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ASYMPTOTIC NUMBERS — ALGEBRAIC OPERATIONS WITH THEM

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The main subject of the present paper is to define the four algebraic operations — addition, subtraction, multiplication and division in the set of the asymptotic numbers A [7] and to deduce the corresponding formulas for the components of the asymptotic number, representing the result as functions of the components of the arguments. The definitions of the operations, in fact, are introduced as a special case of the more general notion of a quasiclassical function — one special class of functions defined on A. The discussion of the algebraic and some other properties of the asymptotic numbers is put off for a next paper.

The set of asymptotic numbers, introduced by the same authors in [7], is a generalization of the system of real (complex) numbers, comprising infinitely small and infinitely large numbers [1], [2]. The reasons for introducing these numbers are connected with concrete problems of the quantum mechanics [5], [6], [8], but it seems to us that they are also interesting for themselves.

The definition of the asymptotic numbers and some of their properties are reminded in the introductory chapter, by which we achieve logical independence of [7].

1. Asymptotic Numbers. Several attempts are known to generalize the notions of the number [1], [2] and the function [3], [4]. In two previous papers we proposed a new generalization introducing the notions of the asymptotic number [7] and the asymptotic function [8] in order to be useful for some problems in quantum mechanics, especially in cases where wave functions appear which do not belong to the Hilbert space [5], [6]. One makes use often of the generalized functions of Soboljev-Schwartz but they are not always the appropriate tool because one cannot multiply them. One is dealing also with the so-called wave packets but it is only a not elaborated idea. Our aim is just to propose a general and complete scheme for work with packets. For that purpose we propose one modification in the definition of the generalized functions $f(x)$. As it is known they can be introduced as equivalent classes in a given set S_s of sequences of functions $f_s(x)(-\infty<x<\infty,\ 0<s<s_1)$, the classes being determined by one and the same limit

$$f\cdot\varphi=\lim_{s\to 0} f_s\cdot\varphi, \tag{1.1}$$

$$f_s\cdot\varphi=\int_{\infty}^{\infty} f_s(x)\varphi(x)dx \tag{1.2}$$

for every $\varphi(x)$ from another set S^*. The algebraic operations $f(x)=F(f_n(x)$ with generalized functions are defined by means of the same operation

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with the representatives $f_{sn}(x)$ of $f_n(x)$, provided the sequence of functions $f_s(x)=F(f_{sn}(x))$ belongs to S_s and different choices of the representatives $f_{sn}(x)$ by given $f_n(x)$ lead to sequences which are representatives of one and the same generalized function $f(x)$. According to this, it turns out that for example the sum $f(x)=f_1(x)+f_2(x)$ is defined but the product $f(x)=f_1(x).f_2(x)$ — not always. In order to achieve possibility for multiplication, we replace by another appropriate (narrower) set $\bar{S}_s$ and $\varphi(x)$ from S^* — by sequence of functions $\varphi_s(x)$ from another appropriate set $\bar{S}^*$. It turns out that $f_s.\varphi_s$ is of the form

$$\sum_n a_n s^n + o_\nu(s), \quad \left(\mu \leqq n \leqq \nu\,;\ \lim_{s\to 0} \frac{o_\nu(s)}{s^\nu} = 0\right). \tag{1.3}$$

We suppose then that two sequences of functions from $\bar{S}_s$ are equivalent and represent the same asymptotic function $f(x)$ if for every $\varphi_s(x)$ from $\bar{S}_s^*$ they lead to expressions (1.3) with the same a_n and ν. Then we get that (i) the product is defined, (ii) the Dirac function and its derivatives are included, (iii) the equivalent classes are quite large (although narrower than those by Soboljev-Schwartz), (iv) the representatives of a given asymptotic function have all necessary properties to be considered as one wave packet.

As a simple example let us choose

$$f_{s1}(x)=\frac{1+\sqrt{s}}{\sqrt{\pi}\, s}\ \exp\left(-(x-a)^2/s^2\right),\ f_{s2}(x)=\frac{2(1+\sqrt{s})}{\Gamma(1/4)s}\ \exp\left(-(x-a)^4/s^4\right).$$

Then, according to (1.2), we find $f_{s1}.\varphi=\varphi(a)+o_0(s)$, $f_{s2}.\varphi=\varphi(a)+o_0(s)$ so that $f_{s1}(x)$ and $f_{s2}(x)$ can be considered as representatives of the Dirac function. But we can put them in square $f_{s1}^2(x)=(1/\pi s^2)\exp\left(-2(x-a)^2/s^2\right)$, $f_{s2}^2(x)=4/\Gamma^2(1/4)s^2\ \exp\left(-2(x-a)^4/s^4\right)$ and we get

$$f_{s1}^2.\ \varphi=\frac{1}{\sqrt{2\pi}s}\varphi(a)+o_{-1}(s)\,;\ f_{s2}^2.\varphi=\frac{2^{3/4}}{\Gamma^2(1/4)s}\ \varphi(a)+o_{-1}(s).$$

One sees (i) that the functionals are of the form (1.3) and (ii) so that in order to reach multiplication we need a finer classification than the one by Soboljev and Schwartz.

It is convenient for expressions of the form (1.3) fixed a_n and ν to be considered as numbers — we name them asymptotic numbers — then by given $f(x)$ we could consider (1.2) as functionals like by the functions of Soboljev-Schwartz. The asymptotic numbers comprise ordinary as well as infinitely small and infinitely large numbers of polynomial type and given accuracy ν [1], [2]. For us they are an auxiliary tool but it seems that they exhibit interesting properties as algebraic objects and for that reason we shall investigate them more widely than is necessary for the applications.

