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THE BANDS OF A FINITE SYMMETRIC SEMIGROUP

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1. Introduction. As it is known, one of the basic problems of the theory of semigroups, central part of which are the semigroups of transformations [4], is the study of the bands, i. e., the semigroups of idempotents ([1], 1.8). In this paper, bands of right and left zeros of the finite symmetric semigroup are studied according to the structure of partitions of the transformation set and to the structure of the set of fixed points respectively. The indispensability for investigation on semigroups, the elements of which have one-sided zeros has been substantiated by Lyapin ([2], 11.5). In this paper the cardinal numbers of the above mentioned bands are defined, which gives a partial answer to problem 46 of [5].

In terminology we follow [1], so that everywhere in this paper X denotes a finite set (for example, the set of the numbers $1, \ldots, n$), T_X — the semigroup of the full transformations on the set X, π_α — the kernel equivalence of $\alpha \in T_X$, π_i — a fixed partition of the set X, A_i — the set of all idempotents of the semigroup T_X, the kernel equivalence of which coincides with π_i.

Let us remind that according to [1] the transformation α on the set X is idempotent if, and only if, it is the identical mapping, when restricted to range $X\alpha$ ([1], 1.1). In other words, if $\pi_\alpha = \alpha \circ \alpha^{-1}$ denotes the kernel equivalence of the transformation α on the set X, then the pair of components $(\pi_\alpha, X\alpha)$ defines α completely as an idempotent, provided $X\alpha$ is a "cross section" of the partition π_α, and all the points of $X\alpha$ are fixed for the transformation α.

From all that has been said (see also [1], 2.2 and [3]) follows

Lemma 1. *For the idempotents α_1, α_2 of the semigroup T_X holds:*

1. $\alpha_1\alpha_2 = \alpha_2$ *if and only if* $\pi_{\alpha_1} \subset \pi_{\alpha_2}$;
2. $\alpha_1\alpha_2 = \alpha_1$ *if and only if* $X\alpha_1 \subset X\alpha_2$;
3. $\alpha_1\alpha_2 = \alpha_2\alpha_1 = \alpha_1$ *if and only if* $\pi_{\alpha_2} \subset \pi_{\alpha_1}$ *and* $X\alpha_1 \subset X\alpha_2$.

Proposition 1. *Let α, β be two idempotents of the semigroup T_X, and let $\gamma = \alpha\beta$. The transformation γ is an idempotent if and only if*

(*) *for any class $\bar{y} \in X/\ker\beta$, for which $\bar{s} = \bar{y} \cap X\alpha \neq \emptyset$ the embedding $\bar{y}\beta \subset \bar{s}\alpha^{-1}$ holds true.*

Here $\bar{s}\alpha^{-1}$ denotes the set of all elements $x \in X$, for which $x\alpha \in \bar{s}$.

Proof. Let $\bar{x}, \bar{y}, \bar{z}$ (with or without indices) denote the elements of the factorsets $X/\ker\alpha$, $X/\ker\beta$, $X/\ker\gamma$ respectively. It is immediately clear that $\pi_\alpha \subset \pi_\gamma$ and $\bar{z} = \bigcup_1^m \bar{x}_i$ if and only if

$$\bar{x}_i a \subset \bar{y} \ (i=1, \ldots, m) \text{ and } \bar{y} \cap (X\alpha \setminus \{\bar{x}_1 a, \ldots, \bar{x}_m a\} = \emptyset.$$

Thus, if the condition $(*)$ is satisfied and if

$$\bar{s} = \{\bar{x}_1 a, \ldots, \bar{x}_m a\}, \quad \bar{z} = \bigcup_{i=1}^{m} \bar{x}_i,$$

then

$$\bar{z}\gamma = (\bigcup_1^m \bar{x}_i)\alpha\beta = \bar{y}\beta \subset \bar{s}\alpha^{-1} = \bar{z},$$

i. e. γ is an idempotent.

