Serdica

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Сердика

Българско математическо списание

HAUSDORFF DERIAVTIVES IN F_Δ

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A transformation in the class F_Δ of point sets in R^2, called Hausdorff differentiation is introduced and studied. In particular F_Δ includes complete graphs of bounded functions. For such graphs the transformation is in some sense similar to the common operation of differentiation. The inverse transformation is also defined and some of its properties are discussed.

1. Denote by F_Δ the class of all point subsets of the real plane R^2 which are bounded, closed and convex with respect to the y-axis and whose projections on the x-axis coincide with the interval Δ on the x-axis.

For $F \in F_\Delta$ denote $F(a)$ the intersection of F with the line $x=a$ and $I_F(x)=\inf\{y \mid (x,y)\in F(x)\}$, $S_F(x)=\sup\{y \mid (x,y)\in F(x)\}$, so that $F(x)=\{(x,y) \mid y\in[I_F(x), S_F(x)]\}$, i. e. each $F\in F_\Delta$ can be defined by the pair of functions I_F and S_F.

Let f be a bounded function with graph $\boldsymbol{f}$ and let $\mathfrak{f}$ be its complete graph, defined by $\mathfrak{f}=\{\cap p : p\in F_\Delta,\ p\supset \boldsymbol{f}\}$. Then $I_{\mathfrak{f}}(x)$ and $S_{\mathfrak{f}}(x)$ coincide with the Baire functions for f given by $I_{\mathfrak{f}}(x)=\liminf_{t\to x} f(t)$, $S_{\mathfrak{f}}(x)=\limsup_{t\to x} f(t)$.

Definition. Let $h>0$. The H-derivative $F^\cdot$ of the set $F\in F_\Delta$ is defined by the pair $I_{F^\cdot}(x)$, $S_{F^\cdot}(x)$, where

$$(1)\qquad \begin{aligned} I_{F^\cdot}(x) &= \liminf_{\substack{x',\, x''\to x \\ (x',y'),\,(x'',y'')\in F}} \frac{y''-y'}{x''-x'}\left(1+h\left|\frac{y''-y'}{x''-x'}\right|\right)^{-1}, \\ S_{F^\cdot}(x) &= \limsup_{\substack{x',\, x''\to x \\ (x',y'),\,(x'',y'')\in F}} \frac{y''-y'}{x''-x'}\left(1+h\left|\frac{y''-y'}{x''-x'}\right|\right)^{-1}. \end{aligned}$$

Let us first restrict our considerations to graphs of differentiable functions. If $\boldsymbol{f}$ is the graph of a differentiable function f, then from (1) it follows that $(x,y)\in\boldsymbol{f}^\cdot$ implies $y=f'/(1+h|f'|)$, what can also be stated in the form

$$(2)\qquad \boldsymbol{f}^\cdot(x)=f'/(1+h|f'|)=f^\cdot(x).$$

Let

$$d_h(A,F)=\min\{\varrho_h(A,B) \mid B\in F\}$$

is the distance between the point A and the set $F \in F_{\Delta}$, where the distance between the points $A(x_A, y_A)$ and $B(x_B, y_B)$ in R^2 is

$$\varrho_h(A, B) = \max(|x_A - x_B|/h, \ |y_A - y_B|/H). \tag{3}$$

Consider then the function $f \div g$ defined for any two continuous functions f and g by

$$f(x) \div g(x) = \operatorname{sign}(f(x) - g(x)) \max(d_h(f(x), g), \ d_h(g(x), f)), \quad x \in \Delta \tag{4}$$

and called H-difference between f and g. As it is shown later (Prop. 2) the H-difference is closely related to the Hausdorff metric [1].