First we shall remind the definition. Here it is slightly simplified, remaining equivalent to the primary one.

D e f i n i t i o n. 1. *Let s be a real parameter, varying in the interval $0<s<s_1$, where s_1 is a fixed positive number and $A(s)$ — the set of all*

real (complex) sequences; $a(s)$ — functions of variable s, for each of them one can find an integer (positive or negative) number μ so that*

$$\lim_{s\to 0} |a(s)| : s^{\mu} < \infty. \tag{1.4}$$

The number μ can depend on the choice of $a(s)$. It is not uniquely determined — if (1.4) holds for $\mu = \mu_1$ it will hold for any $\mu < \mu_1$, too.

Theorem 1. *Every sequence $a(s)$ from $A(s)$ can be written in the form*

$$a(s) = \sum_{n}^{N} a_n s^n + o_\nu(s). \tag{1.5}$$

Here ν is an integer number — accuracy of the representation (1.5), N is a spectrum of integer numbers not greater than ν, a_n are real (complex) coefficients and $o_\nu(s)$ is real (complex) residual term satisfying the condition

$$\lim_{s\to 0} o_\nu(s) : s^n = 0 \tag{1.6}$$

for all integer n not larger than ν, $\nu \geqq N$.

The sum

$$p(s) = \sum_{n}^{N} a_n s^n \tag{1.7}$$

is the main part representation (1.5) of $a(s)$.

Without restriction we shall suppose $a_n \neq 0$.

For a given sequence $a(s)$ the representation (1.5) is not unique. It can be reduced to

$$a(s) = \sum_{n}^{N'} a'_n s^n + o'_{\nu'}(s)$$

by any $\nu' < \nu$. Obviously, we must have $a'_n = a_n$ for $n \in N'$, where $N' = N_{(\nu')}$. Here and in the remaining $N_{(\varkappa)}$, $N^{(\varkappa)}$ and $N^{[\varkappa]}$ are the lower and the upper parts of N (where $n \leq \varkappa$, $n > \varkappa$ and $n \geqq \varkappa$ respectively). In the case when $o_\nu(s)$ in (1.5) is of the same form (1.5) with some other μ, ν, N and a_n the representation (1.5) of $a(s)$ can be extended up to a larger ν.

Definition 2. *If $a(s)$ is a given sequence, the largest μ for which (1.4) holds is the power of $a(s)$. If (1.4) holds for all μ, the power of $a(s)$ by definition is ∞.*

The power μ of every given sequence $a(s)$ is uniquely determined.

Definition 3. *If $a(s)$ is a given sequence from $A(s)$ the largest value ν_a of ν for which (1.5) holds, is the accuracy of $a(s)$. If there is no such (finite) largest ν, the accuracy ν_a should be ∞.*

For $\nu = \nu_a = \infty$ the condition (1.6) is meaningful too. (That was the reason not to write (1.6) in the simpler form $\lim_{s\to 0} o_\nu(s) : s^\nu = 0$.) For every in-

* We shall call the functions $a(s)$ from $A(s)$ sequences because they play part of the sequences (with continual index s) in our theory.

teger ν not larger than the accuracy ν_a the sequence $a(s)$ can be written in the form (1.5). If the accuracy is infinite, we can generalize (1.5) for $\nu=\infty$. It is easy to obtain N and $a_n(n\in N)$ for $\nu=\nu_a=\infty$, because with increasing ν new coefficients appear, and once appeared they do not change. We have only to separate $a(s)$ in $p(s)$ and $o_\nu(s)$ taking into account that if $\nu=\nu_a=\infty$ $p(s)$ can be one infinite and even divergent series. We have to determine $p(s)$ so that, adding to it an appropriate function of the type $o_\infty(s)$, to obtain the given sequence $a(s)$. By this rule $p(s)$ is defined up to a function $o_\infty(s)$, too.

Definition 4. *If for $\nu=\infty$ N remains finite or if it is infinite but (1.7) has radius of convergence $s_0\geq s_1$, we can take (1.7) as a definition of $p(s)$, too. If $0<s_0<s_1$, we put for example*

$$p(s)=\begin{cases}\sum\limits_{n}^{N} a_n s^n & \text{for} \quad s<s_0,\\ 0 & \text{for} \quad s_0\leq s<s_1.\end{cases} \tag{1.8}$$

If $s_0=0$, we can choose

$$p(s)=\sum_{n}^{N} a_n s^n\,(1-\exp(-1/|a_n|\,s_1^n\sqrt{s})) \tag{1.9}$$

([9], 9.2). The series (1.9) is convergent (and analytic) for every choice of a_n. The determination (1.9) can be also chosen for $s_0>0$ and even for $s_0>s_1$, and for finite N but in these cases, of course, (1.7) and (1.8) are simpler.

Definition 5. *We say that $a'(s)$ is subordinate to $a(s)$ if the sequence*

$$\Delta a(s)=a(s)-a'(s)$$

satisfies (1.6) for $\nu=\nu_a$, ν_a being the accuracy of $a(s)$.

