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DETERMINATION BY NOMOGRAMS OF THE ZEROS OF A CROSS PRODUCT OF DERIVATIVES OF BESSEL FUNCTIONS

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The method given in L. Salchev & V. Popov (1973) is applied to the determination by nomograms of the zeros of the equation $I'_\nu(\alpha\beta)\,Y'_\nu(\alpha)-I'_\nu(\alpha)\,Y'_\nu(\alpha\beta)=0$ for real ν and $\beta>0$.

Several papers [1, 2, 3, 4, 5, 6] deal with questions concerning the zeros of the equation

$$I'_\nu(\alpha\beta)Y'_\nu(\alpha)-I'_\nu(\alpha)Y'_\nu(\alpha\beta)=0, \tag{1}$$

where $I'_\nu(\alpha)$ and $Y'_\nu(\alpha)$ are the first derivatives with respect to α of the Bessel functions of the first and second kind ($\beta>0$). In some of these publications [1, 2, 7, 8] are given tables or graphs for finding the zeros of the equation (1) for several positive integer values of ν and for different values of the parameter β. The most extended table can be found in F. Bridge and S. Angrist [8]. J. McMahon [9] gives an asymptotic expression for the zeros of the equation (1). V. Smorgonski and V. Ilarionov [10] give some bounds for them depending on the zeros of the Bessel functions.

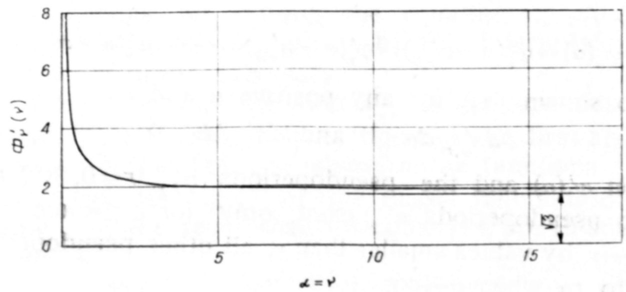

Fig. 1

It is shown in [11] that the solution of the equation (1) for any non-negative values of ν and positive β can be treated in the same way like that used in [12] for solving a cross product equation containing Bessel functions. In order to make this paper selfconsistent let us recall the results of [11]:

a) The function

(2) $$\varphi'_\nu(\alpha) = Y'_\nu(\alpha)/I'_\nu(\alpha), \quad \nu > 0$$

is introduced. Then (1) becomes

(3) $$\varphi'_\nu(\alpha) = \varphi'_\nu(\alpha\beta).$$

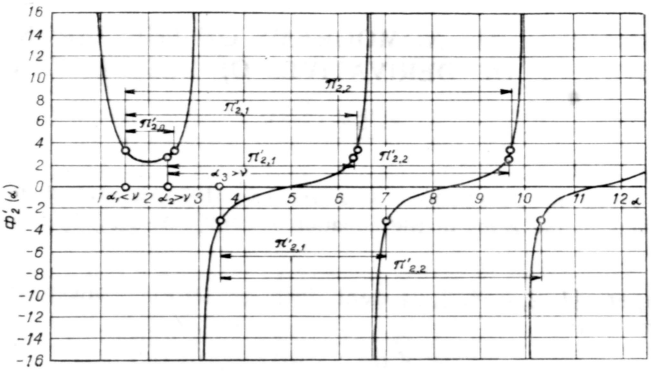

Fig. 2

It is established that $\varphi'_\nu(\alpha)$ is a continuous decreasing function of α for $0 < \alpha < \nu$ and a continuous increasing function of α for $\alpha > \nu$ and has a single relative minimum at $\alpha = \nu$ as $d\varphi'_\nu(\alpha)/d\alpha = 0$ and $d^2\varphi'_\nu(\alpha)/d\alpha^2 = 4\pi\alpha^2 I'^2_\nu(\alpha) > 0$ at $\alpha = \nu$. This minimum as a function of ν is shown in Fig. 1.

