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REMARKS ON THE WIENER-PALEY-SCHWARTZ THEOREM*

TODOR G. GENČEV

A fairly simple proof of the Wiener-Paley-Schwartz theorem (L. Schwartz, 1951) is outlined. A necessary and sufficient condition for an entire function in C^n to have the form

$$f(z)=\int\limits_{S_\sigma} e^{-i(z,\,t)}\,\mu(dt)$$

s derived n the framework of the theory of distributions.

Being a central result in the theory of entire functions of exponential type, the theorem of Wiener-Paley-Schwartz [5] implies easily some important and complicated theorems obtained a long time before its discovery. That is why it is desirable to have a simple proof of this theorem. It turns out that not only this proof but simple proofs of almost all known resuls of Wiener-Paley's type [1, p. 103—107] can be given by means of the simple lemmas stated below. For convenience, we begin with some definitions.

Let $C^n=R^n_x+iR^n_y$ be the space of the complex variables $(z_1, z_2, \ldots, z_n)$ $z_j=x_j+iy_j$, $j=1, 2, \ldots, n$; we set $z=(z_1, z_2, \ldots, z_n)=x+iy$, where $x=\operatorname{Re} z$ $=(x_1, x_2, \ldots, x_n)$, $y=\operatorname{Im} z=(y_1, y_2, \ldots, y_n)$, $|z|^2=\sum\limits_{j=1}^{n} x_j^2+\sum\limits_{j=1}^{n} y_j^2$

$$=|\operatorname{Re} z|^2+|\operatorname{Im} z|^2.$$

The linear space of the entire functions in C^n of exponential type $\leq\sigma$, i. e. the entire functions such that $|f(z)|\leq A_\varepsilon e^{(\sigma+\varepsilon)|z|}$, $z\in C^n$, $\varepsilon>0$, will be denoted by E^n_σ and its subspace, consisting of the functions bounded on R^n_x, by B^n_σ. Further, if Ω is a domain in the Euclidean space R^n with generic point $t=(t_1, t_2, \ldots, t_n)$, by $C^\infty(\Omega)$ we denote the set of the functions defined in Ω, whose partial derivatives of any order exist and are continuous in Ω. The set of the functions in $C^\infty(\Omega)$ with compact support in Ω will be denoted by $C^\infty_0(\Omega)$ and the set of the distributions with compact support in R^n — by $\mathcal{E}'(R^n)$.

Now we are in a position to state the Wiener-Paley-Schwartz theorem (WPS theorem), which is known in two forms:

1. (Strong form). *A function $f(z)$ in E^n_σ satisfies the condition $f(x)$ $=O(|x|^m)$, when $|x|\to\infty$ on R^n_x, if and only if it is the Fourier transform*

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of a distribution in $\mathcal{E}'(R^n)$ *with support in the sphere* $|t|^2=\sum_{j=1}^{n} t_j^2 \leq \sigma^2$.

2. (Weak form). *A function* $f(z)$ *in* E_σ^n *satisfies the estimate* $|f(z)| \leq M(1+|z|)^m e^{\sigma|\operatorname{Im} z|}$, $M=\text{const}$, *if and only if it is the Fourier transform of some* $u \in \mathcal{E}'(R^n)$ *with* $\operatorname{supp} u \subset \{|t| \leq \sigma\}$.

While a short and natural proof of the theorem in its weak form is well known [3], the proof of the strong version is rather complicated. By means of the following lemmas one can easily prove that the condition $f(x)=O(|x|^m)$, as $|x| \to \infty$ on R_x^n, implies the inequality $|f(z)| \leq M(1+|z|)^{2m} \times e^{\sigma|\operatorname{Im} z|}$ and therefore to reduce the strong form of the theorem to the weak one.

Lemma 1. *Let* $f(z) \in E_\sigma^1$ *and let on* R_x^1 *the inequality*

$$|f(x)| \leq M(1+|x|)^m, \quad M>0, \tag{1}$$

$m \geq 0$ — *constants, be satisfied. Then the inequality*

$$|f(z)| \leq M_1(1+|z|)^m e^{\sigma|\operatorname{Im} z|}, \quad M_1 \leq 2^{m/2} M \tag{2}$$

holds.