Theorem 2. *The relation of subordinance is reflexive and transitive but not symmetric. (It is symmetric only in the subsets $A_\nu(s)$ of sequences $a(s)$ with given accuracies ν.) This subordinance is partial — not for every choice of the sequences $a(s)$ and $a'(s)$ — one of them is subordinate to the other.*

Definition 6. *The set of all sequences $a'(s)$ subordinate to a given sequence $a(s)$ represents an asymptotic number a.*

Definition 7. *The representatives $a'(s)$, to which $a(s)$ is subordinate too, are proper representatives of a. If not — $a'(s)$ is a supplementary one.*

Theorem 3. *For every choice of the accuracy ν (integer number or ∞) the spectrum N (of integer numbers bound from below and not larger than ν) and the (real or complex) coefficients a_n ($n\in N$) corresponds one and only one asymptotic number a, so that we can put*

$$a=(\nu, N, a_n) \quad (n\in N). \tag{1.10}$$

Theorem 4. *For given a its components ν, N, a_n are uniquely determined.*

Theorem 5. *The power μ of the asymptotic number a is the power of the leading term in its main part, i. e. the first element in the spectrum N. By empty N we put $\mu=\nu$.*

Hence every asymptotic number a, every sequence $a(s)$ representing a and every representation (1.5) of $a(s)$ have their uniquely determined accuracies ν, spectra N and coefficients a_n. There are connections between these quantities associated with the number a, the sequences $a(s)$ representing it and their representations (1.5), but in general they do not coincide.

Definition 8. *Using the main part and the accuracy of a given asymptotic number a besides (1.10) we can indicate:*

$$a=(p(s))_\nu=\left(\sum_n^N a_n s^n\right)_\nu . \tag{1.11}$$

Teorem 6. *A given sequence $a(s)=\sum_n^N a_n s^n+o_{\nu_a}(s)$ is proper representative of one and only one asymptotic number $a=(\nu, N, a_n)$ with $\nu=\nu_a$ but it is supplementary representative of an infinity of asymptotic numbers a_ν each of them characterized by its accuracy $\nu<\nu_a$. The spectra N_ν and coefficients $a_{\nu n}$ of them are given by $N_\nu=N_{(\nu)}$ and $a_{\nu n}=a_n$ for $n\in N_\nu$.*

Theorem 7. *If $a(s)$ is a representative of two numbers $a\neq a'$, the accuracies of a and a' must be different and all representatives of the number with the higher accuracy are representatives of the number with the lower one too.*

This theorem is a consequence of the previous one.

Definition 9. *The relative accuracy and the relative spectrum are*

$$\lambda=\nu-\mu, \quad M=N-\mu. \tag{1.12}$$

Here $N-\mu$ is the spectrum which one obtains subtracting μ from each element of N. For numbers with nonempty spectrum the leading coefficient and the relative coefficients are

$$\alpha=a_\mu, \quad \alpha_m=a_{\mu+m}:a_\mu. \quad (m\in M). \tag{1.13}$$

If the spectrum is empty α is arbitrary and there are no α_m.

Then denoting

$$o_\lambda(s)=o_\nu(s):\alpha s^\mu, \tag{1.14}$$

instead of (1.5), (1.10) and (1.11), we can write

$$\begin{gathered} a(s)=\alpha s^\mu\left(\sum_m^M \alpha_m s^m+o_\lambda(s)\right), \\ a=(\lambda, \mu, M, \alpha, \alpha_m), \quad a=\alpha s^\mu\left(\sum_m^M \alpha_m s^m\right)_{(\lambda)} . \end{gathered} \tag{1.15}$$

Definition 10. *The asymptotic numbers zero and one are the numbers whose representatives are differences, respectively ratios of the representatives of one and the same number a, the ratio being arbitrarily* *defined for the values of s where the denominator vanishes. (In the definition of 1 the number a cannot be 0.)*

Theorem 8. *The zero and one, thus defined, depend on a, namely on its accuracy. We have an infinity of asymptotic zeros o_ν and asymptotic ones 1_λ, represented by*

$$(1.16)\qquad o_\nu(s)\quad (\nu=0, \pm 1, \ldots, \infty),\quad 1_\lambda(s)=1+o_\lambda(s)\quad (\lambda=0, 1, \ldots, \infty).$$

It is useful in some cases to remove entirely or partly the condition all coefficients to be not zero.

Definition 11. *Removing the condition $a_n \neq 0$, we get the free spectrum of the asimptotic number a. Putting only the condition the first coefficient, if there is a first coefficient, i. e. if the spectrum is not empty, to be different from zero, we obtain the nonsingular spectrum. In order to distinguish these two spectra from the spectrum introduced in the beginning, we will name it primary spectrum. By the representation (1.15), if the relative spectrum is not empty, the first element of M is always 1, so that M can be only primary or nonsingular.*

A given asymptotic number a has an unique primary spectrum, but several free and nonsingular ones. One can obtain them by adding to the primary spectrum supplementary elements n such that $n \leqq \nu$ or $\mu < n \leqq \nu$ respectively.

According to the definition of the power μ in the case when we are dealing with free spectra N, μ must be not simply the first element of N but the first one for which $a_\mu \neq 0$. If $N=\varnothing$ or $N\neq\varnothing$ but all $a_n=0$, we put $\mu=\nu$, $M=\varnothing$, $a_0=a_\mu$ — arbitrary.

Definition 12. *The set of all asymptotic numbers, corresponding to the set of all sequences $a(s)\in A(s)$, will be denoted by A.*

The elements of A are characterized by ν, N, $a_n (n \in N,\ a_n \neq 0)$ (1.10) or λ, μ, M, α, $\alpha_m (\alpha_0=1,\ \alpha_m \neq 0,\ m\in M^{(0)},\ \alpha\neq 0$ for $M^{(0)}\neq\varnothing)$ (1.15).