Proposition. 1. *If f is differentiable in Δ, for $x \in \Delta$ holds*

$$\lim_{\tau \to 0} \frac{f(x+\tau) \div f(x)}{\tau} = f^{\cdot}(x) = \frac{f'(x)}{1 + h\,|f'(x)|}. \tag{5}$$

Proof. Denote $f(x+\tau) = f_\tau(x)$. One has then $f_\tau(x) \div f(x) = \operatorname{sign}(f_\tau(x) - f(x)) \max(d_h(\boldsymbol{f}(x), \boldsymbol{f}_\tau), \ d_h(\boldsymbol{f}_\tau(x), \boldsymbol{f})) = \operatorname{sign}(f_\tau(x) - f(x)) \max(|B'P'|, \ |B''P''|)$ (see Fig. 1). We have used the notations $O = \boldsymbol{f}(x)$, $O^* = \boldsymbol{f}_\tau(x)$, $O' = \boldsymbol{f}_\tau(x+\tau)$, $O'' = \boldsymbol{f}(x+\tau)$, $B' = \boldsymbol{f}_\tau(x')$, $B'' = \boldsymbol{f}(x'')$.

a) suppose $|B'P'| \geq |B''P''|$. In this case $f_\tau(x) \div f(x) = \operatorname{sign}[f_\tau(x) - f(x)]\,|B'P'| = \overline{P'B'}$ where $\overline{P'B'}$ denotes the algebraic value of the segment $P'B'$ with respect to the y-axis. Denote $\lambda = \overline{P'B'}/\overline{O'P'} = (f_\tau(x) - f(x))/(x' - (x - \tau)) = (f(x'+\tau) - f(x))/(x' + \tau - x)$. The convergence of τ to zero implies $x' + \tau \to x$ and hence $\lambda \to f'(x)$. From $\lambda = \overline{P'B'}/\overline{O'P'} = \overline{P'B'}/(\tau - h\,|P'B'|)$ we

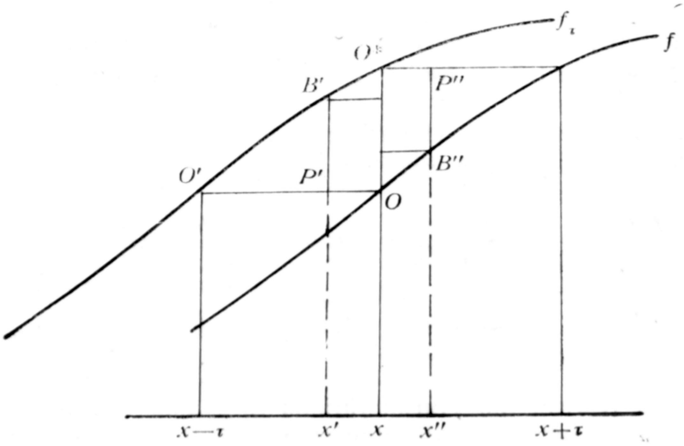

Fig. 1

have $\overline{P'B'} = \lambda\tau/(1 + h\,|\lambda|)$ and $\lim_{\tau \to \infty}[f(x+\tau) f(x)] : /\tau = \lim_{\tau \to \infty} \overline{P'B'} = f'(x) / (1 + h\,|f'(x)|)$.

b) suppose now $|B'P'| < |B''P''|$. Similarly $f_\tau(x) \div f(x) = \overline{B''P''}$. Denote $\lambda = \overline{B''P''}/\overline{P''O''} = [f(x+\tau) - f(x'')]/(x + \tau - x'')$. When τ tends to zero we have again $\lambda \to f'(x)$. From $\lambda = \overline{B''P''}/\overline{P''O''} = \overline{B''P''}/(\tau - h|\overline{P''B''}|$ follows $\overline{B''P''} = \lambda\tau/(1+h|\lambda|)$ and hence $\lim_{\tau\to\infty}[f(x+\tau) \dot{-} f(x)]/\tau = f'(x)/(1+h|f'(x)|)$, which proves the proposition.

The Hausdorff distance $r_h(f, g)$ between two continuous functions generated by (3) can be expressed easily by means of the H-difference (4).

