sphere and* $\dim X < \infty$. *If* $l = \max\{\dim_{Z_2} \Phi(x) \mid x \in X\}$, *then* $\dim \{x \in X \mid F(x) \cap F(\Phi(x)) \neq \emptyset\} \geq n - k - l$.

By Hurewitz-Sklyarenko theorem [12] we obtain

$$\dim \{x \in X \mid F(x) \cap F(\Phi(x)) \neq \emptyset\} \geq \dim_{Z_2} B - l.$$

As soon as $s_n(\Gamma(\Phi)) \neq 0$ (Lemma 22), it follows from Lemmas 20, 21 and 22 that $s_{n-k}(B) \neq 0$, hence $\dim_{Z_2} B \geq n - k$.

It follows from Corollary 14 the following theorem.

Theorem 5. *Let a paracompact space X be an n-cohomological sphere and* $\dim X < \infty$. *If X is a free Z_2-space and* $F: X \to R^k$ *an acyclic mapping, then*

$$\dim \{x \in X \mid F(x) \cap F(Tx) \neq \emptyset\} \geq n - k.$$

9. Single-valued mappings of a Z_2-space in the k-dimensional Euclidean space R^k. Let a locally compact Hausdorff space X be a Z_2-

space and $F=\{x\in X \mid Tx=x\}$ the fixed point set of X. The space $U=X\setminus F$ is a locally compact Hausdorff free Z_2-space. By $\widetilde{U}$ we shall denote the orbit space of U.

Suppose that A is a closed subset of the space X such that $A\cup T(A)=X$. The set A contains the set F, also the set $T(A)$ contains F. The set $B=A\cap T(A)$ is closed in the space X, hence it is a locally compact space.

Let us consider the set $V=B\setminus F$. It is a locally compact free Z_2-space. The identical inclusion $j: V\to U$ is proper mapping, i. e., for every compact set $K\subset U$, the set $j^{-1}(K)$ is a compact one (it follows from $j^{-1}(K)=B\cap K$).

The mapping $j: V\to U$ induces the mapping $\widetilde{j}:\widetilde{V}\to\widetilde{U}$ of the orbit space of V in the orbit space of U. As soon as the mapping j is a proper one, the mapping $\widetilde{j}$ is proper too.

Let us consider the set $C=U\cap A$. This set has the following properties: a) C is a closed subset of the space A, b) $C\cup T(C)=U$, c) $C\cap T(C)=V$.

It follows from Lemma 7 that the diagram

$$\begin{array}{ccc} H^l_c(F) & \xrightarrow{s_{l,k}(X)} & H^{k+l}_c(\widetilde{U}) \\ \| & & \uparrow \Delta^{k+l-1} \\ H^l_c(F) & \xrightarrow{s_{l,k-1}(B)} & H^{k+l-1}_c(\widetilde{V}) \end{array}$$

is commutative, i. e., the following lemma is true

Lemma 23. $s_{l,k}(X)=\Delta^{k+l-1}s_{l,k-1}(B)$.

Corollary 15. *If* $s_{l,k}(X)\neq 0$, *then* $s_{l,k-1}(B)\neq 0$.

Lemma 24. *Let a locally compact Hausdorff space X be a Z_2-space and* $s_{l,m}(X)\neq 0$. *If $f: X\to R^k$ is a continuous single-valued mapping of the space X in the k-dimensional Eucludean space R^k, then for* $k\leq m-1$, $s_{l,m-k}(A)\neq 0$, *where* $A=\{x\in X \mid f(x)=f(Tx)\}$.

Let $X_0=X$, $X_i=\{x\in X \mid f_s(x)=f_s(Tx),\ s=1,\dots,i\}$ and $A_i=\{x\in X_{i-1} \mid f_i(x)=f_i(Tx)\}$. Here $f=(f_1,\dots,f_k): X\to R^k$.

By the assumptions of Lemma 24, $s_{l,m}(X)\neq 0$. The set A_1 is a closed one and $A_1\cup T(A_1)=X$. It follows by Corollary 15 that $s_{l,m-1}(X_1)\neq 0$. By induction with respect to i we obtain $s_{l,m-k}(A)\neq 0$.

Suppose that a locally compact Hausdorff space X is a free Z_2-space and $\dim_{Z_2}X<\infty$. Let also X be an n-cohomological sphere, i. e., the cohomology $H^*_c(X)$ is isomorphic to $H^*(S^n)$. It follows from [6, Ch. 3, § 4, Corollary 4.5] that the fixed point set F of X is an r-cohomological sphere for some $-1\leq r\leq n$. The case $F=\emptyset$ is considered in 7. Now we shall assume that $F\neq\emptyset$, i. e., F is an r-cohomological sphere and $0\leq r\leq n$.

