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ON THE INVARIANCE OF DOMAIN

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Let E be a Banach space and let $\Phi: E \to E$ be an acyclic, compact ε-strong vector field. Then $\Phi(E)=E$.

Let R^{n+1} be the $(n+1)$-dimensional Euclidean space. In [1] Brouwer proved that for a homeomorphism $f: R^{n+1} \to R^{n+1}$ the set $f(R^{n+1})$ is an open set in R^{n+1}. This famous theorem is one of the main propositions of the dimension theory and has important applications. In [2] the following generalization of Brouwer's theorem about the invariance of domain was proved by Borsuk: if $f: R^{n+1} \to R^{n+1}$ is an ε-map (for a given $\varepsilon>0$) then the set $f(R^{n+1})$ is open in R^{n+1}. We remind that for metric spaces X and Y and $f: X \to Y$ being a single valued continuous mapping, f is called an ε-map if the diameter of the set $f^{-1}(y)$ is smaller than ε for every point $y \in f(X)$. One sufficient condition for $f(R^{n+1})=R^{n+1}$ is given by Borsuk: if f is a strong ε-map, then $f(R^{n+1})=R^{n+1}$. The map $\varphi: X_1 \to X_2$ is said to be a strong ε-map if there exists a positive real number η such that $\varrho_1(x_1, x_2)<\varepsilon$ when $\varrho_2(y_1, y_2)<\eta$ (here $y_i=\varphi(x_i)$ and ϱ_i is the metric of the space X_i, $i=1, 2$). In the case when X_1 and X_2 are compact spaces, every ε-map is a strong ε-map (but not in general case). The theorem of Borsuk about the invariance of domain was obtained for the first time in [2] from an application of the Antipoden Satz [3].

The results of the theory of the invariance of domain are of some interest in connection with the problem of the existence of solutions of various classes of equations. To illustrate this, let us consider the equation $y=f(x)$, where f is a map of X in Y. Let us suppose also that $f(X)$ is an open set in Y and that for $y_0 \in Y$ this equation has a solution. In this case the above equation is solvable for every y, which is sufficiently close to y_0.

In [4] there is given a new treatment of the theory of Fredholm's integral equations, in which the more general case of equations $y=x-Ax$ (A being a linear compact operator of the Banach space E) is considered. The general case of a nonlinear A is more complicated and just in this case the results of the theory of invariance of domain are used.

In [6] is given the following generalization of the above mentioned theorem of Brouwer of the invariance of domain. The image of a Banach space under the mapping $f=I-A: E \to E$ is an open set in E under the condition that A is compact and one-to-one (here, as usually, I is the identity mapping of the space E). In the case when A is compact and ε-map this theorem is proved in [5]. In connection with this last generalization the notion of degree of a map, first introduced in [7], is used as well as a generalization of Borsuk's Antipodensatz [8].

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It is a question of interest: are the theorems about invariance of domain valid in the case of multivalued maps? In [9] the case of a multivalued acyclic ε-map $F: R^{n+1} \to R^{n+1}$ is considered. For such a map the set $F(R^{n+1})$ is an open set in R^{n+1}.

The purpose of this paper is to prove this theorem in the case of infinite dimensional linear spaces.

1. Preliminaries. Here we shall recall some definitions and notations.

Definition 1. *A multivalued map $\Phi: X \to Y$ is called acyclic if it is upper semicontinuous and for every $x \in X$ the set $\Phi(x)$ is a connected compact one and $H^i(\Phi(x))=0$ for $i \geqq 1$.*

In this paper we shall use Grothendieck-Godement cohomology with Z_2 coefficients [10].

Definition 2. *A multivalued map $\Phi: X \to Y$ is said to be compact if the set $\Phi(X)$ has a compact closure in the space Y.*

The closure of the set A in the space Y we shall denote by $\operatorname{cl} A$. By $\operatorname{Fr} A$ we shall denote the frontier of the set A in the space Y and by $\operatorname{Int} A$ — the interior of the set A in Y.

