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SUPERNILPOTENT AND UNDERIDEMPOTENT RADICAL CLOSURES OF RIGHT IDEALS

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The concept of radical closure was introduced for the first time by V. A. Andrunakievič and Yu. M. Riabuchin (1974). Their aim was to get some results concerning the structure of the right ideals in a given associative ring, similar to the concept of radical. We shall show that the concepts: supernilpotent and underidempotent, special radical can be defined for radical closures' right ideals. The results are similar to those of V. A. Andrunakievič (1958).

1. Preliminary remarks. Let R be an associative algebra (not necessarily with a unit) over a commutative ring Φ with unit. We shall denote by $\mathfrak{P}(R)$ the set of all right ideals of R, and with $\mathfrak{U}(R)$ — the class of all pairs (A, B), where $A, B\in\mathfrak{P}(R)$ and $A\supset B$. If $(A, B)\in\mathfrak{U}(R)$ and C is a right ideal of R, for which $B\subset C\subset A$, we shall call the pair (C, B) a subpair of (A, B) and the pair (A, C) a factor-pair of (A, B). A pair (A, B) is called a zero pair if $A=B$. Further we shall suppose that every subclass of the class $\mathfrak{U}(R)$ contains all zero pairs.

Definition 1. [3] *The mapping $\varrho:\mathfrak{U}(R)\to\mathfrak{P}(R)$ will be called radical closure if the following conditions hold*:

ϱ. 1. *For every pair $(A, B)\in\mathfrak{U}(R): B\subset\varrho(A, B)\subset A$.*

ϱ. 2. *For every pair $(A, B)\in\mathfrak{U}(R): \varrho(A, \varrho(A, B))=\varrho(A, B)$ and $\varrho(\varrho(A, B), B)=\varrho(A, B)$.*

ϱ. 3. *If $B\subset C\subset A$ are right ideals of R, then $\varrho(A, C)\supset\varrho(A, B)$ and $\varrho(C, B)\subset\varrho(A, B)$.*

With every radical closure ϱ one can connect the following two classes of pairs

$$\mathfrak{R}(\varrho)=\{(A, B)\in\mathfrak{U}(R)\mid\varrho(A, B)=A\}$$

$$\mathfrak{S}(\varrho)=\{(A, B)\in\mathfrak{U}(R)\mid\varrho(A, B)=B\}$$

It is clear [3] that for every pair $(A, B)\in\mathfrak{U}(R)$ holds $\varrho(A, B)=\bigcap T_\alpha=\Sigma Q_\beta$, where $\{T_\alpha\}$ is the set of all right ideals of R, for which $(A, T_\alpha)\in\mathfrak{S}(\varrho)$ and $A\supset T_\alpha\supset B$, and $\{Q_\beta\}$ is the set of all right ideals of R, for which $A\supset Q_\beta\supset B$ and $(Q_\beta, B)\in\mathfrak{R}(\varrho)$. This shows that the radical closure ϱ is completely determined if one of the classes $\mathfrak{R}(\varrho)$ or $\mathfrak{S}(\varrho)$ is given.

Definition 2. [3] *Let ϱ be a radical closure. A pair (A, B) will be called ϱ-radical if $\varrho(A, B)=A$ and ϱ-semi-simple if $\varrho(A, B)=B$.*

Definition 3. [3] *The class $\mathfrak{F}\subset\mathfrak{U}(R)$ will be called a radical class if there exists a radical closure ϱ, such that $\mathfrak{F}=\mathfrak{R}(\varrho)$. The class $\mathfrak{C}\subset\mathfrak{U}(R)$ will be called a semi-simple one if there exists a radical closure ϱ, such that $\mathfrak{C}=\mathfrak{S}(\varrho)$.*

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The following characteristics of radical and semi-simple classes are given in [3].

Proposition 1. *A class of pairs* $\mathcal{F}\subset\mathfrak{U}(R)$ *is a radical one iff*:

(i) *If* $(A, B)\in\mathcal{F}$, *then for every right ideal* D *of* R *holds* $(A+D, B+D)\in\mathcal{C}$.

(ii) *If for every non-zero factor-pair* (A, C) *of the pair* (A, B) *there exists a non-zero subpair* $(D, C)\in\mathcal{F}$, *then* $(A, B)\in\mathcal{F}$.

