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CONFORMAL INVARIANTS OF A RIEMANNIAN MANIFOLD

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Some new conformal invariants for a n-dimensional Riemannian manifold are given.

Let the Riemannian manifold (M, g) of dimension n be transformed conformally into a Riemannian manifold $(\bar{M}, \bar{g})$. In the case $M \equiv M$ the differentiable manifold M is provided with two Riemannian metrics g and $\bar{g}$ [2]. We have $\bar{g} = e^{2\sigma}g$. Consider the following statements:

1. *If C is the tensor of conformal curvature for (M, g), then $\bar{C} = e^{2\sigma}C$.*

2. *If $\lambda, \mu > 1, \nu > 1$ are integers with $\mu + \nu = \lambda \leq n$ and the tangent space E_λ is the direct sum of the orthogonal tangent subspaces E_μ, E_ν, then*

$$e^{2\sigma}\left[\frac{\bar{K}(E_\lambda)}{\lambda - 1} - \frac{\bar{K}(E_\mu)}{\mu - 1} - \frac{\bar{K}(E_\nu)}{\nu - 1}\right] = \frac{K(E_\lambda)}{\lambda - 1} - \frac{K(E_\mu)}{\mu - 1} - \frac{K(E_\nu)}{\nu - 1}.$$

Here $K(E_\lambda)$ is the curvature of the λ-dimensional subspace E_λ at a point p of (M, g) [1].

3. *If $\lambda, \lambda_1 > 1 \ldots, \lambda_k > 1$ are integers with $\lambda_1 + \ldots + \lambda_k = \lambda \leqq n$ and the tangent subspace E_λ is the direct sum of the orthogonal tangent subspaces $E_{\lambda_1}, \ldots, E_{\lambda_k}$ then*

$$e^{2\sigma}\left[\frac{\bar{K}(E_\lambda)}{\lambda - 1} - \sum_{i=1}^{k} \frac{\bar{K}(E_{\lambda_i})}{\lambda_i - 1}\right] = \frac{K(E_\lambda)}{\lambda - 1} - \sum_{i=1}^{k} \frac{K(E_{\lambda_i})}{\lambda_i - 1}.$$

4. *If $2 \leq \lambda \leq n - 2$ and $\perp E_\lambda$ is the orthogonal complement of the tangent subspace E_λ, then*

$$e^{2\sigma}\left[\frac{\bar{S}(p)}{n - 1} - \frac{\bar{K}(E_\lambda)}{\lambda - 1} - \frac{\bar{K}(\perp E_\lambda)}{n - \lambda - 1}\right] = \frac{S(p)}{n - 1} - \frac{K(E_\lambda)}{\lambda - 1} - \frac{K(\perp E_\lambda)}{n - \lambda - 1}.$$

5. *If $2 \leq \lambda \leq n - 2$ then*

$$e^{2\sigma}\left[(n - 2)\bar{K}(E_\lambda) - (\lambda - 1)\sum_{i=1}^{\lambda} \bar{\varrho}_i + \frac{\lambda(\lambda - 1)}{n - 1}\bar{S}(p)\right]$$

$$= (n - 2)K(E_\lambda) - (\lambda - 1)\sum_{i=1}^{\lambda} \varrho_i + \frac{\lambda(\lambda - 1)}{n - 1} S(p)$$

where $\varrho_1 + \cdots + \varrho_\lambda$ is the sum of the Ricci-curvatures of the λ-orthogonal directions in E_λ.