In the case when X is a metric space with metric ϱ, and $B \subset X$ we shall denote by $O_\varepsilon B$ the set $\{y \in X \setminus \varrho(B, y) < \varepsilon\}$ (here $\varrho(B, y) = \inf \varrho(x, y)$, $x \in B$).

Definition 3. *Suppose that X and Y are metric spaces and $\Phi: X \to Y$ is a multivalued map. The map Φ is called a strong ε-map if there exists a positive number η such that $\varrho(x_1, x_2) < \varepsilon$ as soon as $O_\eta \Phi(x_1) \cap \Phi(x_2) \neq \emptyset$ (here ϱ is the metric in the space X).*

We shall consider the Banach space E with a metric given by its norm.

Definition 4. *Given an acyclic compact map $F: E \to E$, the map $\Phi = I - F: E \to E$ of the space E is said to be a compact acyclic vector field (for $x \in E$, $\Phi(x) = x - F(x)$).*

Definition 5. *A compact acyclic vector field $\Phi = I - F$ is called a strong ε-field if F is a strong ε-map.*

Suppose $\theta: X \to Y$ is a multivalued map. By $\Gamma(X, \theta)$ we shall denote the graph of the map, i. e., $\Gamma(X, \theta) = \{(x, y) \in X \times Y \mid y \in \theta(x)\}$. There is a projection $p(\theta): \Gamma(X, \theta) \to X$ given by $p(\theta)(x, y) = x$ for $(x, y) \in \Gamma(X, \theta)$.

We shall use the notion of a degree of compact acyclic vector field given in [11].

Suppose that V is an open convex and bounded set in the Banach space E, containing the origin of E. Given a compact acyclic vector field $\Psi = I - F: V \to E$ for which $\Psi(\operatorname{Fr} V)$ does not contain the origin of E, the degree $d[\Psi, \operatorname{Fr} V]$ of the map Ψ on the boundary $\operatorname{Fr} V$ of the set V is defined as follows.

The set $\Psi(\operatorname{Fr} V)$ is a closed subset of the space E which does not contain the origin of the space E. Let W be an open convex neighbourhood of the origin of E such that $W \cap \Psi(\operatorname{Fr} V) = \emptyset$. Consider the compact set $F(\operatorname{Fr} V)$. There exists a single valued map $P: F(\operatorname{Fr} V) \to S$ of $F(\operatorname{Fr} V)$ in the finite dimensional linear subspace S of the space E such that $Py - y \in W$ for $y \in F(\operatorname{Fr} V)$. Let $V' = S \cap V$, then $\operatorname{Fr} V' = \operatorname{Fr} V \cap S$ is the $(\dim S - 1)$-dimensional sphere.

Let us consider $\Phi_1: \operatorname{Fr} V' \to E$, $\Phi_1 = \Psi \mid \operatorname{Fr} V'$ and denote by $\Gamma(\operatorname{Fr} V', \Phi_1)$ its graph. It follows from the Vietoris-Begle theorem that the homomorphism $p(\Phi_1)^*: H^*(\operatorname{Fr} V') \to H^*(\Gamma(\operatorname{Fr} V', \Phi_1))$ induced by $p(\Phi_1)$ is an isomorphism [10].

The map $c:\Gamma(\mathrm{Fr}\,V', \Phi_1) \to S\setminus\{0\}$ given by $c(x, y)=x-Py$ induces a homomorphism $c^*: H^*(S\setminus\{0\}) \to H^*(\Gamma(\mathrm{Fr}\,V', \Phi_1))$. The degree of homomorphism $(p(\Phi_1)^*)^{-1}c^*$ is called a degree of the map Ψ on $\mathrm{Fr}\,V$ (in our notations $d[\Psi, \mathrm{Fr}\,V]$).

2. Main theorem. *Let E be Banach space and let $\Phi: E \to E$ be an acyclic compact ε-strong vector field. Then $\Phi(E)=E$.*

In order to prove this theorem we shall need some lemmas.