In the case $m=0$ this lemma is well-known [1]. In the general case the proof follows the same lines.

Lemma 2 can be obtained by using lemma 1 in a suitable way.

Lemma 2. *Let* $f(z) \in E_\sigma^n$ *and let on* R_x^n *the inequality*

$$|f(x)| \leq M(1+|x|)^m, \quad |x|^2=\sum_{j=1}^{n} x_j^2, \tag{3}$$

hold. Then we have in C^n

$$|f(z)| \leq M(1+|z|)^{2m} e^{\sigma|\operatorname{Im} z|}. \tag{4}$$

Of course lemma 2 is not so precise as lemma 1, but for our purpose it is quite sufficient.

In order to illustrate the significance of the WPS theorem, we shall deduce from it the following classic result of Plancherel and Polya [4].

Theorem 1. *Let* Z *be the lattice of the vectors* $\nu=(\nu_1, \nu_2, \ldots, \nu_n) \in R_x^n$, ν_j *being integers*, and let* Π_ν *be the* n*-dimensional cube defined by the inequalities* $\nu_j \leq x_j \leq \nu_j+1$, $j=1, 2, \ldots, n$. *If* $f(z) \in E_\sigma^n$, $\sigma<\pi$ *then the inequalities*

a) $$\left(\sum_{\nu \in Z} |f(\nu)|^p\right)^{1/p} \leq C_1\left(\int_{R_x^n} |f(x)|^p dx\right)^{1/p}, \quad p \geq 1,$$

b) $$\left(\int_{R_x^n} |f(x)|^p dx\right)^{1/p} \leq C_2\left(\sum_{\nu \in Z} |f(\nu)|^p\right)^{1/p}, \quad p \geq 1,$$

c) $$\left(\sum_{\nu \in Z} \max_{\Pi_\nu} |f(x)|^p\right)^{1/p} \leq C_3\left(\sum_{\nu \in Z} |f(\nu)|^p\right)^{1/p}, \quad p \geq 1,$$

are equivalent. The constants C_k, $k=1, 2, 3$ *depend only on* σ.

* In what follows Z always will have the same meaning.

The proof of this theorem which we want to outline is based on three lemmas, the last two of them being simple consequences of the WPS theorem but, nevertheless, are of some interest in themselves and are useful in many problems.

Lemma 3. *Let* $h(t) \in C_0^\infty(R^n)$ *and let*

$$\hat{h}(z) = \int_{R^n} e^{-i(z,t)} h(t)dt, \text{ where } (z,t) = \sum_{j=1}^{n} z_j t_j \tag{5}$$

is its Fourier transform. Then on R_x^n *we have*

$$\sum_{\nu \in Z} |\hat{h}(x-\nu)| \leq M, \quad M = \text{const.} \tag{6}$$

Proof. Since the sum of the series (6) is a periodic function of x it is sufficient to consider only the case when $0 \leq x_j \leq 1$, $j = 1, 2, \ldots, n$. Integrating (5) by parts several times we get the inequality

$$|\hat{h}(x)| \leq A/(1+|x|)^{n+1}, \ x \in R_x^n, \ A = \text{const}$$

and the lemma follows.

Lemma 4. *Let* $f(z) \in E_\sigma^n$ *and let on* R_x^n *we have* $f(x) = O(|x|^m)$, *as* $|x| \to \infty$. *Then the equality*

$$f(z) = \frac{1}{(2\pi)^n} \int_{R^n} \hat{h}(z-t) f(t) dt,$$

where $h(t) \in C_0^\infty(R^n)$ *and* $h(t) = 1$ *in some neighbourhood of the sphere* $|t| \leq \sigma$, *holds.*

Lemma 5. *Let* $f(z)$ *be the function from lemma 4 and let* $\sigma < \pi$. *If* $h(t) \in C_0^\infty(\Omega)$, Ω *being the cube* $|t_j| < \pi$, $j = 1, 2, \ldots, n$, *is such that* $h(t) = 1$ *in a neighbourhood of the sphere* $|t| \leq \sigma$, *then the equality*

$$f(z) = \frac{1}{(2\pi)^n} \sum_{\nu \in Z} f(\nu) \hat{h}(z-\nu)$$

is verified.