P r o p o s i t i o n 2. *For every two continuous functions f and g holds*

$$r_h(f, g) = \max_{x\in\Delta} |f(x) \dot{-} g(x)|.$$

P r o o f. $r_h(f, g) = r_h(\boldsymbol{f}, \boldsymbol{g}) = \max\{\max_{x\in\Delta} d_h(\boldsymbol{f}(x), \boldsymbol{g}), \max_{x\in\Delta} d_h(\boldsymbol{f}, \boldsymbol{g}(x))\} = \max_{x\in\Delta}\{\max[d_h(\boldsymbol{f}(x), \boldsymbol{g}), d_h(\boldsymbol{f}, \boldsymbol{g}(x))]\} = \max_{x\in\Delta}|f(x) \dot{-} g(x)|$.

By means of (2) the following relations for the H-derivative can be obtained. In what follows f and g are assumed continuous and c is a constant:

i) $$(cf)^{\cdot} = cf^{\cdot}/(1+h(|c|-1)|f^{\cdot}|),$$

ii) $$(f+g)^{\cdot} = \frac{f^{\cdot}+g^{\cdot}-h(|g^{\cdot}|f^{\cdot}+|f^{\cdot}|g^{\cdot})}{1+h(f^{\cdot}+g^{\cdot}-|f^{\cdot}|-|g^{\cdot}|)+h^2(|f^{\cdot}||g^{\cdot}|-|f^{\cdot}|g^{\cdot}-f^{\cdot}|g^{\cdot}|)},$$

iii) $$(fg)^{\cdot} = \frac{f^{\cdot}g+fg^{\cdot}-hf^{\cdot}g^{\cdot}(f+g)}{1+h(f^{\cdot}g+fg^{\cdot}-f^{\cdot}-g^{\cdot})+h^2f^{\cdot}g^{\cdot}(1-f-g)},$$

iv) $$(f/g)^{\cdot} = (\beta f^{\cdot}g - \alpha fg^{\cdot})/[\alpha\beta g^2 + h(\beta f^{\cdot}g - \alpha fg^{\cdot})],$$ wherein

$$\alpha = 1 - h|f^{\cdot}|, \quad \beta = 1 - h|g^{\cdot}|,$$

v) $$(f\circ g)^{\cdot} = (f^{\cdot}\circ g)g^{\cdot}/\{[1-h(f^{\cdot}\circ g)](1-hg^{\cdot})+h(f\circ g)g^{\cdot}\},$$ where $f\circ g$ denotes the composition of f and g,

(vi) $$c^{\cdot} = 0,$$

(vii) $$[c_1x/(1-h|c|)+c_2]^{\cdot} = c_1,$$

viii) $$(-h^{-1}x - h^{-2}\ln|1-hx|)^{\cdot} = x,$$

ix) $$f^{\cdot} = f \quad \text{implies} \quad f\exp(-kf) = \exp x.$$

Let us now obtain the H-derivatives of the graphs of some nondifferentiable functions. Consider the consecutive H-derivatives of the graph $\boldsymbol{m}$ of $m(x) = |x|$, $x\in\Delta$, $0\in\Delta$.

i) From (1) we get $I_{m^{\cdot}}(x) = -1/(1+h)$ for $x\leq 0$, $I_{m^{\cdot}}(x) = 1/(1+h)$ for $x\geq 0$ and $S_{m^{\cdot}}(x) = -1/(1+h)$ for $x<0$, $S_{m^{\cdot}}(x) = 1/(1+h)$ for $x\geq 0$, i. e. $\boldsymbol{m}^{\cdot}$ is the complete graph of the function $(1+h)^{-1}\sigma(x)$, where $\sigma(x) = -1$ for $x\leq 0$, $\sigma(x) = 1$ for $x>0$.

ii) For the H-derivative of the graph $\boldsymbol{\sigma}$ of σ by means of (1) we get $I_{\sigma^{\cdot}}(x) = 0$ and $S_{\sigma^{\cdot}}(x) = 0$ for $x\neq 0$, $S_{\sigma^{\cdot}}(x) = h^{-1}$ for $x<0$, i. e. $\boldsymbol{\sigma}^{\cdot}$ is the complete graph of the function $h^{-1}\delta(x)$ where $\delta(x) = 0$ for $x\neq 0$, $\delta(x) = 1$ for $x = 0$.

iii) The H-derivative of δ is the set $\boldsymbol{\kappa} = \boldsymbol{\delta}^{\cdot}$ consisting of the points of the interval Δ plus the points of the segment A_1A_2 where $A_i = (0, (-1)^i h^{-1})$. There is no function for which $\boldsymbol{\kappa}$ is its complete graph.