Let V be open, convex, bounded set in the space E and $\Phi: \mathrm{cl}\,V \to E$ be an acyclic compact vector field. Suppose that the point $y_0 \in E$ does not belong to the set $\Phi(\mathrm{Fr}\,V)$. We shall assume also that the set V contains the origin O of the space E. As usual, we shall consider the map $\Psi=\Phi-y_0$ given by $\Psi(x)=\Phi(x)-y_0$ where $x \in \mathrm{Fr}\,V$. It is clear that Ψ is an acyclic compact vector field and $\Psi(\mathrm{Fr}\,V)$ does not contain the origin O of E. The degree $d[\Psi, \mathrm{Fr}\,V]$ of the map Ψ on $\mathrm{Fr}\,V$ is called a degree of Φ at the point y_0. We shall denote it by $d[\Phi, y_0, \mathrm{Fr}\,V]$.

We shall count up some properties of $d[\Phi, y_0, \mathrm{Fr}\,V]$.

1. *The map $\Phi|\mathrm{Fr}\,V: \mathrm{Fr}\,V \to E$ is sufficient to define $d[\Phi, y_0, \mathrm{Fr}\,V]$.*

2. *$d[\Phi, y_0, \mathrm{Fr}\,V]$ is homotopic invariant, i. e., if $\theta: \mathrm{Fr}\,V\times I \to E$ is an acyclic compact vector field and $y_0 \notin \theta(x, t)$ for every $x \in \mathrm{Fr}\,V$ and $0 \leq t \leq 1$, then*

$$d[\theta(x, t), y_0, \mathrm{Fr}\,V]=d[\theta(x, 1), y_0, \mathrm{Fr}\,V]$$

3. *If $y_0 \notin \Phi(\mathrm{cl}\,V)$, then $d[\Phi, y_0, \mathrm{Fr}\,V]=0$.*

We shall use the notations given in Section 1, where we reminded the definition of a degree of compact vector field.

We shall suppose that a) the identity inclusion $i: \mathrm{Fr}\,V' \to V'$ induces the inclusion $j:\Gamma(\mathrm{Fr}\,V', \Psi) \to \Gamma(V', \Psi)$, b) for a subset $A \subset Y'$ the map $c(A): \Gamma(A, \Psi) \to S\setminus\{0\}$ is given by $c(A)(x, y)=x-Py$ for $(x, y) \in \Gamma(A, \Psi)$.

It follows from the commutative diagram

$$\begin{array}{ccccc}
V' & \xleftarrow{p(\Psi)} & \Gamma(V', \Psi) & \xrightarrow{c(V')} & S\setminus\{0\} \\
i\uparrow & & \uparrow j & & \| \\
\mathrm{Fr}\,V' & \xleftarrow{p(\mathrm{Fr}\,V')} & \Gamma(\mathrm{Fr}\,V', \Psi) & \xrightarrow{c(\mathrm{Fr}\,V')} & S\setminus\{0\}
\end{array}$$

that $d[\Psi, \mathrm{Fr}\,V]=0$, but $d[\Phi, y_0, \mathrm{Fr}\,V]=d[\Psi, \mathrm{Fr}\,V]$ hence $d[\Phi, y_0, \mathrm{Fr}\,V]=0$.

Corollary 1. *If $d[\Phi, y_0, \mathrm{Fr}\,V]\neq 0$, then there exists $x_0 \in \mathrm{cl}\,V$ such that $y_0 \in \Phi(x_0)$.*