It follows from lemma 7 in [13] that $s_{r,n-r}(X)\neq 0$.

Theorem 6. *Let a locally compact Hausdorff space X be a Z_2-space and $F\neq\emptyset$ be the fixed point set of X. Let X be an n-cohomological sphere and F an r-cohomological sphere. If $f: X\to R^k$ is a continuous single-valued mapping and $k\leq n-r-1$ then*

$$\dim_{Z_2}\{x\in X\setminus F \mid f(x)=f(Tx)\}\geq n-k.$$

We have $s_{r,n-r}(X)\neq 0$. It follows from Lemma 24 that $s_{r,n-r-k}(A)\neq 0$ where $A=\{x\in X \mid f(x)=f(Tx)\}$. Therefore $H^{n-k}_c(\widetilde{A\setminus F})\neq 0$ ($\widetilde{A\setminus F}$ is the orbit space of the Z_2-space $A\setminus F$). From this we obtain Theorem 6.

C o r o l l a r y 16. *Let a metrizable locally compact Hausdorff Z_2-space X be an n-cohomological sphere and the fixed point set F of X be an r-cohomological sphere, $0 \leq r \leq n$. If $f: X \to R^k$ is a continuous single-valued mapping, $k \leq n-r-1$ and* $\dim X < \infty$, *then*

$$\dim \{x \in X \setminus F \mid f(x) = f(Tx)\} \geqq n-k.$$

C o r o l l a r y 16′. *Let T be an orthogonal involution of S^n with the fixed point set S^r. If $f: S^n \to R^k$ is a continuous single-valued mapping and $k \leq n-r-1$, then*

$$\dim \{x \in S^n \setminus S^r \mid f(x) = f(Tx)\} \geqq n-k.$$

In this case we can apply Yang's theorem. Let

$S^n = \{x \in R^{n+1} \mid \| x \| = 1\}$ and

$S^r = \{x = (x_1, \ldots, x_{n+1}) \in S^n \mid x_{r+1} = \cdots = x_n = 0\}$. Let us consider

$S^{n-r-1} = \{x = (x_1, \ldots, x_{n+1}) \in S^n \mid x_1 = \cdots = x_r = 0\}$.

The involution $T: S^{n-r-1} \to S^{n-r-1}$, which is induced by T is the antipodal mapping. For the mapping $\varphi = f \mid S^{n-r-1}$ in the case $k \leq n-r-1$ we obtain $\dim \{x \in S^{n-r-1} \mid \varphi(x) = \varphi(-x)\} \geq n-k-r-1$.

Corollary 16′ gives a stronger result.

10. Acyclic mappings of a Z_2-space in the k-dimensional Euclidean space R^k. In this section the space X shall be a locally compact Hausdorff Z_2-space and F shall be the fixed point set of X. Suppose that $\Phi: X \to R^k$ is an acyclic mapping from the space X in the k-dimensional Euclidean space R^k. The mapping Φ is called a simple acyclic mapping if the mapping $\Phi|_F: F \to R^k$ is a single-valued one.

In what follows Φ shall be a single acyclic mapping.

Let $\psi: X \to R^k \times R^k$ be the mapping given by $\psi(x) = \Phi(x) \times \Phi(Tx)$ for every $x \in X$. The mapping ψ is an acyclic one and the graph $\Gamma(\psi)$ of ψ is a locally compact Hausdorff space,

$$\Gamma(\psi) = \{(x, u, v) \in X \times R^k \times R^k \mid u \in \Phi(x),\ v \in \Phi(Tx)\}.$$

The projection $p: \Gamma(\psi) \to X$ ($p(x, u, v) = x$, for $(x, u, v) \in \Gamma(\psi)$) is a closed mapping and $p^{-1}(x)$ is an acyclic compact set for every $x \in X$.

The space $\Gamma(\psi)$ is a Z_2-space: $T: \Gamma(\psi) \to \Gamma(\psi)$ is given by $T(x, u, v) = (T(x), v, u)$. Let F_1 be the fixed point set of $\Gamma(\psi)$ ($F_1 = \{(x, u, v) \in \Gamma(\psi) \mid x \in F,$ $u = v = \varphi(x)\}$). The mapping $p|_{F_1}: F_1 \to F$ is a homeomorphism and $p^{-1}(F) = F_1$ (here we use that Φ is a single-valued mapping).