These examples show that the H-derivative can be considered as an extension of the concept of derivative over a larger class of functions. Hence some theorems like those belonging to Fermat, Rolle etc. can be generalized. We give a generalization of the Lagrange mean value theorem as an example.

Proposition 3. *Let f be a bounded function in the interval $\Delta=[a, b]$. There exists then a $x\in\Delta$ such that*

$$I_{f^{\cdot}}(x)\leq\varphi(a, b)=\frac{f(b)-f(a)}{b-a}\left(1+h\left|\frac{f(b)-f(a)}{b-a}\right|\right)^{-1}\leq S_{f^{\cdot}}(x)$$

Proof. Denote by (x, z) such a point from the complete graph $\mathbf{f}$ of f which has the greatest possible distance to the segment $\boldsymbol{l}$ with ends $(a, f(a))$ and $(b, f(b))$. In particular (x, z) might lie on $\boldsymbol{l}$, in which case $\mathbf{f}=\boldsymbol{l}$ and Prop. 3 is obviously true. If $x'<x<x''$ and (x', y'), $(x'', y'')\in\mathbf{f}$ denote $\operatorname{tg}\beta=[f(b)-f(a)]/(b-a)$, $\operatorname{tg}\beta'=(z-y')/(x-x')$ and $\operatorname{tg}\beta''=(y''-z)/(x''-x)$.

We have either $\beta''\leq\beta\leq\beta'$ or $\beta'\leq\beta\leq\beta''$. Consider the case $\beta''\leq\beta\leq\beta'$ (the other case is handled similarly), then $(y''-z)/(x''-x)\leq[f(b)-f(a)]/(b-a)\leq(z-y')/(x-x')$. Hence

$$\frac{y''-z}{x''-x}\left(1+h\left|\frac{y''-z}{x''-x}\right|\right)^{-1}\leq\varphi(a, b)\leq\frac{z-y'}{x-x'}\left(1+h\left|\frac{z-y'}{x-x'}\right|\right)^{-1}.$$

This implies

$$\liminf_{\substack{x', x''\to x\\ (x',y'),(x'',y'')\in f}}\frac{y'-y''}{x'-x''}\left(1+h\left|\frac{y'-y''}{x'-x''}\right|\right)^{-1}$$

$$\leq\varphi(a, b)\leq\limsup_{\substack{x', x''\to x\\ (x',y'),(x'',y'')\in f}}\frac{y'-y''}{x'-x''}\left(1+h\left|\frac{y'-y''}{x'-x''}\right|\right)^{-1},$$

what was to be proved.

2. Looking for physical applications of the H-derivative we might note that the H-derivative can be interpreted as some sort of velocity. It might be interesting to point out that for smooth functions the H-derivative relates to the usual derivative in the same way as the so-called apparent velocity [2, p. 178], [3, p. 135] relates to the usual velocity. Indeed between the apparent velocity $v_a=f^{\cdot}$ and the standard velocity $v=f'$ the relation (2) holds, where $1/h$ should be interpreted as the velocity of light $(1/h=c)$.

3. Here we give an application of the H-derivative in the theory of approximation. The following theorem for linear positive operators is well-known [4, p. 21].

Theorem (Korovkin). *Let $\{L_n\}$ be a sequence of linear positive operators such that the so-called Korovkin conditions* $\lim_{n\to\infty}\max_{x\in\Delta}|L_n(t^i, x)-x^i|=0$, $i=0, 1, 2$ *hold. Then for every function f continuous in the interval Δ one has* $\lim_{n\to\infty}\max_{x\in\Delta}|L_n(f, x)-f(x)|=0$.