4. *If $a \in E$ is sufficiently small vector, then*

$$d[\Phi+a, y_0, \mathrm{Fr}\,V]=d[\Phi, y_0, \mathrm{Fr}\,V]$$

We have $0 \notin \Psi(\mathrm{Fr}\,V)$, therefore there exists an open convex neighbourhood W of the origin, such that $W \cap \Psi(\mathrm{Fr}\,V)=\emptyset$. Let S be a finite dimensional linear subspace of E such that $S+W \supset F(\mathrm{Fr}\,V)$ and let $P: F(\mathrm{Fr}\,V) \to S$ be a single valued map for which $Py-y \in W/2$ for every point $y \in F(\mathrm{Fr}\,V)$. Suppose that $a \in W/2$ and $\Psi_1=\Psi-a$. There is a map $k:\Gamma(\mathrm{Fr}\,V', \Psi_1) \to \Gamma(\mathrm{Fr}\,V', \Psi)$ given by $k(x, y)=(x, y-a)$ for $(x, y) \in \Gamma(\mathrm{Fr}\,V', \Psi_1)$. If $\bar{c}:\Gamma(\mathrm{Fr}\,V', \Psi_1) \to S\setminus\{0\}$ is the map $\bar{c}(x, y)=x-py$, the following diagram is commutative:

$$\begin{array}{ccccc} \mathrm{Fr}\, V' & \xleftarrow{p(\psi)} & \Gamma(\mathrm{Fr}\, V', \Psi) & \xrightarrow{c(\mathrm{Fr}\, V')} & S\setminus\{0\} \\ \| & & \uparrow k & & \| \\ \mathrm{Fr}\, V' & \xleftarrow{p(\Psi_1)} & \Gamma(\mathrm{Fr}\, V', \Psi_1) & \xrightarrow{\bar{c}} & S\setminus\{0\}. \end{array}$$

It follows from this diagram that $d[\Phi+a, y_0, \mathrm{Fr}\, V]=d[\Phi, y_0, \mathrm{Fr}\, V]$.

5. *If* $y_0 \in \mathrm{Fr}\,(\Phi(\mathrm{cl}\, V))$, *then* $d[\Phi, y_0, \mathrm{Fr}\, V]=0$.

In order to prove this, let us consider an open, convex neighbourhood W of the origin O such that $2W \cap \Psi(\mathrm{Fr}\, U)=\emptyset$. The origin O belongs to the frontier of the set $\Psi(\mathrm{cl}\, V)$, hence there is a point $a \in W$ such that $a \notin \Psi(\mathrm{cl}\, V)$. From 4 we have $d[\Phi-a, y_0, \mathrm{Fr}\, V]=d[\Phi, y_0, \mathrm{Fr}\, V]$. It follows from the choice of the point a that $y_0 \notin (\Phi-a)(\mathrm{cl}\, V)$, hence from 3 we obtain $d[\Phi-a, y_0, \mathrm{Fr}\, V)=0$.

Corollary 2. *If* $d[\Phi, y_0, \mathrm{Fr}\, V]\neq 0$ *then* $y_0 \in \mathrm{Int}\, \Phi(\mathrm{cl}\, V)$.

3. Proof of the theorem. We shall prove that if $y_0 \in \Phi(E)$, then $O_\eta y_0 \subset \Phi(E)$. Without loss of generality we may suppose that $y_0 \in \Phi(0)$.

By S_ε we shall denote the boundary of the set $O_\varepsilon=\{z \in E \mid \|z\| \leq \varepsilon\}$. First of all we shall prove that $d[\Phi, y_0, S_\varepsilon]\neq 0$.

We shall use the notations given in the definition of the degree of a compact acyclic vector field (see section 1), putting $V=\{x \in E \mid \|x\| \leq \varepsilon\}$. Let us denote the dimension of the linear space S by $n+1$. Then V' is $(n+1)$-dimensional ε-ball D^{n+1} with center in the origin O of E and $\partial V'$ is a n-dimensional sphere S^n.

Let us consider

a) the set $X=\{(x, y) \in D^{n+1} \times D^{n+1} \mid \|x-y\|=\varepsilon\}$. This set is homeomorphic to the set $S^n \times D^{n+1}$;

b) the compact spaces

$$Y=\{(x, y, u, v) \in X \times E \times E \mid v \in \Psi(x),\ v \in \Psi(y)\}$$
$$Z=\{(x, y, u) \in X \times E \mid u \in \Psi(x)\}, \quad \text{here} \quad \Psi=\Phi-y_0;$$

c) the map $g: Y \to Z$ given by $g(x, y, u, v)=(x, y, u)$ for $(x, y, u, v) \in Y$;

d) the map $j: \Gamma(\mathrm{Fr}\, V', \Psi) \to Y$ given by $j(x, u)=(x, O, u, O)$ for $(x, u) \in \Gamma(\mathrm{Fr}\, V', \Psi)$;