Conversely, let

$$\gamma^2 = (\alpha\beta)^2 = \alpha\beta = \gamma,$$

$$\bar{s} = \bar{y} \cap X\alpha = \{a_1, \ldots, a_m\} \neq \emptyset, \quad \bar{x}_i\alpha = a_i$$

and $\bar{z} = \bigcup_{i=1}^{m} \bar{x}_i$.

Now, we have to prove that $\bar{y}\beta \subset \bar{z}$. Indeed, on the one hand,

$$\bar{z}\gamma = \bar{z}(\alpha\beta) = (\bigcup_1^m \bar{x}_i)\alpha\beta = ((\bigcup_1^m \bar{x}_i)\alpha)\beta = \bar{s}\beta = \bar{y}\beta,$$

and, on the other hand, since γ is an idempotent, then $\bar{z}\gamma \subset \bar{z}$ for any element $\bar{z} \in X/\ker\gamma$. Hence, $\bar{y}\beta = \bar{z}\gamma \subset \bar{z}$.

Corollary 1. *For any two idempotents α, β of the semigroup T_X, for which $\pi_\beta \subset \pi_\alpha$, the products $\alpha\beta$ and $\beta\alpha$ are also idempotents.*

Proof. On the one hand, by Lemma 1 for the idempotents α, β holds $\beta\alpha = \alpha$. On the other hand, let $\bar{x} \in X/\ker\alpha$, $\bar{y} \in X/\ker\beta$, $\bar{x}\alpha = a$ and $\bar{y} \subset \bar{x}$. Obviously, $|\bar{y} \cap X\alpha| \leq 1$. If $a \in \bar{y} \cap X\alpha$, then

$$\bar{y}\beta \subset \bar{y} \subset \bar{x} = a\alpha^{-1}.$$

The last relations show by Proposition 1 that the product $\alpha\beta$ is also an idempotent.

2. Basic results. Lemma 2. *Let π_0 be an arbitrary partition of the set X, and let A_0 be its corresponding set of idempotents. Then:*

i) *Any subset of the set A_0 is a right zero semigroup.*

ii) *The cardinal number of the band A_0 is equal to the product of the number of elements of all equivalence classes of the set X by* mod π_0.

The proof follows immediately from Lemma 1.

Corollary 2. *The maximal right zero bands A and B of the semigroup T_X are isomorphic if and only if $\pi|\bar{x}| = \pi|\bar{y}|$ for all $\bar{x} \in X/\ker\alpha$ and $\bar{y} \in X/\ker\beta$, where $\alpha \in A$ and $\beta \in B$.*

Corollary 3. *Any number*

$$\omega = p_1^{\alpha_1} p_2^{\alpha_2} \ldots p_m^{\alpha_m},$$

where $p_1, p_2, \ldots, p_m$ *are different prime numbers and* $\alpha_1, \alpha_2, \ldots, \alpha_m$ *are natural numbers, such that*

$$\alpha_1 p_1 + \alpha_2 p_2 + \ldots + \alpha_m p_m = n = |X|$$

is a cardinal number of some maximal right zero band of the semigroup T_X.

Proof. It follows immediately from Lemma 2 that such will be the band, corresponding to a partition of α_i $(i=1, \ldots, m)$ classes each of which has p_i elements.

Lemma 3. *Let* π_1, π_2 *be two different partitions of the set* X; A_1, A_2 — *their corresponding bands*; A — *an arbitrary subset of the set* A_1. *Then* $A \cup A_2$ *is a band of the semigroup for which*:

1) A_2 *is a two-sided ideal.*

2) *The elements of the semigroup* A_2 *are right zeros if and only if* $\pi_1 \subset \pi_2$.

Proof. The necessity follows immediately from 2) and lemma 1.

Conversely, let $\pi_1 \subset \pi_2$ and $\alpha \in A$, $\alpha_2 \in A_2$. By Lemma 1 $\alpha_1\alpha_2 = \alpha_2 \in A_2$, and by Corollary 2 $\alpha_2\alpha$ is also an idempotent and $\pi_2 \subset \pi_{\alpha_2\alpha}$. Since $\pi_1 \subset \pi_2$, then $|\bar{x} \cap X\alpha_2| \leq 1$ for any class $\bar{x} \in X/\pi_1$. Hence, $\pi_{\alpha_2\alpha} = \pi_2$ and $\alpha_2\alpha \in A_2$.