By means of these lemmas a fairly simple proof of theorem 1 can be given by reasonings similar, to some extent, to those of Wiener [6, ch. 2, lemma 6].

Theorem 1 has some corollaries. One of them is the following

Theorem 2. *A function* $f(z)$ *belong to* $E_\sigma^n \cap L_p(R_x^n)$, $p \geq 1$, *if and only if it is the Fourier transform of a distribution* $u(\varphi)$ *with support in the sphere* $|t| \leq \sigma$, *and such that the estimate*

$$|u(\varphi)| \leq C \left(\sum_{\nu \in Z} |\hat{\varphi}(\nu)|^q \right)^{1/q}, \quad \frac{1}{p} + \frac{1}{q} = 1, \ C = \text{const}$$

holds for every $\varphi(t) \in C_0^\infty(D)$, *where* D *is the sphere* $|t| < \pi$.

Particularly, $f(z) \in {}'B_\sigma^n$, *if and only if the inequality*

$$(7)\qquad |u(\varphi)| \leq C \sum_{\nu \in Z} |\hat{\varphi}(\nu)|, \quad \varphi \in C_0^\infty(D),$$

is satisfied.

Let us note that $\hat{\varphi}(\nu)$ is nothing but the νth Fourier coefficient of $\varphi(t)$ multiplied by $(2\pi)^n$.

Now we shall give a direct proof of this theorem in the case $p=\infty$.

Proof. Suppose firstly that $f(z) \in B_\sigma^n$, $\sigma<\pi$ and $\varphi(t) \in C_0^\infty(D)$. By the WPS theorem $f(z)$ is the Fourier transform of a distribution $u \in \mathcal{E}'(R^n)$, $\operatorname{supp} u \subset \{|t| \leq \sigma\}$ and, as is well-known [3] $f(z)=u_t(e^{-i(z,t)})$.

Now let $h(t) \in C_0^\infty(D)$ be such that $h(t)=1$ in a neighbourhood of $\operatorname{supp}\varphi \cup \operatorname{supp} u$. Developing $\varphi(t)$ in Fourier series we obtain

$$(8)\qquad \varphi(t)=(2\pi)^{-n} \sum_{\nu \in Z} \hat{\varphi}(\nu) e^{i(\nu, t)},$$

which is uniformly convergent in R^n together with any of the series obtained from (8) by partial differentiation. Hence, each of the sequences $\{D^\alpha S_m(t)\}$, where $S_m(t)=h(t)(2\pi)^n \sum_{|\nu| \leq m} \hat{\varphi}(\nu) e^{i(\nu, t)}$, $\alpha=(\alpha_1, \alpha_2, \ldots, \alpha_n)$ $D^\alpha=D_1^{\alpha_1} D_2^{\alpha_2} \ldots D_n^{\alpha_n}$ is uniformly convergent in R^n. Since, on the orher hand, $\operatorname{supp} S_m \subset D$, we have $\lim_{m\to\infty} u(S_m)=u(h\varphi)=u(\varphi)$ and in virtue of the inequality

$$|u(S_m)|=(2\pi)^{-n}\Big|\sum_{|\nu| \leq m} \hat{\varphi}(\nu) f(-\nu)\Big| \leq \operatorname{supp}_{R_x^n} |f(x)| \sum_{\nu \in Z} |\hat{\varphi}(\nu)|$$

the necessity of the condition (7) is proved.

Now suppose $u \in \mathcal{E}'(R^n)$, $\operatorname{supp} u \subset \{|t| \leq \sigma\}$ is such that (7) is satisfied. We have to prove that the function $f(z)=u_t(e^{-i(z,t)})$ is bounded on R_x^n, since it obviously belongs to E_σ^n. If $h(t) \in C_0^\infty(D)$ is such that $h(t)=1$ in a neighbourhood of $\operatorname{supp} u$, we have $f(z)=u_t(\varphi_z(t))$, where $\varphi_z(t)=h(t) e^{-i(z,t)}$ and consequently

$$|f(x)|=|u_t(\varphi_x(t))| \leq C \sum_{\nu \in Z} |\hat{\varphi}_x(\nu)|=C \sum_{\nu \in Z} |\hat{h}(x+\nu)|, \quad x \in R^n$$

and the desired conclusion follows immediately from lemma 3.