Proposition 2. *A class of pairs* $\mathcal{C}\subset\mathfrak{U}(R)$ *is a semi-simple one iff*:

(iii) *If* $(A, B)\in\mathcal{C}$, *then for every right ideal* C *of* R *holds*: $(A\cap C, B\cap C)\in\mathcal{F}$.

(iv) *If for every non-zero subpair* (C, B) *of the pair* (A, B) *there, exists a non-zero factor-pair* $(C, D)\in\mathcal{C}$, *then* $(A, B)\in\mathcal{C}$.

Definition 4. [3] *If* ϱ_1 *and* ϱ_2 *are two radical closures, then we shall say that* $\varrho_1\leq\varrho_2$, *if for every pair* (A, B): $\varrho_1(A, B)\subset\varrho_2(A, B)$.

It is clear, that the following three conditions are equivalent: (a) $\varrho_1\leq\varrho_2$; (b) $\mathcal{R}(\varrho_1)\subset\mathcal{R}(\varrho_2)$; (c) $\mathcal{S}(\varrho_1)\supset\mathcal{S}(\varrho_2)$.

2. Hereditary radical closures. First we shall prove the following

Proposition 3. *If for the class* $\mathcal{C}_1\subset\mathfrak{U}(R)$ (iii) *holds, then a radical closure* ϱ^t *exists, such that* $\mathcal{C}_1\subset\mathcal{S}(\varrho_1)$ *and for every radical closure* η, *for which* $\mathcal{C}_1\subset\mathcal{S}(\eta)$ *we have* $\varrho^t\geq\eta$.

Proof. Let $\mathcal{C}$ be the class of all pairs $(A, B)\in\mathfrak{U}(R)$, such that for every non-zero subpair (C, B) or (A, B) there exists a non-zero factor-pair $(C, D)\in\mathcal{C}_1$. It is obvious that $\mathcal{C}_1\subset)\mathcal{C}$.

We shall prove that for $\mathcal{C}$ (iii) and (iv) hold. Let $(A, B)\in\mathcal{C}$ and C be a right ideal of R. If $A\cap C=B\cap C$ then according to the agreement already made at the beginning of this paper: $(A\cap C, B\cap C)\in\mathcal{C}$. Let $A\cap C\neq B\cap C$ and $(D, B\cap C)$ be a non-zero subpair of $(A\cap C, B\cap C)$. Then $(D+B, B)$ is a non-zero subpair of (A, B). (If $D+B=B$, then $D\subset B$ and as $D\subset A\cap C\subset C$, then $D\subset B\cap C$, which is not true). Since $(A, B)\in\mathcal{C}$, then non-zero factor-pair $(D+B, E)\in\mathcal{C}_1$ of $(D+B, B)$ exists. Now it follows that $((D+B)\cap D, E\cap D)=(D, E\cap D)$ is a non-zero factor-pair of $(D, B\cap C)$ and $(D, E\cap D)\in\mathcal{C}_1$. Therefore $(A\cap C, B\cap C)\in\mathcal{C}$ or $\mathcal{C}$ has the property (iii). One checks that (iv) holds directly. But then by proposition 2 there exists a radical closure ϱ^t, such that $\mathcal{C}=\mathcal{S}(\varrho^t)$. Finally let η be a radical closure, for which $\mathcal{C}\subset\mathcal{S}(\eta)$. Then $\mathcal{C}=\mathcal{S}(\varrho^t)\subset\mathcal{S}(\eta)$ and therefore $\eta\leqq\varrho^t$. Thus the prcposition is proved.

The radical closure ϱ^t shall be called upper radical closure, determined by $\mathcal{C}_1$.

Definition 5. *The radical closure* ϱ *is called hereditary, if the following condition holds*:

(v) *If* $\varrho(A, B)=A$, *then every pair* (C, B), *which is a subpair of the pair* (A, B) *one has* $\varrho(C, B)=C$.

Lemma 1. *The radical closure* ϱ *is hereditary iff*:

(vi) *For any pair* $(A, B)\in\mathfrak{U}(R)$ *and for any subpair* (C, B) *of* (A, B) *we have* $\varrho(C, B)=\varrho(A, B)\cap C$.