Obviously,

$$|A_1 \cup A_2| = |A_1| + |A_2|.$$

Theorem 1. *To each maximal chain of partitions of the set* X *there corresponds a maximal band, the kernel equivalences of the elements of which coincide with one of these partitions, and any two elements* α, β *of which satisfy the condition*

1) $\alpha\beta = \beta$ *or* $\beta\alpha = \alpha$.

Conversely, the kernel equivalences of each maximal band, the elements of which satisfy the condition 1), form a maximal chain of partitions of the set X.

Proof. Let

$$\pi_1 \subset \pi_2 \subset \ldots \subset \pi_m \tag{1}$$

be a maximal chain of partitions of the set X and let A_i be a band, corresponding to the partition π_i $(i=1, \ldots, m)$. By Lemma 3, the set $A = \bigcup_1^m A_i$ is a band, and if for $\alpha_i \in A_i$, $\alpha_j \in A_j$ we have $i<j$, then $\alpha_i\alpha_j = \alpha_j$ and $\alpha_j\alpha_i \in A_j$.

Conversely, let A be a maximal band, the elements of which satisfy the condition 1). That means (by Lemma 1 and Lemma 2), that the kernel equivalences of any two elements of A are comparable to each other. Hence, the kernel equivalences corresponding to the elements of A form a chain and it is a maximal one, since the band A is maximal with property 1.

From the proof of the theorem is clear that if A_i is a right zero band, corresponding to the partition π_i of the chain (1), $(i=1, \ldots, m)$, then for any subset $\{A_{i_1}, \ldots, A_{i_s}\}$ $(1 \leq k \leq i_k < i_{k+1} \leq m)$ of the set $\{A_1, \ldots, A_m\}$ the union $A \cup A_{i_2} \cup \ldots \cup A_{i_s}$, where $A \subset A_{i_1}$ is a band; A_{i_s} is its ideal, and each element of A_{i_s} is its right zero. Moreover,

$$(2)\qquad |A_1 \cup A_2 \cup \ldots \cup A_{i_s}| = |A| + |A_{i_2}| + \ldots + |A_{i_s}|.$$

Corollary 4. *Let $|X| = n$ and let t, k are natural numbers, such that $t^k < n < t^{k+1}$ and $n = n_i t^i + t_i$ $(0 \leq t_i < t^i,\ i = 1, \ldots, k)$. At that, it will be assumed that for any rational number a, $t_i a =$ if $t_i = 0$. With these denotations, any number satisfying*

$$1 \leq \omega \leq 1 + \sum_{r=1}^{k} t^{(n_{r-1} - t)^{r+1)+r}} t_{-1} \frac{t^{[r-t(r-1)]nr} - 1}{t^{r-t(r-1)} - 1} + \sum_{r=1}^{k} (t^r)^{n_r} t_r + n$$

is a cardinal number of some band with property (1).

Proof. Let $\overline{n}_1 = n_1$ if $t_1 = 0$, and let $\overline{n}_1 = n_1 + 1$ otherwise. Consider the set

$$P_1 = \{\pi_1^{(1)}, \pi_2^{(1)}, \ldots, \pi_{\overline{n}_1}^{(1)}\}$$

of such partitions of the set X, for which one only of the classes of $X/\pi_i^{(1)}$ $(i = 1, \ldots, \overline{n}_1)$ unites one-element classes of X/π_{i-1}, and the rest of its elements coincide with those of X/π_{i-1}, i. e.

$$X/\pi_0^{(1)} = X, \quad X/\pi_1^{(1)} = \{\{1, \ldots, t\},\ t+1, \ldots, n\}$$

and for any $i = 2, \ldots, n_1$:

$$X/\pi_i^{(1)} = \{\overline{x}_{i1}^{(1)}, \overline{x}_{i2}^{(1)}, \ldots, \overline{x}_{in-i(t-1)}^{(1)}\},$$

where $|\overline{x}_{ii}| = t$, $\overline{x}_{ij} = \overline{x}_{i-1j}$ if $j = 1, \ldots, i-1$, and $|\overline{x}_{ij}| = 1$ if $j = i+1, \ldots, n_1$