The function $\varphi'_\nu(\alpha)$ is pseudoperiodical in the sense of [12], i. e. there exist numbers $\pi'_{\nu,r}$, $r = 0, 1, \ldots$, called pseudoperiods of order r, such that we have

(4) $$\varphi'_\nu(\alpha) = \varphi'_\nu(\alpha + \pi'_{\nu,0}) = \varphi'_\nu(\alpha + \pi'_{\nu,1}) = \cdots = \varphi'_\nu(\alpha + \pi'_{\nu,r}).$$

b) It is also shown that for any positive α and ν

(5) $$d\pi'_{\nu,r}/d\alpha < 0 \text{ and } d\pi'_{\nu,r}/d\nu > 0.$$

The function $\varphi'_2(\alpha)$ and the pseudoperiods $\pi'_{2,r}$, $r = 0, 1, 2$ are shown in Fig. 2. The zero pseudoperiods $\pi'_{\nu,0}$ exist only for $\alpha < \nu$ and while they go to zero when $\alpha \to \nu$ by values smaller than ν, all other pseudoperiods $\pi'_{\nu,r} > r\pi$, $r = 1, 2, \ldots$ go to $r\pi$ when $\alpha \to \infty$.

Proposition. *The pseudoperiods* $\pi'_{-\nu,r}$ *of the functions* $\varphi'_{-\nu}(\alpha)$ *are equal to the pseudoperiods* $\pi'_{\nu,r}$ *of the functions* $\varphi'_\nu(\alpha)$.

The proof of this proposition is a replication of the proof given in [12].

According to (3) and using the pseudoperiodical properties of the functions $\varphi'_\nu(\alpha)$ given by (4), we can write that

(6) $$\alpha|\beta - 1| = \pi'_{\nu,r}.$$

Then for the smaller argument $\bar{\alpha}_r$ (equal to α_r if $\beta > 1$ or $\beta\alpha_r$ if $\beta < 1$) of the function $\varphi'_\nu(\alpha)$ from equation (6) we get

(7) $$\bar{a}_r = \pi'_{\nu,r}/(\beta - 1) \quad \text{for} \quad \beta > 1$$
or
(7′) $$\bar{a}_r = \pi'_{\nu,r}/(\beta^{-1} - 1) \quad \text{for} \quad \beta < 1.$$

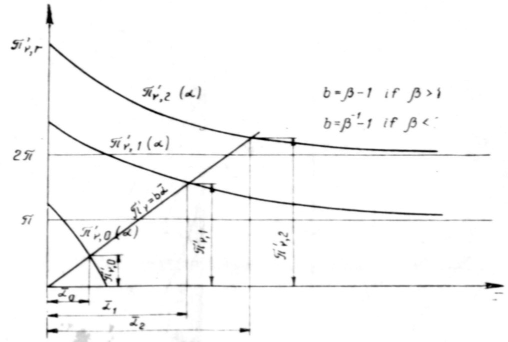

Fig. 3

From equations (7) and (7′) it follows that the abscissa of the cross-point of the curve $\pi'_{\nu,r} = \pi'_{\nu,r}(\alpha)$ with the line $\pi'_{\nu,r} = (\beta - 1)\bar{\alpha}$ for $\beta > 1$, or with the line $\pi'_{\nu,r} = (\beta^{-1} - 1)\bar{\alpha}$ for $\beta < 1$ is the value of the smaller argument $\bar{\alpha}_r$ (α_r or $\alpha_r\beta$) in the equation (1) (Fig. 3).

Let us note also that the pseudoperiods $\pi'_{\nu,r}$ in (6) are the ordinates of the same cross points (Fig. 3).

If we dispose with a sufficient number of curves of the kind $\pi'_{\nu,r} = \pi'_{\nu,r}(\alpha)$ and straight lines $\pi'_{\nu,r} = b\alpha$, where $b = \beta - 1$ if $\beta > 1$ or $b = \beta^{-1} - 1$ if $\beta < 1$, we can construct nomograms for solving the equation (1). According to the properties of the pseudoperiods $\pi'_{\nu,r}$ given by (5), interpolation can be used in such nomograms whenever necessary.

In order to draw curves of the kind $\pi'_{\nu,r} = \pi'_{\nu,r}(\alpha)$ one has to use tables of the first derivatives of the Bessel functions of the first and second kind according to equations (3) and (4). As tables of the functions $I'_\nu(\alpha)$ and $Y'_\nu(\alpha)$ are scarce [13] one can use tables of the Bessel functions of the first and second kind. Indeed, by the recurrence formulae [14, pp. 45 and 66] equation (3) can be written as

(8) $$\frac{Y_{\nu+1}(\alpha) - Y_{\nu-1}(\alpha)}{I_{\nu+1}(\alpha) - I_{\nu-1}(\alpha)} = \frac{Y_{\nu+1}(\alpha\beta) - Y_{\nu-1}(\alpha\beta)}{I_{\nu+1}(\alpha\beta) - I_{\nu-1}(\alpha\beta)}.$$