Definition 13. *The discrete parameters in (1.10) or (1.15) represent the structure index ϱ of a: $\varrho=(\nu, N)=(\lambda, \mu, M)$.*

Definition 14. *The subsets A_ϱ of A one obtains varying the coefficients a_n by conserved structure index ϱ represented primary, elementary or nonsingular sets corresponding to the three types of spectra N — primary, free or nonsingular. For subsets A_n consisting only of zero, we put $\varrho=(\nu, \varnothing)=(0, \mu, \varnothing)$, $\mu=\nu$.*

2. Quasiclassical Asymptotic Functions

Definition 15. *The general notion of asymptotic function*

$$(2.1)\qquad y=F(x_i)\quad (i=1, 2, \ldots, m)$$

requires to every choice of the asymptotic variables x_i, running over a given subset X in the space of m-tuples of asymptotic numbers A^m and each characterized by the parameters (ν_i, N_i, a_{in}) (1.10), to correspond by a given rule value of the asymptotic variable y, represented by (ν, N, a_n) (1.10) again. We shall restrict ourselves with a narrower class — the quasiclassical asymptotic functions, defined, defined as follows.

Definition 16. *Let*

$$(2.2)\qquad \eta=f(\xi_i, s)\quad (i=1, 2, \ldots, m)$$

be an ordinary function defined on a set of points X^* in the $m+1$-dimenisonal space of the parameter s and the real (complex) ordinary variables ξ_i. Denoting by x_i and y asymptotic variables, let $a_i(s)$ be arbitrarily given values a_i of x_i. Let

$$(2.3)\qquad a(s)=f(a_i(s), s)$$

be the sequence (function of parameter s) one obtains replacing ξ_i in (2.2) by $a_i(s)$ and determining it arbitrarily on the set S^* of values of s in the interval $0<s<s_1$, for which the point $(a_i(s), s)$ does not belong to X^*, i. e. for which $f(a_i(s), s)$ is not defined directly by (2.3). The value of the quasiclassical asymptotic function

$$(2.4)\qquad y=f(x_i)$$

corresponding to the classical function (2.2) for $x_i=a_i$ is the asymptotic number a with the highest accuracy to the representatives of which belong all sequences $a(s)$ one obtains from (2.3) (varying $a_i(s)$ by given a_i and defining each $a(s)$ (2.3) arbitrarily over S^*).

According to theorem 7, if there is a number a^* to the representatives of which belong all $a(s)$ (2.3), there must be an infinity of such numbers a^*, each characterized by its accuracy ν^* but one and only number a between them with highest accuracy ν. The existence of numbers a^* is not always assured so that it can happen $f(x_i)$ to be defined not for all x_i.

Definition 7. *a_i of x_i for which there exist numbers a^* and consequently one number a with the indicated properties determine the set of points X in the set A^m for which the function (2.4) is defined.*

Let us stress that two data are necessary to be given for the determination of the value

$$(2.5)\qquad a=f(x_i)_{x_i=a_i}$$

of the quasiclassical asymptotic function $f(x_i)$ (4): (*i*) the classical function $f(\xi_i\, s)$ (2.2), defined on a given set X^* in the space of ξ_i and s and (ii) the values a_i of the arguments x_i, for which one wants to find the value of $f(x_i)$ (2.4). The problem to determine the set X, where the function $f(x_i)$ (2.4) corresponding to $f(\xi_i\, s)$ (2.2) is defined, is a problem which has to be solved for each function separately.

Definition 8. *If the sequences $f(a_i(s), s)$, obtained from (2.3) after arbitrary continuation on S^* and corresponding to all representatives $a_i(s)$ of the given values a_i of x_i, cover completely the set of representatives of a number of the type a^*, we say that the point a_i of X is perfect. Otherwise it is imperfect.*

Theorem 9. *The perfect number of the type a^*, if it exists, coincides with the result a (2.5).*

3. Algebraic Operations

Definition 9. *The elementary algebraic operations — addition, subtraction, multiplication and division*

$$(3.1)\qquad x=x'+x'',\quad x=x'-x'',\quad x=x'.x'',\quad x=x':x''$$

are the quasiclassical functions, which according to Definition 16, correspond to

(3.2) $$\eta=\xi'+\xi'', \quad \eta=\xi'-\xi'', \quad \eta=\xi'.\xi'', \quad \eta=\xi':\xi''.$$

For the case of addition equation (2.3) reads

(3.3) $$a(s)=a'(s)+a''(s).$$

According to (1.10) and (1.5)

(3.4) $$a'=(\nu', N', a'_n), \quad a''=(\nu'', N''_1 \, a''_n), \quad a=(\nu, N, a_n),$$

(3.5) $$a'(s)=\sum_n^{N'} a'_n s^n + o'_{\nu'}(s), \quad a''(s)=\sum_n^{N''} a''_n s^n + o''_{\nu''}(s),$$

$$a(s)=\sum_n^{N} a_n s^n + o_\nu(s).$$

Substituting (3.5) in (3.3) we get first

(3.6) $$\nu=\min(\nu', \nu'')$$

and then

(3.7) $$N=N'_{(\nu)} \cup N''_{(\nu)},$$

(3.8) $$a_n=a'_n+a''_n.$$

Writing (3.8) we have supposed $a'_n=0$ for $n \bar{\in} N'$ and $a''_n=0$ for $n \bar{\in} N''$. So we can formulate

Theorem 10. *The sum $x(1)$ is defined for all asymptotic arguments x' and x'' and according to (3.4) is given by (3.6)—(3.8).*