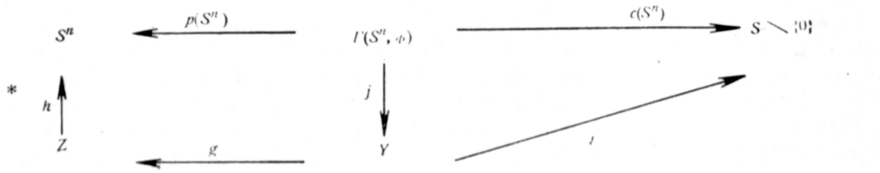
e) the map $f: Y \to S\setminus\{0\}$ given by

$$f(x, y, u, v)=x-Pu-y+Pv$$

for $(x, y, u, v) \in Y$;

f) $h: Z \to \mathrm{Fr}\, V'$ given by $h(x, y, u)=x$ for $(x, y\ u) \in Z$.

It is straightforward to check that the following diagram is commutative.



Фиг. 1

The space Z is the graph of the acyclic map $\alpha: X \to E$ given by $\alpha(x, y) = \Psi(x)$. As above, $H^*(X)$ is isomorphic to $H^*(Z)$ and, therefore, Z has the cohomology as a n-dimensional sphere. Also, Y is a compact space, which has the same cohomology as the n-dimensional sphere, because the map g induces an isomorphism $g^*: H^*(Z) \to H^*(Y)$.

We have an involution $T: Y \to Y$ which acts on Y without fixed points. The involution T is given by $T(x, y, u, v) = (y, x, v, u)$ for $(x, y, u, v) \in Y$.

In $S \setminus \{0\}$ we shall consider the antipodal map. In this case the map $f: Y \to S \setminus \{0\}$ is an equivariant map, i. e. $fT = -f$. By the well-known theorem (see for example [12]) it follows that $\deg f \equiv 1 \pmod 2$. From the diagram (*) we obtain that $d[\Psi, S_\varepsilon] \neq 0$. Therefore, $d[\Phi, y_0, S_\varepsilon] \neq 0$. It follows from Corollary 2 that $y_0 \in \operatorname{Int} \Phi(\operatorname{cl} O_\varepsilon)$. Therefore $\Phi(E)$ is an open set in the space E.

Let $O_\eta y_0 = \{z \in E \mid \| y_0 - z \| < \eta\}$. Suppose that $\Phi(S_\varepsilon) \cap O_\eta y_0 \neq \emptyset$. Then there exist points $x, y \in E$ such that a) $x \in S_\varepsilon$, i. e. $\| x - x_0 \| = \varepsilon$; b) $y \in \Phi(x)$; c) $\| y - y_0 \| < \eta$. As Φ is a strong ε-field, it follows from b) and c) that $\| x - x_0 \| < \varepsilon$, which contradicts to a). Hence $\Phi(S_\varepsilon) \cap O_\eta y_0 = \emptyset$.

Let us consider the homotopy $\theta(x, t): S_\varepsilon \times I \to E$, given by $\theta(x, t) = \Phi(x) - [ty + (1-t)y_0]$, where $x \in S_\varepsilon$, $y \in O_\eta y_0$, $0 \leq t \leq 1$. The homotopy θ is an acyclic compact vector field and $0 \notin \theta(S_\varepsilon \times I)$, then $d[\theta(x, 0), S_\varepsilon] = d[\theta(x, 1), S_\varepsilon]$. Therefore, $d[\Phi, y, S_\varepsilon] \neq 0$ for every $y \in O_\eta y_0$. From corollary 2, $O_\eta y_0 \subset \operatorname{Int} \Phi(O_\varepsilon)$. This gives us $\Phi(E) = E$. The theorem is proved.

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