As tables of the Bessel functions $Y_\nu(\alpha)$ for ν not an integer are also scarce [14], when ν is not an integer it is convenient to use the recurrence formulae [14, p. 4] and the fact established in Proposition 1, namely that the functions $Y'_\nu(\alpha)/I'_\nu(\alpha)$ and $I'_{-\nu}(\alpha)/I'_\nu(\alpha)$ have one and the same pseudoperiod $\pi'_{\nu,r}$. The necessary pseudoperiods can be determined from (4) and

(9) $$\frac{I_{-\nu-1}(\alpha) - I_{-\nu+1}(\alpha)}{I_{\nu-1}(\alpha) - I_{\nu+1}(\alpha)} = \frac{I_{-\nu-1}(\alpha\beta) - I_{-\nu+1}(\alpha\beta)}{I_{\nu-1}(\alpha\beta) - I_{\nu+1}(\alpha\beta)}$$

or from (4) and

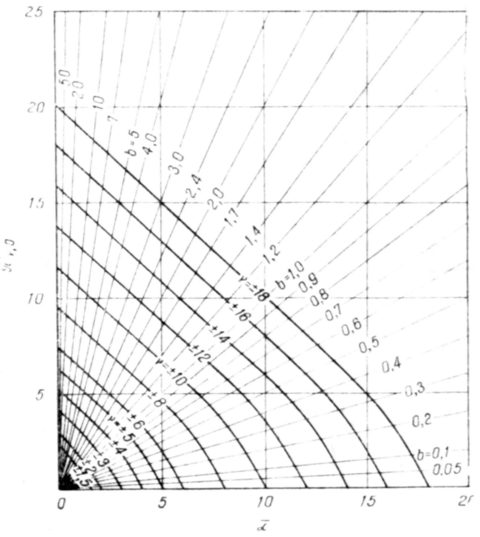

Fig. 4

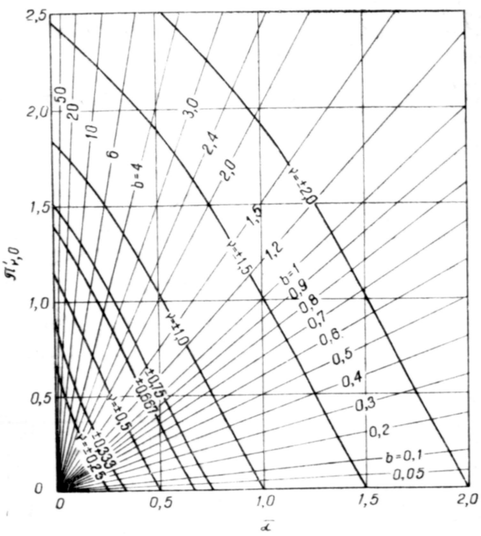

Fig. 5

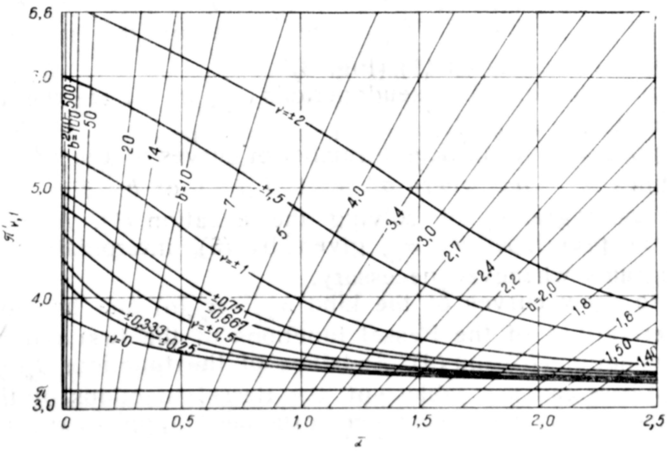

Fig. 6

$$\frac{\nu I_{-\nu}(\alpha)/\alpha + I_{-\nu+1}(\alpha)}{\nu I_{\nu}(\alpha)/\alpha - I_{\nu-1}(\alpha)} = \frac{\nu I_{-\nu}(\alpha\beta)/\alpha\beta + I_{-\nu+1}(\alpha\beta)}{\nu I_{\nu}(\alpha\beta)/\alpha\beta - I_{\nu-1}(\alpha\beta)} \tag{10}$$

instead of using (4), (3) or (8).