We shall add a formula for the power μ of the sum. If

$$\mu' \neq \mu'' \text{ or } \mu'=\mu'', \ N'^{[\mu']}=\varnothing \quad \text{or} \quad \mu'=\mu'', \ N''^{[\mu'']}=\varnothing$$

$$\text{or} \quad \mu'=\mu'', \ N'^{[\mu']}, \ N''^{[\mu'']} \neq \varnothing, \quad a'_{\mu'}+a''_{\mu''} \neq 0 \text{ we have}$$

$$\mu=\min(\mu', \mu'').$$

In the opposlte case when

$$\mu'=\mu'', \ N'^{[\mu']}, \ N''^{[\mu'']} \neq \varnothing, \quad a'_{\mu'}+a''_{\mu''}=0$$

we say that there arises annihilation and we have to look for the next nonvanishing terms with indices μ'_1 and μ''_1. Then we find that if

$$\mu'_1 \neq \mu''_1 \text{ or } \mu'_1=\mu''_1, \ N'^{[\mu']}=\varnothing \text{ or } \mu'_1=\mu''_1, \ N''^{[\mu''_1]}=\varnothing$$

$$\text{or } \mu'_1=\mu''_1, \ N'^{[\mu']}, \ N''^{[\mu'']}=\varnothing, \ a'_{\mu'}+a''_{\mu''} \neq \varnothing$$

we have

$$\mu=\min(\mu'_1, \mu''_1).$$

In the opposite case we have a double annihilation, etc. Generalizing we reach

Theorem 11. *If*

$$(3.9)\qquad \mu'_j=\mu''_j,\ N'^{[\mu'_j]},\ N''^{[\mu''_j]}\neq\varnothing,\quad a'_{\mu'_j}+a''_{\mu''_j}=0$$

$$(j=0, 1,\ldots, i-1;\quad \mu'_0=\mu',\ \mu''_0=\mu'')$$

but

$$(3.10)\qquad \mu'_i\neq\mu''_i \text{ or } \mu'_i=\mu''_i,\ N'^{[\mu'_i]}=\varnothing \text{ or } \mu'_i=\mu''_i,\ N''^{[\mu''_i]}=\varnothing \text{ or}$$

$$\mu'_i=\mu''_i,\ N'^{[\mu'_i]},\ N''^{[\mu''_i]}\neq\varnothing,\ a'_{\mu'_i}+a''_{\mu''_i}\neq 0$$

we have i-tuple annixilation and in this case

$$(3.11)\qquad \mu=\min(\mu'_i,\ \mu''_i).$$

Theorem 12. *For the difference*

$$(3.12)\qquad a=a'-a''$$

the formulas giving the components (1.10) of a as functions of the components of a' and a'' are analogous. Only in (3.8), (3.9) and (3.10) instead of $a'_n+a''_n$, $a'_{\mu'_j}+a''_{\mu''_j}$ *and* $a'_{\mu'_i}+a''_{\mu''_i}$ *will appear* $a'_n-a''_n$, $a'_{\mu'_j}-a''_{\mu''_j}$, *and* $a'_{\mu'_i}-a''_{\mu''_i}$.

Now one can prove

Theorem 13. *If we have three asymptotic numbers a, b, c one of which is the sum or the difference of the two others, the accuracies of two of them, standing on the different hand sides of the equality, must be equal and the accuracy of the third one must be not lower. We can transmit the number with the not lower accuracy from the one to the other hand side of the equality without losing its validity.*

So this theorem gives us the conditions by which the pairs at equalities $a=b-c$ and $b=a+c$ or $c=b-a$ and $a=b-c$ are equivalent. To prove Theorem 13 we have to realize that the connections between the components of a, b, c induced by these equalities are the same — first for the accuracies, which is clear from (3.6), and then for the spectra and the coefficients, which is obvious, because then the statement is brought back to the same statement for polynomials.

From Theorem 13 it follows

Theorem 14. *The sum and the difference are perfect for all choice of the arguments.*

Let us take the sum

$$(3.13)\qquad a=a'+a''$$

and let us suppose $\nu'\geq\nu''$. Then (3.13) according to Theorem 13 can be written in the form

$$(3.14)\qquad a''=a-a'.$$

Let us choose the residual terms of a and a' arbitrarily and let us determine the residual term of a'' from the equality

$$a''(s)=a(s)-a'(s)$$

which is always allowed. But since (3.14) is equivalent to (3.13), it turns out that by every choice of the residual term of the sum a in (3.13), it can be obtained by appropriate choice of the residual term of a'' (and by arbitrary choice of the residual term of a'). In the case $\nu'\leq\nu''$ the roles of a' and a'' have to be exchanged.

The proof of Theorem 14 for the difference $a=a'-a''$ is analogous — instead of (3.14) we have to start from $a'=a+a''$ for $\nu''\geq\nu'$ or from $a''=a'-a$ for $\nu''\leq\nu'$.

The spectra N' and N'' in all formulas concerning the addition and subtraction can be chosen to he primary, free or nonsingular, but the spectrum of the result N given by (3.7) is a free one. So if we want to deal only with one type of spectra, we have to suppose that N' and N'' are free too.

