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ENFLO-ROSENTHAL BASIS SETS IN BANACH SPACES

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We define coefficient functionals and partial sum operators for the ER-basis sets introduced by Enflo and Rosenthal. We show that the coefficient functionals are continuous and if the ER-basis set is normalized then they are also uniformly bounded. Some usual non-separable Banach spaces have no ER-basis set. The partial sum operators need not be uniformly bounded, but have a weaker uniform boundedness property. We give a characterization of reflexivity of Banach spaces with an ER-basis set.

1. Introduction. The problem of extension of the notion of a basis $\{x_n\}$ of a separable Banach space E to the case when E is not assumed to be separable, replacing the sequence $\{x_n\}$ by a family $\{x_i\}_{i\in I}$ which is not assumed countable, presents the main difficulty of finding for this case a suitable generalization of conditional convergence of series. Recently P. Enflo and H. P. Rosenthal [1] have suggested to overcome this difficulty in the following natural way, which they feel is perhaps the "correct" generalization of the notion of a basis to spaces which need not be separable:

Definition 1. *A family $\{x_i\}_{i\in I}$ of elements in a Banach space E is called an ER-basis set for E if: (a) $E=[x_i]_{i\in I}$, the closed linear span of $\{x_i\}_{i\in I}$ and (b) every countable subfamily of $\{x_i\}_{i\in I}$ has one ordering under which it is a basic sequence (i. e., a basis of its closed linear span).*

Enflo and Rosenthal [1] have used the term Schauder basis set, but we prefer the above terminology (in order to avoid confusion between "Schauder basis" and "Schauder basis set" when $\{x_i\}_{i\in I}$ is countable); here, of course, ER stands for "Enflo-Rosenthal".

In [1] it was observed that if $\{x_i\}_{i\in I}$ is an ER-basis set for a Banach space E, then for every separable subspace G of E there exists a separable subspace F of E with $F\supset G$, such that some countable subfamily $\{x_i\}_{i\in I_0}$ of $\{x_i\}_{i\in I}$ has an ordering under which it is a basis of F. Hence, in particular, $\{x_i\}_{i\in I}$ is an ER-basis set for a separable Banach space E iff $\{x_i\}_{i\in I}$ is countable and has an ordering under which it is a basis of E; thus, a separable Banach space has an ER-basis set if and only if it has a basis.

It is natural to ask to what extent the known properties of bases remain valid for ER-basis sets. In 2 of the present paper we shall show that one can define coefficient functionals for ER-basis sets and that these functionals are still continuous; moreover if the ER-basis set is normalized, then they are also uniformly bounded. We shall also obtain that many usual non-separable Banach spaces have no ER-basis set. In 3 we shall define partial sum operators for ER-basis sets and we shall show that in general they need not be uniformly bounded and need not have even a certain weaker property, but for each countable subset of an ER-basis set one can select an ordering of that subset which makes it a basic sequence, in such a way that the norms of these basic sequences are uniformly bounded. Finally, in **4** we shall define shrinking and boundedly complete ER-basis sets and use them to give a characterization of reflexivity of Banach spaces with ER-basis sets which generalizes the characterization given by R. C. James [3] for separable spaces with usual bases.

The proofs of the above results are not standard (since the methods of the separable case cannot be extended for non-separable spaces) and actually make use of the corresponding results for separable spaces.

The terminology and notations will be the usual ones, see e. g. [7].

Finally, we wish to express our thanks to T. Figiel (Warsaw) for reading the manuscript and making valuable remarks which have simplified some parts of this paper.

2. The coefficient functionals. Non-separable spaces which have no ER-basis set. Let $\{x_i\}_{i\in I}$ be an ER-basis set for a Banach space E. Then, clearly, every finite subset of $\{x_i\}_{i\in I}$ is linearly independent and hence we can define a family of linear functionals $\{f_i\}_{i\in I}$ on $\operatorname{lin}\{x_i\}_{i\in I}$ (the linear span of $\{x_i\}_{i\in I}$) by

$$f_i\Big(\sum_{k=1}^{n}\alpha_{i_k}x_{i_k}\Big)=\begin{cases}\alpha_i \text{ if } i\in\{i_k\}_{k=1}^{n}\\ 0 \text{ if } i\in I\setminus\{i_k\}_{k=1}^{n}\end{cases}\quad\Big(\sum_{k=1}^{n}\alpha_{i_k}x_{i_k}\in\operatorname{lin}\{x_i\}_{i\in I}\Big). \tag{1}$$