It is clear that all the values α_r or $\pi'_{\nu,r}$ satisfying the equation (3) would satisfy equations (8), (9) and (10) as well. Then using the available tables of Bessel functions [13, 14, 15, 16, 17] and the equation (3), (4), (9) and (10), values of the pseudoperiods $\pi'_{\nu,r}$ can be found and graphs of the

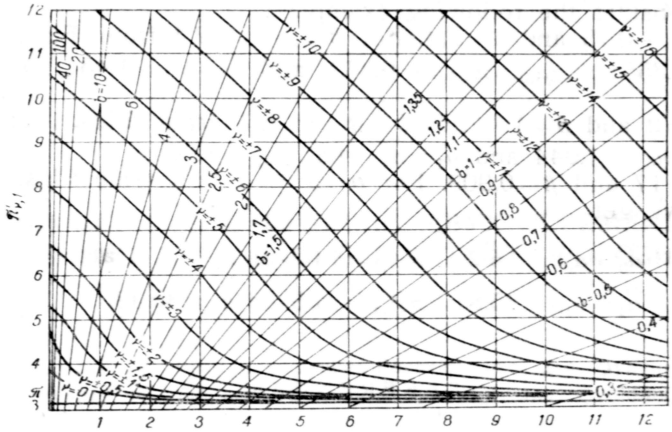

Fig. 7

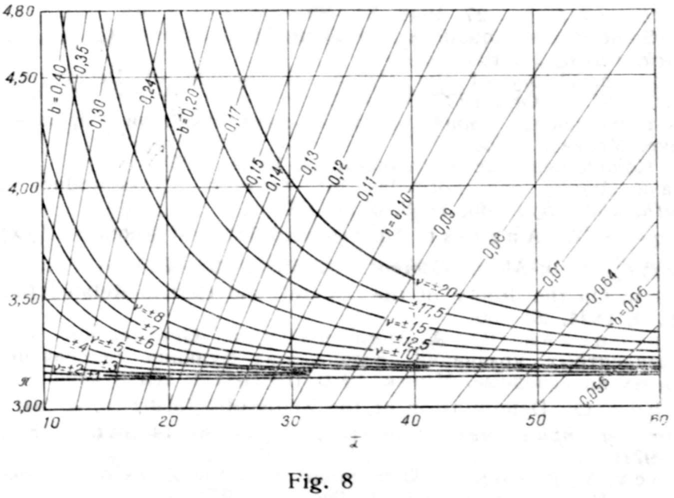

Fig. 8

curves $\pi'_{\nu,r}=\pi'_{\nu,r}(\alpha)$ can be drawn. They are shown in Figs. 4, 5, 6 and 7 For big values of α the graphs of the pseudoperiods $\pi'_{\nu,r}$ can be drawn using tables of the zeros of the functions $I'_\nu(\alpha)$ and $Y'_\nu(\alpha)$ [18], as evidently in this case we have

(11) $$\pi'_{\nu,r}(j'_{\nu,s})=j'_{\nu,s+r}-j'_{\nu,s}$$

as well as

(12) $$\pi'_{\nu,r}(y'_{\nu,s})=y'_{\nu,s+r}-y'_{\nu,s}\,.$$

These graphs are shown in Fig. 8.

Finally consider the following examples:

a) Let $\nu=-0.75$ and $\beta=3.4$. For $b=\beta-1=2.4$ from Figs. 1 and 3 we get $\bar{\alpha}_0=\alpha_0=0.355$; $\pi_{-3/4,0}=\pi_{3/4,0}=0.852$ and $\bar{\alpha}_1=\alpha_1=1.45$; $\pi_{-3/4,1}=\pi_{3/4,1}=3.48$.

b) Let $\nu=2$ and $\beta=1.7$. For $b=\beta-1=0.7$ from Figs. 1 and 4 we get $\bar{\alpha}_0=\alpha_0=1.49$; $\pi_{3,0}=1.044$ and $\bar{\alpha}_1=\alpha_1=4.8$; $\pi_{3,1}=3.36$.

c) Let $\nu=17$ and $\beta=0.909$. For $b=\beta^{-1}-1=0.1$ from Figs. 2 and 1 by interpolation we get

$$\bar{\alpha}_0=\alpha_0\beta=16.2;\quad \pi_{17,1}=1.62 \text{ or } \alpha_0=17.8 \text{ and}$$

$$\bar{\alpha}_1=\alpha_1\beta=35.8;\quad \pi_{17,1}=3.58 \text{ or } \alpha_1=39.38.$$

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