The theorem just proved gives rise to the following question: from the proof of the WPS theorem [3] we know that if $u \in \mathcal{E}'(R^n)$, $\operatorname{supp} u \subset \{|t| \leq \sigma\}$, is subject to the condition $|u(\varphi)| \leq C \sum_{|\alpha| \leq m} \max_K |D^\alpha \varphi|$, $|\alpha|=\sum_{j=1}^n \alpha_j$, where K is some compact neighbourhood of $\operatorname{supp} u$, then we have

$$|f(z)|=|u_t(e^{-i(z,t)})| \leq C(1+|z|)^m e^{\sigma|\operatorname{Im} z|}.$$

Thus in the case $m=0$ the function $f(z)$ is bounded on R_x^n. In other words the condition $|u(\varphi)| \leq C \max_K |\varphi|$ is sufficient but, according to theorem 2, not necessary for the boundedness of $f(z)$ on R_x^n. It is natural to look for a characteristic of the functions in B^n_σ which are Fourier transforms of distributions such that the estimit

$$|u(\varphi)| \leqq C \max_{S_\sigma} |\varphi(x)|, \quad S_\sigma = \{|t| \leqq \sigma\} \tag{9}$$

holds. The answer to this question is given by the following

T e o r e m 3. *A function $f(z)$ in B^n_σ is the Fourier transform of a distribution $u \in \mathcal{E}'(R^n)$ with support in the sphere S_σ for which (9) holds, if and only if it has the form*

$$f(z) = \int_{S_\sigma} e^{-i(z,t)} \mu(dt), \tag{10}$$

where $\mu(dt)$ is a regular countably additive measure in the sphere S_σ.

P r o o f. Suppose $f(z) = u(e^{-i(z,t)})$, where $u \in \mathcal{E}'(R^n)$, $\operatorname{supp} u \subset \{|t| \leqq \sigma\}$ and (9) holds. Since according to the Riesz representation theorem [2, p. 265] we have $u(\varphi) = \int_{S_\sigma} \varphi(t)\mu(dt)$, $S_\sigma = \{|t| \leqq \sigma\}$, the necessity of the condition (10) is proved. The sufficiency of (10) follows immediately from the trivial estimate

$$\left|\int_{S_\sigma} \varphi(t)\mu(dt)\right| \leq \max_{S_\sigma} |\varphi(t)| \operatorname{Var}_{S_\sigma} |\mu(dt)|.$$

C o r o l l a r y 1. *There exist functions in B^n_σ which cannot be represented in the form (10). For example such a function is the Fourier transform of the distribution*

$$u_0(\varphi) = \int_{R^n} h(t) \frac{\varphi(t_1, t_2, \ldots, t_n) - \varphi(-t_1, t_2, \ldots, t_n)}{t_1} dt,$$

where $h(t) \in C_0^\infty(S_\sigma)$ is equal to 1 in some neighbourhood of $t=0$.

Indeed, setting $z = (z_1, z')$, $t = (t_1, t')$ we have

$$f_0(z) = u_0(e^{-i(z,t)}) = \int_{S_\sigma} h(t) \frac{e^{-iz_1t_1} - e^{iz_1t_1}}{t_1} e^{-i(z',t')} dt,$$

which is obviously bounded on R^n_x, but, as is easily seen, the estimate (9) is not verified.

C o r o l l a r y 2. *It follows from the classic Wiener-Paley theorem, that the functions in B^n_σ of the form (10) are dense in B^n_σ. Now we may assert that the functions in B^n_σ which cannot be represented in the form (10) are also dense in B^n_σ.* Indeed, if $f(z) \in B^n_\sigma$ has the form (10) then $f_\varepsilon(z) = f(z) + \varepsilon f_0(z)$ cannot be represented in such a way and univormly tends to $f(z)$ on the bounded sets in C^n, when $\varepsilon \to 0$.

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Centre for Research and Training
in Mothematics and Machanics
1000 Sofia P. O. Box 373

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