For example the Bernstein polynomials $B_n(f; x)=\sum_{k=0}^{n}\binom{n}{k}f(k/n)\,x^k(1-x)^k$ in the interval $[0, 1]$ are linear positive operators. For these polynomials we have [5, p. 251] $\lim_{n\to\infty}\max_{x\in[0,1]}|B'_n(f; x)-f'(x)|=0$ if f' is continuous. Let us also mention that $B'_n(f; x)=B_{n-1}(n(f(t+1/n)-f(t)); x)$.

Let $\{L_n(f, x)\}$ be a sequence of linear positive operators which satisfy the Korovkin conditions and

(6) $$L'_n(f\,;\,h)=L_{n-1}(F_n\,;\,x),$$

wherein $F_n(x)=[f(\alpha_n x+\beta_n)-f(\alpha_n x)]/\beta_n$, with $\alpha_n\to 1$, $\beta_n\to 0$, for example

$$F_n(x)=n(f(x+\frac{1}{n})-f(x)\quad\text{or}\quad F_n(x)=n[f(\frac{n-1}{n}x+\frac{1}{n})-f(\frac{n-1}{n}x)].$$

Such operators are for example the Bernstein operators and the generalized operators of Baskakov

$$B_n^*=\sum_{k=0}^{\infty}(-1)^k\frac{\varphi_n^{(k)}(x)x^k}{k!}f(k/n),$$

where $\varphi_n^{(k)}(x)$ satisfy certain conditions.

We prove the following

Proposition 4. *If under the forementioned conditions for the function f holds*

(7) $$\lim_{n\to\infty}r_h(L_n(f), f)=0,$$

then

(8) $$\lim_{n\to\infty}r_h(L_n^{\cdot}(f), f^{\cdot})=0.$$

Lemma. *Let $\{L_n(f)\}_1^{\infty}$ be a sequence of linear positive operators that satisfy the Korovkin conditions. Then for every* $\varepsilon>0$ and $\delta>0$, $x_0\in[a, b]$ there exists a n_0 such that by $n>n_0$ (n_0 depends only on $\max|f|$ but not on f) we have $m_\delta(x)-\varepsilon\leq L_n(f\,;x)\leq M_\delta(x_0)+\varepsilon$; $x_0-\delta/2\leq x\leq x_0+\delta/2$ where, $M_\delta(x_0)=\max_{|x-x_0|<\delta}f(x)$; $m_\delta(x_0)=\min_{|x-x_0|<\delta}|f(x)|$.

The lemma follows from the theorem of Korovkin.

We prove now that (7) implies (8) by the assumptions (6) for $\{L_n\}$.

Let $(x, y)\in f^{\cdot}$. We have to show that for arbitrary small $\varepsilon>0$ and sufficiently large n there exist points $(x_n, y_n)\in L_n^{\cdot}(f)$ such that $|x-x_n|<\varepsilon$, $|y-y_n|<\varepsilon$. The definition of $f^{\cdot}$ implies that there exist points (x', y'), $(x'' y'')\in f$ such that

(9) $$\left|y-\frac{y'-y''}{x'-x''}(1+h\left|\frac{y'-y''}{x'-x''}\right|)^{-1}\right|<\frac{\varepsilon}{4}.$$

From (7) it follows that there exist sequences of points $(x'_n, y'_n)\in L_n(f)$ and $(x''_n, y''_n)\in L_n(f)$, such that $x'_n\to x'$, $x''_n\to x''$, $y'_n\to y'$, $y''_n\to y''$. Then (9) implies that for sufficiently large n

(10) $$\left|y-\frac{y'_n-y''_n}{x'_n-x''_n}(1+h\left|\frac{y'_n-y''_n}{x'_n-x''_n}\right|)^{-1}\right|<\frac{\varepsilon}{2}.$$

Proposition 3 asserts that there exist points $(x_n, y_n)\in L_n^{\cdot}(f)$ such that $x''_n\leq x_n\leq x''_n$ or $x''_n\leq x_n\leq x'_n$ and

(11) $$y_n = \frac{y'_n - y''_n}{x'_n - x''_n}\left(1 + h\left|\frac{y'_n - y''_n}{x'_n - x''_n}\right|\right)^{-1}.$$

From (10) and (11) follows the wanted $|x - x_n| < \varepsilon$, $|y - y_n| < \varepsilon$.