Theorem 1. *Let $\{x_i\}_{i\in I}$ be an ER-basis set for a Banach space E. Then the functionals f_i defined by* (1) *are continuous on $\operatorname{lin}\{x_i\}_{i\in I}$ and hence they admit unique extensions to $f_i\in F^*(i\in I)$. Moreover, there exists a constant M such that*

$$1\leq\|x_i\|\,\|f_i\|\leq M\quad(i\in I). \tag{2}$$

Proof. We may assume (replacing (x_i, f_i) by $(x_i/\|x_i\|,\ \|x_i\|f_i)$), that $\|x_i\|=1\ (i\in I)$. We shall show that there exists a constant M such that

$$|f_i(x)|\leq M\|x\|\quad(x\in\operatorname{lin}\{x_i\}_{i\in I},\ i\in I), \tag{3}$$

which will complete the proof. Assume that (3) does not hold. Then there exist a sequence $\{i_n\}\subset I$ and a sequence $\{y_n\}\subset\operatorname{lin}\{x_i\}_{i\in I}$ with $\|y_n\|=1$ $(n=1, 2, \ldots)$ such that

$$\lim_{n\to\infty}f_{i_n}(y_n)=\infty. \tag{4}$$

Since $[x_{i_n}, y_n]$ is a separable subspace of E, there exists, by the remark of Enflo and Rosenthal mentioned in the introduction, a countable subfamily $\{x_i\}_{i\in I_0}$ of $\{x_i\}_{i\in I}$ such that $[x_{i_n}, y_n]\subset[x_i]_{i\in I_0}$ and that $\{x_i\}_{i\in I_0}$ has an ordering under which it is a basic sequence. Let $\{\varphi_i\}_{i\in I_0}\subset([x_i]_{i\in I_0})^*$, $\varphi_i(x_j)=\delta_{ij}(i, j\in I_0)$. Then, by $\{i_n\}\subset I_0$ and a result of Banach, see e. g. [7, Ch. I, § 3, theorem 3.1],

$$\sup_n|f_{i_n}(y_n)|=\sup_n|\varphi_{i_n}(y_n)|\leq\sup_n\|\varphi_n\|\leq\sup_{i\in I_0}\|\varphi_i\|=\sup_{i\in I_0}\|x_i\|\,\|\varphi_i\|<\infty,$$

which contradicts (4). Thus, we have (3), which completes the proof.

Now we can give

Definition 2. *Let* $\{x_i\}_{i\in I}$ *be an ER-basis set for a Banach space* E. *The family* $\{f_i\}_{i\in I}\subset E^*$ *of theorem 1 above is called the family of coefficient functionals associated to the BR-basis set* $\{x_i\}_{i\in I}$.

The term "coefficient functionals" is motivated by formula (1) and by

Corollary 1. *Let* $\{x_i\}_{i\in I}$ *be an ER-basis set for a Banach space* E *and let* $\{f_i\}_{i\in I}\subset E^*$ *be the associated family of coefficient functionals. Then for each* $x\in E$ *there exists a unique family of scalars* $\{\alpha_i\}_{i\in I}$ *such that the set* $S=\{i\in I\mid \alpha_i\neq 0\}$ *is countable and has an ordering* $\{i_n\}$ *such that*

$$x=\sum_{n=1}^{\infty}\alpha_{i_n}x_{i_n}; \tag{5}$$

namely

$$\alpha_i=f_i(x)\quad (i\in I). \tag{6}$$

Proof. Let $x\in E$. Since $[x_i]_{i\in I}=E$, there exists a countable subset I_0 of E such that $x\in[x_i]_{i\in I}$. Then $\{x_i\}_{i\in I_0}$ is a basic sequence under some ordering $\{i_0^n\}$ of I_0, so there is a sequence of scalars $\{\alpha_{i_n^0}\}$ such that

$$x=\sum_{n=1}^{\infty}\alpha_{i_n^0}x_{i_n^0}. \tag{7}$$

Define now

$$\alpha_i=0\quad (i\in I\setminus I_0). \tag{8}$$

We shall show that $\{\alpha_i\}_{i\in I}$ has the required properties. Indeed, by (8), $S=\{i\in I\mid \alpha_i\neq 0\}\subset I_0$ (so S is countable) and hence, by (7), we have (5) under the ordering $\{i_n\}$ of S induced by $\{i_n^0\}$. Finally, by $f_i\in E^*$, (7), (8) and the biorthogonality of $(x_i,f_i)_{i\in I}$ we have (6), which completes the proof of corollary 1.