In the case when $t_1 \neq 0$ put $x/\pi_{\overline{n}_1} = \{\overline{x}_{\overline{n}_1 1}, \ldots, \overline{x}_{\overline{n}_1 \overline{n}_1}\}$, where $\overline{x}_{\overline{n}_1 j} = \overline{x}_{n_1 j}$ $(j = 1, \ldots, n_1)$ and $\overline{x}_{\overline{n}_1 \overline{n}_1} = \bigcup_{j=n_1}^{n} \overline{x}_{n_1 j}$. Obviously, P_1 is linearly ordered by embedding, and if $A_1^{(1)}$ $(i = 0, 1, \ldots, \overline{n}_1)$ are the corresponding to these partitions bands, then $A^{(1)} = \bigcup_{i=0}^{\overline{n}_1} A_i^{(1)}$ is also a band, since $|A_i^{(1)}| = t^i$ and

$$\omega_1 = |A^{(1)}| = \sum_{i=0}^{n_1} t^i + t^{n_1} t_1 = \frac{t^{n_1+1} - 1}{t - 1} + t^{n_1} t_1.$$

According to the proof of this theorem (see also Lemma 3), any number $\omega: 1 \leq \omega \leq \omega_1$ is a cardinal number of some band with property 1.

Inductively, knowing the set P_{r-1}, the set

$$P_r = \{\pi_1^{(r)} \ldots, \pi_{\overline{n}_r}^{(r)}\},$$

where $\overline{n}_r = n_r$ if $t_r = 0$, and $\overline{n}_r = n_r + 1$ otherwise, can be constructed by analogy with the construction of the set P_1, only with regard to the set $X_{r-1} = X/\pi_{\overline{n}_{r-1}}^{(r-1)}$:

$$X_{r-1}/\pi_i^{(r)} = \{\overline{x}_{i1}^{(r)}, \ldots, \overline{x}_{i\,n_{r-1}-i(t-1)}^{(r)}\} \quad (i = 1, \ldots, \overline{n}_r),$$

where

$$\bar{x}_{11}^{(r)} = \bigcup_{j=1}^{t} \bar{x}_{\bar{n}_{r-1}j}^{(r-1)}, \quad \bar{x}_{1j}^{(r)} = \bar{x}_{\bar{n}_{r-1}t+j-1}^{(r-1)} \; (j=2, \ldots, \bar{n}_{r-1}-t+1).$$

When $i=2, \ldots, n_r$, put

$$\bar{x}_{ij}^{(r)} = \bar{x}_{j-1j}^{(r)} \quad \text{for} \quad j=1, \ldots, i-1, \quad \bar{x}_{ii}^{(r)} = \bigcup_{j=i}^{i+t-1} \bar{x}_{i-1j}^{(r)}$$

and

$$\bar{x}_{ij}^{(r)} = \bar{x}_{i-1t+j-1}^{(r)} \quad \text{for} \quad j=i+1, \ldots, \bar{n}_{r-1}-t+1.$$

If $t_r \neq 0$, put

$$\bar{x}_{\bar{n}_r \bar{n}_r}^{(r)} = \bigcup_{j=n_r+1}^{\bar{n}_{r-1}} \bar{x}_{n_r j} \quad \text{and} \quad \bar{x}_{\bar{n}_r j}^{(r)} = \bar{x}_{n_r j} \; (j=1, \ldots, n_r).$$