The formula (2.3) for the multiplication reads:

$$a(s)=a'(s)\,.\,a''(s). \tag{3.15}$$

Then making use of the notations (3.4) and (3.5) we get first

$$\nu=\min(\mu'+\nu'',\ \mu''+\nu') \tag{3.16}$$

and then

$$N=(N'+N'')_{(\nu)}, \tag{3.17}$$

$$a_n=\sum_{n'\in N,\ n''\in N''}^{n'+n''=n} a'_{n'}a''_{n''}\quad (n\in N), \tag{3.18}$$

$N'+N''$ being the spectrum one obtains adding each element of N'' to each elemenf of N'. Using the notations (1.12) and (1.13) we get in an analogous way

$$\lambda=\min(\lambda',\lambda''), \tag{3.19}$$

$$\mu=\mu'+\mu'', \tag{3.20}$$

$$M=(M'+M'')_{(\lambda)}, \tag{3.21}$$

$$\alpha=\alpha'\,.\,\alpha'', \tag{3.22}$$

$$\alpha_m=\sum_{m'\in M',\ m''\in M''}^{m'+m''=m}\alpha'_{m'}\alpha''_{m''}. \tag{3.23}$$

The advantage of these formulas is that they realize maximum separation of the components: λ depends on λ' and λ'', μ — on μ' and μ'', α — on α' and α'' and α_m — on α'_m and α''_m.

If one or both factors a' and a'' are zero, the formulas (3.16)—(3.18) and (3.19)—(3.23) remain but in this case one finds simply

$$a' \,.\, a'' = o_{\mu'+\mu''}\,.$$

By the division **(2.3)** reads

(3.24) $$a(s) = a'(s) : a''(s).$$

We suppose the denominator a'' not to be zero. Then making use of (1.10) we get

(3.25) $$\nu = \mu' - \mu'' + \min(\nu' - \mu',\ \nu'' - \mu''),$$

(3.26) $$N = (N' - \mu'' + (N''^{(\mu'')} - \mu'')^*{}_{(\nu)},$$

(3.27) $$a_n = \sum_{l \geq 0} (-1)^l a_{ln}, \quad (n \in N),$$

(3.28) $$a_{ln} = \frac{1}{a''^{\,l+1}_{\mu''}} \sum_{k} \sum_{l_1, l_2, \ldots, l_p \geq 0}^{l_1+l_2+\cdots+l_p = l} \frac{l!}{l_1!\, l_2! \ldots\ l_p!}\, a'_k\, a''^{\,l_1}_{k_1}\, a''^{\,l_2}_{k_2} \ldots a''^{\,l_p}_{k_p}.$$

Here and in the remainder M^* is the saturation of M, i. e. the spectrum which (i) contains M, (ii) if $k, l \in M^*$ then $k+l \in M^*$ and (iii) M^* is the minimum spectrum with these properties

(3.29) $$M^* = M \cup (M+M) \cup (M+M+M) \cup \ldots$$

The summation in (3.28) is restricted by the condition

(3.30) $$k + k_1 l_1 + k_2 l_2 + \cdots + k_p l_p = l_\mu + n,$$

where $\mu'' < k_1 < k_2 < \cdots < k_p \leq \nu''$ are the elements of $N''^{(\mu'')}$ and p is this number. The summation in (3.27) is finite because the condition $l_1 + l_2 + \cdots + l_p = l$ can be fulfilled only by sufficiently small l.

Making use of the notations (1.9) and (1.10) we get obviously

(3.31) $$\lambda = \min(\lambda', \lambda''),$$

(3.32) $$\mu = \mu' - \mu'',$$

(3.33) $$\alpha = \alpha' : \alpha'',$$

(3.34) $$\alpha_m = \sum_{l \geq 0} (-1)^l{}_{\alpha lm},$$

(3.35) $$\alpha_{lm} = \sum_{k}^{M'(\lambda)} \sum_{l_1, l_2, \ldots, l_p \geq 0}^{l_1+l_2+\cdots+l_p = l} \alpha'_k\, \alpha''^{\,l_1}_{k_1}\, \alpha''^{\,l_2}_{k_2} \ldots \alpha''^{\,l_p}_{k_p}.$$

Here $0 < k_1 < k_2 < \cdots < k_p \leq \lambda''$ are the elements of $M''^{(0)}$ and the condition (3.30) reads now

(3.36) $$k + k_1 l_1 + \cdots + k_p l_p = n.$$

The formulas (3.25)—(3.28), (3.30) and (3.31)—(3.36) remain also if a' is zero $o_{\nu'}$, but in this case we find simply $a' : a'' = o_{\nu' - \mu''}$. If a'' is zero, the ratio $a' : a''$ does not exist, even not for $a' = o_{\nu'}$.

So we see

Theorem 15. *The product and the ratio of two asymptotic numbers a' and a'' are defined for all a' and a'' with the natural exception when the denominator of the ratio is a zero o_ν.*

By the demonstration of these formulas we have to take into account that the representatives $a''(s)$ of the denominator a'' can vanish for a set S^* of values of s where the values of (3.24) are chosen arbitrarily. Since a'' is not zero, for every $a''(s)$ exists one value s_0 of s so that $s \in S^*$ only for $s \geqq s_0$. But the arbitrariness in the definition of (3.24) for $s \in S^*$ does not violate the behaviour in the neighbourhood of $s=0$ which is only of significance.

The formulas (3.25)—(3.28), (3.30) are valid for all three types of spectra N' and N'' — primary, free and nonsingular, and (3.31)—(3.36) — for primary or nonsingular. The result is given with a free spectrum if it is obtained through (3.25)—(3.28), (3.30) and N' and N'' are free or it is with a nonsingular spectrum in all other cases. So if we want to deal with only one type of spectra by the multiplication and division, we have to choose the nonsingular ones.

Here we can prove two theorems analogous to Theorem 13 and Theorem 14.

Theorem 16. *If we have three not simultaneously vanishing asymptotic numbers a, b, c, one of which is the product or the ratio of the two others, the relative accuracies at two of them, standing on the different hand sides of the equality, must be equal and the relative accuracy of the third one must be not lower. We can transmit the number with the not lower relative accuracy from the one to the other hand side of the equality without losing its validity.*

The first part of this theorem is an immediate consequence of (3.19) and (3.31). For the second part we have to take into account also the next equation (3.20)—(3.23) and (3.32)—(3.36), which simply reproduce the Cauchy rule for multiplication and division of asymptotic series and according to which one can easily transmit factors from one to the other hand side.