We recall that a family $\{x_i\}_{i\in I}$ of elements in a Banach space E is called an M-basis of E if $[x_i]_{i\in I}=E$ and if there exists a (clearly unique) total family $\{f_i\}_{i\in I}\subset E^*$ such that $(x_i, f_i)_{i\in I}$ is biorthogonal (i. e., $f_i(x_j)=\delta_{ij}$ for $i,j\in I$).

Corollary 2. *Every ER-basis set* $\{x_i\}_{i\in I}$ *for a Banach space* E *is an M-basis of* E.

Proof. Let $\{f_i\}_{i\in I}\subset E^*$ be the family of coefficient functionals associated to the ER-basis set $\{x_i\}_{i\in I}$. Then $(x_i, f_i)_{i\in I}$ is biorthogonal and thus, since $[x_i]_{i\in I}=E$, it remains to prove that $\{f_i\}_{i\in I}$ is total on E. Let $x\in E$, $f_i(x)=0$ $(i\in I)$. Then, since $[x_i]_{i\in I}=E$, there exists a countable subset I_0 of E such that $x\in[x_i]_{i\in I_0}$. Also, there exists an ordering $\{i_n\}$ of I_0 such that $\{x_{i_n}\}$ is a basic sequence. Let $\{\varphi_i\}_{i\in I_0}\subset([x_i]_{i\in I_0})^*$, $\varphi_i(x_j)=\delta_{ij}$ $(i,j\in I_0)$. Then $x=\sum_{n=1}^{\infty}\varphi_{i_n}(x)x_{i_n}=\sum_{n=1}^{\infty}f_{i_n}(x)x_{i_n}=0$ (since $\varphi_{i_n}(x)=f_{i_n}(x)$ by corollary 1), which completes the proof.

As a consequence, we obtain that many usual non-separable Banach spaces have no ER-basis set. We recall that a Banach space E is called a Grothendieck space if every $\sigma(E^*,E)$-convergent sequence $\{g_n\}$ in E^* is $\sigma(E^*,E^{**})$-convergent. For example, it is known that $E=l^\infty$ is a non-reflexive Grothendieck space [2] (it is clear that every reflexive space is a Grothendieck space).

Corollary 3. *No non-reflexive Grothendieck space E admits an ER-basis set.*

Proof. By a theorem of W. B. Johnson [4], no non-reflexive Grothendieck space admits an M-basis. Hence the conclusion follows from corollary 2.

3. The partial sum operators. Definition 3. *Let $\{x_i\}_{i\in I}$ be an ER-basis set for a Banach space E, with coefficient functionals $\{f_i\}_{i\in I}\subset E^*$. The family of operators $\{s_{\{i_k\},n}\}$ on E defined by*

$$s_{\{i_k\},n}(x)=\sum_{k=1}^{n} f_{i_k}(x)x_{i_k} \quad (x\in E), \tag{9}$$

where $\{i_k\}$ is an arbitrary ordering of the countable set $S=\{i\in I \mid f_i(x)\neq 0\}$ as in corollary 1 and where $n=1, 2, \ldots$, is called the family of partial sum operators associated to the ER-basis set $\{x_i\}_{i\in I}$.

By theorem 1, these partial sum operators are continuous on E, so it is natural to ask whether they are uniformly bounded on E. In general, even the answer to the question whether the partial sum operators have the following considerably weaker property, is also negative: Is it possible to select, for each countable subset I_0 of I, an ordering $\{i_k^0\}$ of I_0 such that $\sup_n \|s_{\{i_k^0\},n}\|\leq M(\{i_k^0\})<\infty$? (here $\sup_{I_0}\sup_{\{i_k^0\}} M(\{i_k^0\})=\infty$ is also permitted) Indeed, if $I=\{1, 2, 3, \ldots\}$ and if E is reflexive, this property implies that for every infinite subset I_0 of I we have $[x_i]_{i\in I_0}\oplus[x_j]_{j\in I\setminus I_0}=E$ (e. g. by [5, theorem 8], and hence $\{x_i\}_{i\in I}$ is an unconditional basis of E (by [6]).