Now we shall show inversely that for sufficiently large n all points from $L^{\cdot}_n$ are ε-close to $f^{\cdot}$. Assume that this is not true. Then there exist points $(x_n, y_n) \in L^{\cdot}_n$ and a point (x_0, y_0) such that $x_n \to x_0$, $y_n \to y_0$ and in the square with centre (x_0, y_0) and side ε there are no points from $f^{\cdot}$. We may suppose that $y_0 > 0$. It is then obvious that $y_0 \leq 1/h$. As $f^{\cdot}$ is convex with respect to the y-axis, we might assume that for $x_0 - \varepsilon \leq x \leq x_0 + \varepsilon$, $(x, y) \in f^{\cdot}$ holds $y < y_0 - \varepsilon$. The definition of H-difference implies then that there exists a neighbourhood $(x_0 - \delta, x_0 + \delta)$ of the point x_0 such that for $\tilde{x}$ and $\tilde{\tilde{x}}$ from the interval $[x_0 - \delta, x_0 + \delta]$ we have

(12) $$\left(\frac{f(\tilde{x}) - f(\tilde{\tilde{x}})}{\tilde{x} - \tilde{\tilde{x}}}\right)/\left(1 + h\left|\frac{f(\tilde{x}) - f(\tilde{\tilde{x}})}{\tilde{x} - \tilde{\tilde{x}}}\right|\right) \leq y_0 - \varepsilon/2.$$

From the conditions (6) we have that

(13) $$L^{\cdot}_n(f; x) = L_{n-1}(F_n; x)(1 + h|L_{n-1}(F_n; x)|)^{-1}$$

where $F_n(x) = (f(\alpha_n x + \beta_n) - f(\alpha_n x))/\beta_n$, $\alpha_n \to 1$, $\beta_n \to 0$.

From (12) it follows

$$\frac{f(\tilde{x}) - f(\tilde{\tilde{x}})}{\tilde{x} - \tilde{\tilde{x}}} \leq \frac{y_0 - \varepsilon/2}{1 - h(y_0 - \varepsilon/2)},$$

provided ε is such that $1 - h(y - \varepsilon/2) \neq 0$.

The lemma implies that for sufficiently large n we have $L_{n-1}(F_n; x) \leq (y_0 - \varepsilon/2)/(1 - h(y_0 - \varepsilon/2))$, $|x_0 - x| \leq \min\{\varepsilon/2, \delta/2\}$.

Taking into account this inequality and the monotonicity of $x/(1 + h|x|)$ as a function of $x > 0$, we get from (13)

(14) $$L^{\cdot}_n(f; x) \leq \frac{(y_0 - \varepsilon/2)(1 - h(y_0 - \varepsilon/2))}{(1 - h(y_0 - \varepsilon/2))[1 - h(y_0 - \varepsilon/2) + h(y_0 - \varepsilon/2)]} = y_0 - \varepsilon/2,$$

for $|x_0 - x| \leq \min(\varepsilon/2, \delta/2)$.

But (14) obviously contradicts to the assumption that (x_0, y_0) is a boundary point of $L^{\cdot}_n$. Hence for sequences L_n, which satisfy (6), (7) implies (8). Thus proposition 4 is proved.