Theorem 17. *The product and the ratio are perfect for every choice of the arguments for which they are defined, if the denominator of the ratio is not zero.*

The proof is analogous to that for the sum and the difference. Only by the product the case when all numbers a, b, c are zeros has to be considered separately. Indeed, if we set,

$$o_{\nu'+\nu''}(s)=o'_{\nu'}(s)\,.\,o''_{\nu''}(s), \tag{3.37}$$

$$o'_{\nu'}(s)=s^{\nu'}o(s),\quad o''_{\nu''}(s)=s^{\nu''}o(s), \tag{3.38}$$

we get

$$o(s)=\sqrt{o_{\nu'+\nu''}(s)s^{-\nu'-\nu''}}=o_0(s). \tag{3.39}$$

We put in (3.38) and see that in the case of complex asymptotic numbers (3.37) can be satisfied for every choice of $o_{\nu'+\nu''}(s)$. In the case of real asymptotic numbers we can replace (3.38) by $o'_{\nu'}(s)=s^{\nu'}o(s)$, $o''_{\nu''}(s)=s^{\nu''}\varrho(s)o(s)$ where $\varepsilon(s)=\operatorname{sign} o_{\nu'+\nu''}(s)$. Then instead of (3.39) we get

$$o(s)=\sqrt{o_{\nu'+\nu''}(s) \mid s^{-\nu'\ \nu''}}=o_0(s).$$

So that the reality of $o(s)$, $o'_{\nu'}(s)$ and $o''_{\nu''}(s)$ is ensured.

Thus one sees that all four algebraic operations are defined for every choice of the arguments a' and a'' and the result is perfect with the natural restriction by the division — the denominator must be not zero o_ν (see Definition 10).

4. Remarks on the Quasiclassical Functions and the Algebraic Operations.

Remark 1. The theory of the asymptotic numbers is quite similar but more general than the theory of asymptotic series [9]. Indeed, we are dealing not only with positive but also with negative powers of s. This generalization is essential for our purposes — to define the asymptotic functions, comprising functions analogous to the Dirac's δ-function and its derivatives [8], but it is not connected with mathematical difficulties and more or less is known. More essential point in our definition is the introduction of accuracy ν. If we admit ν to be ∞ we shall come back to the theory of asymptotic series with positive and negative powers. By means of the accuracy we reach more generality and give the theory connection with the notion of local sets (at $s=0$). Although the definition is quite natural, it is just the accuracy that leads to some peculiarities looking like exceptions against the rules of the classical algebra.

Remark 2. Instead of (1.5), without restriction, we could put

$$a(s)=\sum_{n=\mu}^{\nu} a_n s^n + o_\nu(s),$$

i. e. we can avoid the work with the spectra N. We have preferred (1.5), for example, because in this way the correspondence (1.10) is biunivoque and the components ν, N, a_n give us an idea how large the set of asumptotic numbers is. We prefer (1.5) also in connection with conveniences to be seen in the future.

Remark 3. One can introduce the algebraic operations in a simpler way — as numbers, the set of whose representatives coincides with the set of sequences obtained by means of the same operation executed over all admissible pairs of representatives of the arguments (supposing by the division that the obtained sequences are arbitrarily determined for values of s, where the denominator vanishes). In the same way we could introduce also the quasiclassical functions but then we had to agree that functions as $x-x$, x^2, $x:x$ etc. leading to imperfect results are not defined and this would be a disadvantage. It is not possible to determine the result by means of a single pair of representatives of the arguments because we shall obtain a single sequence and it can represent different numbers, so that the result would not be uniquely determined. Just the consistency of the definitions of the algebraic operations and the quasiclassical functions was the reason to introduce the asymptotic numbers by means of a relation of subordinance and not of equivalence and to include the supplementary representatives.

Remark 4. By finite $\mu(\lambda)$ we can say that the main part of the sum and the difference (the product and the ratio) is given by the sum and the difference (the product and the ratio) of the main parts of the terms (factors) cut off up to $\nu(\lambda)$. But by infinite this does not hold because of the conventions (1.8) and (1.9).

Remark 5. With the aid of the notion of the quasiclassical function we can define general algebraic operations between asymptotic and ordinary numbers. We have only to suppose that in (2.2) besides the parameter and the real (complex) arguments ξ_i which are to be replaced by representatives of asymptotic numbers, there are supplementary variables c_j ($j=1, 2, \ldots, n$), which remain ordinary real (complex) numbers:

$$y=f(x_i, c_j, s). \tag{4.1}$$

So we can understand expressions as ax, $(x+a)/(x-a)$, $axy+b, \ldots$ where $a, b, \ldots$ are ordinary and $x, y, \ldots$ — asymptotic numbers. Making use of the existence of s as argument in (2.2) and (4.1), we can make meaningful also expressions like xs^{-1}, $x+as^2$, $axys+bs^{-1}$ etc.