If $A_i^{(r)}$ $(i=1, \ldots, \bar{n}_r)$ is a band, corresponding to the partition $\pi_i^{(r)}$, then for the cardinal number ω_r of the band $A^{(r)} = \bigcup_{i=1}^{\bar{n}_r} A_i^{(r)}$ $(r=1, \ldots, k)$ corresponding to the linearly ordered set of the partitions P_r we find:

$$\omega_0 = |e| = 1,$$

$$\omega_r = |A^{(r)}| = t_{r-1} \sum_{i=1}^{n_r} (t^r)^i (t^{r-1})^{(n_{r-1}-it)} + (t^r)^{n_r t_r},$$

$$\omega_r = t_{r-1} \sum_{i=1}^{n_r} t^{n_{r-1}(r-1)+i(r-rt+t)} + (t^r)^{n_r t_r},$$

$$\omega_r = t^{(n_{r-1}-t)(r-1)+r} t_{r-1} \frac{t^{[r-(r-1)t]n_{r-1}}}{t^{r-(r-1)t}-1} + (t^r)^{n_r t_r}$$

$$(r=1, \ldots, k)$$

$$\omega_{k+1} = n.$$

Then

$$\omega = \sum_{r=0}^{k+1} \omega_r = 1 + \sum_{r=1}^{k} t^{(n_{r-1}-t)(r-1)+r} t_{r-1} \frac{t^{[r-t(r-1)]n_{r-1}}}{t^{r-t(r-1)}-1} + \sum_{r=1}^{k} (t^r)^{n_r t_r} + n.$$

From the proof of theorem 1 (equality (2) respectively) and Lemma 2, it is clear that any number ω belonging to the intervals $[1, |A_k|]$,

$[1, |A_{k-1}|+|A_k|], \ldots, [1, \sum_0^{k+1}|A_i|]$ is a cardinal number for some band of the semigroup T_X.

Obviously, the length m of the so constructed maximal chain is equal to $\sum_1^k n_i+1$.

It is not difficult to check by the given proof that if $n=t^k$, since $n_r=t^{k-r}$, any number

$$1\leq\omega\leq 1+\sum_{r=1}^{k} t^{t(t^{k-r}-1)(r-1)+r}\frac{t^{[r-t(r-1)]t^{k-r}}-1}{t^{r-t(r-1)}-1}$$

is a cardinal number of some band with property 1, and in this case

$$m=n_1+n_2+\ldots+n_k=t^{k-1}+t^{k-2}+\ldots+1=\frac{t^k-1}{t-1}.$$

3. Conclusion. If instead of the component $X\alpha$, π_α is taken as "current" component, we get analogous statements for the left zeros. Thus, let Y_0 be an arbitrary subset of the set X; B_0 — the set of all idempotents $\beta\in T_X$, such that $X\beta=Y_0$, we shall call corresponding to Y_0. The assertions for left zeros, corresponding to those of 2, we are only going to state without giving the proofs.

Lemma 2′. *Let Y_0 be an arbitrary subset of the set X, and let B_0 be its corresponding set of idempotents. Then*:

i) *Any subset of the set B_0 is a semigroup of left zeros.*

ii) *The cardinal number of the band B_0 is equal to m^{n-m}, where $m=|Y_0|$ and $n=|X|$.*

Corollary 3′. *Any number ω satisfying the inequality*

$$1\leq\omega\leq m^{n-m}\quad(m\leq n)$$

is cardinal number of a maximal left zero band of the semigroup T_X.

Lemma 3′. *Let Y_1, Y_2 are two subsets of the set X; B_1, B_2 — their corresponding bands; B — an arbitrary subset of the set B_2. Then $B_1\cup B$ is a band of the semi-group T_X, for which*:

1) *B_1 is an ideal*;

2) *The elements of the band B_1 are its left zeros if and only if $Y_1\subset Y_2$.*

Theorem 1′. *To every maximal chain of subsets of the set X, there corresponds a maximal band, the range of the elements of which coincides with one of these subsets, and any two elements α, β of which satisfy the condition*:

1′) *$\alpha\beta=\alpha$ or $\beta\alpha=\beta$.*

Conversely, the range of any maximal band, the elements of which satisfy the condition 1′, form a maximal chain of subset of the set X.

Corollary 4'. *Any number ω satisfying the inequality*

$$1 \leq \omega \leq \sum_{i=1}^{n} i^{n-i}$$

is a cardinal number of a band with property 1'.

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