However, we shall prove now the following positive result:

Theorem 2. *Let $\{x_i\}_{i\in I}$ be an ER-basis set for a Banach space E. Then for each countable subset I_0 of I one can select an ordering $\{i_k^0\}$ of I_0 such that the corresponding family $\bigcup_{I_0}\{s_{\{i_k^0\},n}\mid [x_i]_{i\in I_0}\}_{n=1}^{\infty}$ is uniformly bounded, i. e. such that*

$$\sup_{I_0} \nu_{\{x_{i_k^0}\}}<\infty, \tag{10}$$

where the sup *is taken over all countable subsets I_0 of I and where $\nu\{x_{i_k^0}\}=\sup_n \|s_{\{i_k^0\},n}\mid [x_i]_{i\in I_0}\|$.*

Proof. Choose, for each countable subset I_0 of I, an ordering $\{i_k^0\}$ of I_0 such that

$$\nu_{\{x_{i_k^0}\}}\leq \inf_{\{i_k\}} \nu_{\{x_{i_k}\}}+1=\nu(I_0)+1, \tag{11}$$

where the inf is taken over all orderings $\{i_k\}$ of I_0 such that $\{x_{i_k}\}$ is a basic sequence. We shall show that this selection satisfies (10). Indeed, if not, then $\sup_{I_0}\nu(I_0)=\infty$, which will lead to a contradiction. Observe that if I_1, I_2 are countable subsets of I with $I_1\subset I_2$, then $\nu(I_1)\leq\nu(I_2)$, since if $\{i_k^2\}$ is an ordering of I_2 such that $\{x_{i_k^2}\}$ is a basis of $[x_i]_{i\in I_2}$, then the subsequence $\{x_{i_k^2}\}_{i_k^2 \in I_1}$ is a basis of $[x_i]_{i \in I_1}$, see e. g. [7, Ch. I, § 4], satisfying, clearly, $\nu_{\{x_{i_k^2}\}_{i_k^2 \in I_1}} \leqq \nu_{\{x_{i_k^2}\}}$. Now, by our assumption, there exists a sequence $\{I_n\}$ of countable subsets of I such that $\lim_{n\to\infty} \nu(I_n)=\infty$. Let $I_0=\bigcup_{n=1}^{\infty} I_n$. Then $I_0 \supset I_n$, whence, by the preceding observation, $\nu(I_0)\geqq\nu(I_n)$ $(n=1, 2, \ldots)$, so $\nu(I_0)=\infty$, which contradicts the assumption that $\{x_i\}_{i\in I}$ is an ER-basis set for E. This completes the proof.

4. Reflexivity of Banach spaces with an ER-basis set. One can give the following natural generaliazations of shrinking and boundedly complete bases:

Definition 3. *An ER-basis set $\{x_i\}_{i\in I}$ for a Banach space E is said to be shrinking (respectively, boundedly complete), if every countable subfamily of $\{x_i\}_{i\in I}$ has an ordering under which it is a shrinking (respectively, a boundedly complete) basic sequence.*

Theorem 3. *A Banach space E with an ER-basis set $\{x_i\}_{i\in I}$ is reflexive if and only if $\{x_i\}_{i\in I}$ is both shrinking and boundedly complete.*

Proof. Since every subspace of a reflexive space is reflexive, the necessity is a consequence of James's theorem [3].

Conversely, assume that the condition is satisfied and let I_0 be a countable subset of I. Then there exists an ordering $\{i_n^0\}$ of I_0 such that $\{x_{i_n^0}\}$ is a shrinking basic sequence and another ordering $\{i_n\}$ of $I_0=\{i_n^0\}$ such that $\{x_{i_n}\}$ is a boundedly complete basic sequence. Clearly, $\{x_{i_n}\}$ is also shrinking (since so is $\{x_{i_n^0}\}$ and since $\{\varphi_{i_n}\}$ is a reordering of $\{\varphi_{i_n^0}\}$, where

$$\{\varphi_i\}_{i\in I_0}\subset([x_i]_{i\in I_0})^*, \quad \varphi_i(x_j)=\delta_{ij} \text{ for } i, j \in I_0)$$

and hence, by James's theorem [3], the subspace $[x_i]_{i\in I_0}$ is reflexive. But by the remark of Enflo and Rosenthal mentioned in the Introduction, every separable subspace G of E is contained in a subspace of E of the form $F=[x_i]_{i\in I_0}$, with $I_0\subset I$ countable. Consequently, every separable subspace of E is reflexive and hence, by Eberlein's theorem, E is reflexive, which completes the proof.

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