Remark 6. In all examples we gave, the results were perfect. There are also examples of imperfect results $y=[x^2]_{x=o_\nu}=o_{2\nu}$, $z=[x^2+y]_{x=o_0,\, y=(-s+s^2)_2}=O_0$, $y=\left[\frac{2x}{x}+3\right]=5_\infty$, $y=[x-x+1]=1_\infty$ (see (1.11). In the first two cases x and y are real asymptotic numbers. Then the sequences $o_\nu(s)^2$ cover obviously only the non-negative representatives of $o_{2\nu}$ and the sequences $o_0^2(s)-s+s^2+o_2(s)$ — only the representatives $a(s)$ of o_0 having the form $-s-o_2(s)\leq a(s)\leq o_0(s)$. In the next two examples x can be real or complex. In each of them one gets only one sequence — $a(s)=5$ and $a(s)=1$, so that the result is 5_∞, resp. 1_∞.

Remark 7. The general definition of quasiclassical function gives us the possibility to define arbitrary rational and transcendental functions for asymptotic arguments. If $(f(\xi_i, s)$ is rational, the corresponding asymptotic function is defined unless the denominator is zero. As an example of transcendental asymptotic function let us take

$$y=\sin x. \tag{4.2}$$

If

$$x=(x_0+x_1 s+x_2 s^2)_2 \tag{4.3}$$

(1.11) we set $y=\left(\sin x_0+x_1\cos x_0 s+\left(x_2\cos x_0-\frac{1}{2}x_1^2\sin x_0\right)s^{-2}\right)_2$. If $x=(x_{-1}s^{-1}+x_0+x_1 s)_1$, $x_{-1}\neq 0$ replacing in (4.2) we do not obtain a sequence from $A(s)$ because at $s=0$ appears an essential singularity. The asymptotic function (4.2) is not defined for arguments with power $\mu<0$. For $\mu\geq 0$ it is defined and its accuracy coincides with the accuracy of x.

As a next example we can take

$$y=\ln x. \tag{4.4}$$

If the power of x is negative, it is not defined. Let x be given by (4.3). If $x_0\neq 0$, we obtain

$$y=\left(\ln x_0+\frac{x_1}{x_0}s+\left(\frac{x_2}{x_1}-\frac{x_1^2}{2x_0^2}\right)s^2\right)_2.$$

If $x_0 = 0$, we do not get a sequence from $A(s)$ — at $s=0$ it has a logarithmic singularity. In general, the function (4.4) is defined only if the power μ of x is 0, the leading coefficient a_μ is not 0. Let now

$$y=\sqrt{x}, \tag{4.5}$$

x being given by (4.3). For $x_0 \neq 0$ we get

$$y=\left(\sqrt{x_0}+\frac{x_1}{2\sqrt{x_0}}s+\left(\frac{x_2}{2\sqrt{x_0}}-\frac{x_1^2}{8x_0\sqrt{x_0}}\right)s^2\right)_2.$$

For $x_0=0$, $x_1 \neq 0$ the function (4.5) is not defined. In general, (4.5) is defined for x with even (positive or negative) power μ and is not defined for odd μ.

The origin of these restrictions for the set of values X, for which functions like (4.2), (4.4) or (4.5) are defined, lies in the circumstance that we are considering asymptotic numbers with polynomial main part. By larger classes of primary sequences $A(s)$ we could push forward the limits of the set X, where the quasiclassical functions can be generalized.

Remark 8. We know that passing from one theory to another (for example, from the ordinary to the vector algebra) some elementary laws (as commutativity, etc.) are violated and some notions are splitted in several notions (different product, etc.). To avoid ambiguities, more sophisticated formulations of the statements and more elaborated notations (as different brackets) are needed. This phenomenon appears also when we pass from the ordinary to the asymptotic numbers and that is why the work with them requires more care.

By classical functions we have two ways for calculating values of functions

$$\eta=f(\xi_i, \zeta_j, s) \quad (i=1, 2, \ldots, m; \quad j=1, 2, \ldots, n) \tag{4.6}$$

of functions

$$\zeta_j=g_j(\xi_i, s). \tag{4.7}$$

The first way requires to replace (4.7) in (4.6) and then to calculate η on the basis of the function $\eta=F(\xi_i, s)=f(\xi_i, g_j(\xi_i, s), s)$.

The second way is to calculate from (4.7) the values of ζ_j for the given values of ξ_i and then to find η from (4.6) as a function of the given ξ_i and the found ζ_j. The results are the same. By the quasiclassical asymptotic functions these two ways are also available but the results (although as exception) can be different. That is why we must introduce different notations for these two procedures. For the first one we shall write simply

$$y=f(x_i, z_j(x_i)) \tag{4.8}$$

and for the second

$$y=f(x_i, [z_j(x_i)]). \tag{4.9}$$

Hence (4.9) represents not a kind but a generalization of the notion of quasiclassical function.

As an example let us take $u=u'=v+w$ where $v=x.z$, $w=y.z$. In the first sense $u=(x+y)z$ and for the accuracy of u according to (3.6), (3.11) and (3.16) we get $\nu'=\min(\mu_{xi}+\nu_z, \mu_{yi}+\nu_z, \mu_z+\nu_x, \mu_z+\nu_y)$, where i is the number of annihilations in $x+y$ and μ_{xi} and μ_{yi} are the ith elements in the spectra of x and y. In the second sense $u=u''=[x.z]+[y.z]$ and[according to (3.16) and (3.6) $\nu''=\min(\mu_x+\nu_z, \mu_y+\nu_z, \mu_z+\nu_y)$.

One sees that if annihilations are present and if

$$\min(\mu_{xi}+\nu_z, \mu_{yi}+\nu_z)<\min(\mu_z+\nu_y, \mu_z+\nu_y)$$

then $\nu'<\nu''$, i. e. $u'\neq u''$. It follows from here that the equality $[x.z]+[y.z=[x+y].z$ showing the law of distributivity in the second sense (4.9) can be violated (through by a special choice of x, y, z).

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