The sets $U = X \setminus F$ and $V = \Gamma(\psi) \setminus F_1$ are free Z_2-spaces. By $\widetilde{U}$ and $\widetilde{V}$ we shall denote the orbit spaces of U and V, respectively. The projection p maps the set U onto the set V and if $q = p \mid U: U \to V$, then the mapping q is a closed and proper one. The mappings p and q induce the mappings $\widetilde{p}: \widetilde{\Gamma}(\psi) \to \widetilde{X}$, $\widetilde{q}: \widetilde{U} \to \widetilde{V}$ of the orbit spaces. The last mappings have the following properties: a) $\widetilde{p}$ and $\widetilde{q}$ are closed and proper mappings, b) the set $\widetilde{p}^{-1}(z)$ is an acyclic compact set for every $z \in \widetilde{X}$, c) the set $\widetilde{q}^{-1}(u)$ is an acyclic compact set for every $u \in \widetilde{V}$.

Applying the Vietoris-Begle theorem to the mappings $\widetilde{p}$ and $\widetilde{q}$ we obtain that the homomorphisms $\widetilde{q}^*: H_c^*(\widetilde{V}) \to H_c^*(\widetilde{U})$ and $\widetilde{p}^*: H_c^*(\widetilde{X}) \to H_c^*(\widetilde{\Gamma}(\psi))$ are isomorphisms.

L e m m a 25. *If* $s_{l,m}(X)\neq 0$, *then* $s_{l,m}(\Gamma(\Phi))\neq 0$.

This lemma follows immediately from Corollary 5.

L e m m a 26. *If* $A=\{x\in X \mid \Phi(x)\cap\Phi(Tx)\neq\emptyset\}$ *and* $s_{l,m}(X)\neq 0$, *then* $s_{l,m-k}(A)\neq 0$ *for* $k\leq m-1$.

Let us consider the single-valued continuous mapping $\varphi:\Gamma(\psi)\to R^k$ given by $\varphi(x,u,v)=u$ for every $(x,u,v)\in\Gamma(\psi)$. The set $B=\{z\in\Gamma(\psi) \mid \varphi(z)=\varphi(Tz)\}$ coincides with the set $\{(x,u,v)\in\Gamma(\psi) \mid u=v\}$. The projection p maps the set B onto the set A. It follows from Lemma 25 that $s_{l,m}(\Gamma(\psi))\neq 0$. The assertion of Lemma 26 follows from Lemma 24.

From Lemma 26 follows the following theorem.

T h e o r e m 7. *Let* X *be a metrizable locally compact* Z_2*-space and* $\dim X<\infty$. *Let* X *be an n-cohomological sphere and the fixed point set* F *of* X *is r-cohomological sphere,* $0\leq r\leq n$. *If* $\Phi:X\to R^k$ *is a simple acyclic mapping and* $k\leq n-r-1$, *then*

$$\dim\{x\in X\setminus F \mid \Phi(x)\cap\Phi(Tx)\neq\emptyset\}\geq n-k.$$

R e m a r k s:

1) We consider in 10. a multi-valued acyclic mapping $\Phi:X\to R^k$ of n-cohomological sphere X into the k-dimensional Euclidean space R^k. On the space X acts a free involution $T:X\to X$.

Theorem 5 is true for admissible multi-valued mappings. The multi-valued mapping $\psi:X\to R^k$ is called admissible if there are n-cohomological spheres X_i and free involutions $T:X_i\to X_i$ and acyclic mappings $\psi_i:X_{i-1}\to X_i$ such that $X_0=X$, $X_n=R^k$, $0\leq i\leq n$ and $\psi=\psi_{n-1}\ldots\psi_0$.

2) In 10. we consider a single acyclic mapping $\Phi:X\to R^k$. The space X is an n-cohomological sphere and the fixed point set F of X is an r-cohomological sphere, $0\leq r\leq n$.

The multi-valued mapping $\psi:X\to R^k$ is called a single admissible mapping if there are n-cohomological spheres X_i with involutions $T:X_i\to X_i$ and single acyclic mappings, $0\leq i\leq n$, $\psi_i:X_{i-1}\to X_i$ such that:

a) the fixed point set F_i of the space X_i is an r-cohomological sphere for every i, $0\leq i\leq n$;

b) $X_0=X$, $X_n=R^k$;

c) $\psi=\psi_{n-1}\ldots\psi_0$;

d) $\psi_i \mid F_i$ are acyclic mappings for $0\leq i\leq n$.

Theorem 7 is true for single admissible mappings $\Phi:X\to R^k$.

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