Let us note that if instead of (3) we use for defining e_h the relation $\varrho_h(A, B) = |x_A - x_B|/h + |y_A - y_B|$, then (5) obtains the form $\lim_{\tau\to 0}[(f_\tau \dot{-} f)/\tau] = \{f'(x)$ if $f' \leq h^{-1}$; h^{-1} if $f' > h^{-1}\}$. Also if we set $\varrho_h(A, B) = [(x_A - x_B)^2 + (y_A - y_B)^2]^{1/2}$ then formula (5) obtains the form $\lim_{\tau\to 0}[(f_\tau \dot{-} f)/\tau] = f'(x)/(1 + h^2 f'^2(x))^{1/2}$

4. Consider now the inverse transformation. Denote by S the class of all bounded Riemann integrable functions in Δ with $\sup_{x\in\Delta}|f(x)| \leq h^{-1}$.

D e f i n i t i o n. If $f \in S$ call the function $I(h, f)$ defined at $x \in \Delta$ by

(15) $$I(h, f; x) = \int_0^x \frac{f(t)}{1 - h|f(t)|}\, dt$$

the Riemann — H-integral of f (if the integral exists).

The following properties of this operator are obvious:
i) the H-integral of a step function f is a polygonal line; moreover if f has a finite number of jumps then $\boldsymbol{I}^{\cdot}(h, f) = \mathbf{f}$;
ii) If $f(x) \geq 0$ then $I(h, f, x) \geq 0$.
iii) If $f \leq c$ for $x \in \Delta$, then also $I^{\cdot}(h, f; x) \leqq c$. (Here obviously $I^{\cdot}(h, f; x)$ denotes the section of $I^{\cdot}(h, f)$ at x.)

Proposition 5. *The equality* $\boldsymbol{I}^{\cdot}(h, f) = \mathbf{f}(x)$ *holds if and only if* $\mathbf{I}_f = \mathbf{f} = \mathbf{S}_f$.

Proof. Suppose $\mathbf{I}_f = \mathbf{f} = \mathbf{S}_f$ holds. Let the step functions φ_n and ψ_n approximate f so that $\psi_n \leq f \leq \varphi_n$, $r_h(\psi_n, \varphi_n) < 1/r$, where r_h is the Hausdorff distance. Property iii) implies that $\psi_n \leq I^{\cdot}(h, f) \leq \varphi'_n$. Hence $r_h(f, I^{\cdot}(h, f)) \leq 1/n$, i. e. $\boldsymbol{I}^{\cdot}(h, f) = \mathbf{f}$.

Assume now that $\mathbf{I}_f \neq \mathbf{S}_f$, i. e. there exists a point (x_0, y_0) say from $\mathbf{S}_f$, and $\varepsilon > 0$ such that in the square with centre (x_0, y_0) and side ε there is no point from $\mathbf{I}_f$. From definition (15) it follows then that for a dense set in the interval $(x_0 - \varepsilon, x_0 + \varepsilon)$ we have $f(x) < y_0 - \varepsilon)$. Consider the function f_ε defined by

$$f_\varepsilon(x) = \begin{cases} f(x) & \text{for } x \bar{\in} (x_0 - \varepsilon, x_0 + \varepsilon) \\ f(x) & \text{for } x \in (x_0 - \varepsilon, x_0 + \varepsilon), \quad f(x) < y_0 - \varepsilon, \\ y_0 - \varepsilon & \text{for } x \in (x_0 - \varepsilon, x_0 + \varepsilon), \quad f(x) \geq y_0 - \varepsilon. \end{cases}$$

Obviously f and f_ε coincide almost everywhere in $[q, b]$ and hence by definition (15) we have $I(h, f_\varepsilon, x) = I(h, f, x)$. Because of $f_\varepsilon \leq y_0 - \varepsilon$ in $(x_0 - \varepsilon, x_0 + \varepsilon)$ and property iii) we have $\boldsymbol{I}^{\cdot}(h, f_\varepsilon, x) = \boldsymbol{I}^{\cdot}(h, f; x) \leq y_0 - \varepsilon$ in the interval $(x_0 - \varepsilon, x_0 + \varepsilon)$ and hence $\boldsymbol{I}^{\cdot}(h, f, x) = \mathbf{f}$ is not possible. This proves the theorem.

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Received 3. 12. 1974, Revised 16. 3. 1976