The main result in this paper is the following

Theorem. *The statements 1—5 are equivalent*

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We shall prove only the implication 3→4. Indeed it follows that the expression

$$\frac{1}{\lambda-1} K_{1,2,\ldots,\lambda_1, i_1, i_2, \ldots, i_{\lambda_2}, \ldots, i_{\lambda_2+\ldots+\lambda_{k-1}+1}, \ldots, i_{\lambda_2+\ldots+\lambda_k}}$$

$$-\frac{1}{\lambda_1-1} K_{1,2,\ldots,\lambda_1} - \frac{1}{\lambda_2-1} K_{i_1, i_2, \ldots, i_{\lambda_2}} - \cdots$$

$$-\frac{1}{\lambda_k-1} K_{i_{\lambda_2+\ldots+\lambda_{k-1}+1}, \ldots, i_{\lambda_2+\ldots+\lambda_k}},$$

where $i_1, i_2, \ldots, i_{\lambda_2+\ldots+\lambda_k}$ are among the numbers $\lambda_1+1, \lambda_1+2, \ldots, \lambda_1+\lambda_2+\cdots+\lambda_k, \ldots, n$ and are mutually different, is a "relative invariant". Adding all such possible "relative invariants" we get the "relative invariant"

$$\left(\frac{1}{\lambda-1}-\frac{1}{\lambda_1-1}\right)\binom{n-\lambda_1}{\lambda_2+\cdots+\lambda_k}\binom{\lambda_2+\cdots+\lambda_k}{\lambda_2}\cdots\binom{\lambda_{k-1}+\lambda_k}{\lambda_{k-1}} K_{1,2,\ldots,\lambda_1}$$

$$+\frac{1}{\lambda-1}\binom{n-\lambda_1-1}{\lambda_2+\cdots+\lambda_k-1}\binom{\lambda_2+\cdots+\lambda_k}{\lambda_2}\cdots\binom{\lambda_{k-1}+\lambda_k}{\lambda_{k-1}} K_{1,2,\ldots\lambda_1;\lambda_1+1,\ldots,n}$$

$$+\frac{1}{\lambda-1}\binom{n-\lambda_1-2}{\lambda_2+\cdots+\lambda_k-2}\binom{\lambda_2+\cdots+\lambda_k}{\lambda_2}\cdots\binom{\lambda_{k-1}+\lambda_k}{\lambda_{k-1}} K_{\lambda_1+1,\ldots,n}$$

$$-\frac{1}{\lambda_2-1}\binom{n-\lambda_1-2}{\lambda_2-2}\binom{n-\lambda_1-\lambda_2}{\lambda_3}\cdots\binom{n-\lambda_1-\lambda_2-\cdots-\lambda_{k-1}}{\lambda_k} K_{\lambda_1+1,\ldots,n}$$

$$-\frac{1}{\lambda_3-1}\binom{n-\lambda_1-2}{\lambda_3-2}\binom{n-\lambda_1-\lambda_3}{\lambda_2}\cdots\binom{n-\lambda_1-\lambda_2-\cdots-\lambda_{k-1}}{\lambda_k} K_{\lambda_1+1,\ldots,n}-\cdots,$$

which is a product of a factor, different from zero and of the expression

$$\frac{S(p)}{n-1}-\frac{K_{1,2,\ldots\lambda_1}}{\lambda_1-1}-\frac{K_{\lambda_1+1,\ldots,n}}{n-\lambda_1-1}.$$

From these results we get the following necessary and sufficient conditions for a conformal flat space ($n \geqq 4$):

2′. $\dfrac{K(E_\lambda)}{\lambda-1}=\dfrac{K(E_\mu)}{\mu-1}+\dfrac{K(E_\nu)}{\nu-1}$,

3′. $\dfrac{K(E_\lambda)}{\lambda-1}=\sum\limits_{i=1}^{k}\dfrac{K(E_{\lambda_i})}{\lambda_i-1}$,

4′. $\dfrac{S(p)}{n-1}=\dfrac{K(E_\lambda)}{\lambda-1}+\dfrac{K(\perp E_\lambda)}{n-\lambda-1}$,

5′. $\dfrac{\lambda(\lambda-1)}{n-1}S(p)=(\lambda-1)\sum\limits_{i=1}^{\lambda}\varrho_i-(n-2)K(E_\lambda)$.

The condition 3′ in the case $\lambda=n$ was established by J. Haantjes [3] and some special cases of his condition — by W. Wrona [4].

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