Proof. It is clear that (v) follows from (vi). Conversely let ϱ be a hereditary radical closure, $(A, B)\in\mathfrak{U}(R)$ and (C, B) be a subpair of (A, B). It is obvious that $\varrho(C, B)\subset\varrho(A, B)\cap C$. But $B\subset\varrho(A, B)\cap C\subset\varrho(A, B)$ and since $\varrho(\varrho(A, B), B)=\varrho(A, B)$, then $\varrho(\varrho(A, B)\cap C, B)=\varrho(A, B)\cap C$. It follows that $\varrho(C, B)\supset\varrho(A, B)\cap C$. Finally we have $\varrho(C, B)=\varrho(A, B)\cap C$.

We see already that if ϱ is a radical closure, then for every pair $(A, B)\in\mathfrak{U}(R)$ it holds $\varrho(A, B)=\bigcap T_\alpha$, where $\{T_\alpha\}$ is the set of all right ideals of R, for which $B\subset T_\alpha\subset A$ and $(A, T_\alpha)\in\mathcal{S}(\varrho)$. We shall be interested in such upper radical closures ϱ^t for which $\varrho^t(A, B)=\bigcap T_\alpha$, where $B\subset T_\alpha\subset A$ and $(A, T_\alpha)\in\mathcal{C}_1$.

Proposition 4. *If $\mathcal{C}_1\subset\mathfrak{U}(R)$ is a class of pairs, satisfying* (iii), *then the conditions:*

(a) *ϱ^t is hereditary;*

(vii) *For any pair $(A, B)\in\mathfrak{U}(R)$ it holds $\varrho^t(A, B)=\bigcap T_\alpha$, where $\{T_\alpha\}$ is the set of all right ideals of R, for which $A\supset T_\alpha\supset B$ and $(A, T_\alpha)\in\mathcal{C}_1$ are equivalent to the condition:*

(viii) *If for the non-zero subpair (C, B) of the pair (A, B) there exists a non-zero factor-pair $(C, D)\in\mathcal{C}_1$, then there is a right ideal T of R, such that $B\subset T\subset A$, T* does not *$\supset C$ and $(A, T)\in\mathcal{C}_1$.*

Proof. Let ϱ^t be a hereditary radical closure and let $\mathcal{C}_1$ satisfy condition (vii). Let still $(A, B)\in\mathfrak{U}(R)$, (C, B) be a non-zero subpair of (A, B) and $(C, D)\in\mathcal{C}_1$ be a non-zero factor-pair of (C, B). Then $\varrho^t(C, B)\neq C$ and since $\varrho^t(C, B)=\varrho^t(A, B)\cap C$ one has C does not $\subset\varrho^t(A, B)$. On the other hand, $\varrho^t(A, B)=\bigcap T_\alpha$, where $\{T_\alpha\}$ is the set of all right ideals of R, for which $B\subset T_\alpha\subset A$ and $(A, T_\alpha)\in\mathcal{C}_1$. But then there exists a right ideal T_α or R, such that $B\subset T_\alpha\subset A$, C does not $\subset T_\alpha$ and $(A, T_\alpha)\in\mathcal{C}_1$.

Conversely let $\mathcal{C}_1$ satisfy condition (viii). First we shall prove that ϱ^t is a hereditary radical closure. Let $(A, B)\in\mathfrak{U}(R)$, $\varrho^t(A, B)=A$ and (C, B) be a non-zero subpair of (A, B). Suppose $\varrho^t(C, B)\neq C$. Then the pair $(C, \varrho^t(C, B))$ is a non-zero one and $(C, \varrho^t(C, B))\in\mathcal{S}(\varrho^t)$. It follows that there exists a right ideal D of R such that $C\supset D\supset\varrho^t(A, B)\supset B$, $C\neq D$ and $(C, D)\in\mathcal{C}_1$. By (viii) there is a right ideal T of R, for which $B\subset T\subset A$, T does not $\supset C$ and $(A, T)\in\mathcal{C}$. But then $T\subset A$, $T\neq A$ and $T=\varrho^t(A, T)\supset\varrho^t(A,B)=A$, which is a contradiction.

Finally let $(A, B)\in\mathfrak{U}(R)$ and let $\{T_\alpha\}|\alpha\in J\}$ be the set of all right ideals of R, satisfying the conditions: $B\subset T_\alpha\subset A$ and $(A, T_\alpha)\in\mathcal{C}_1$. Denote $S=\bigcap T_\alpha$ $(\alpha\in J)$. It is obvious that we have $S=\varrho^t(A, S)\supset\varrho^t(A, B)$. Suppose $S\supset\varrho^t(A, B)$, $S\neq\varrho^t(A, B)$. First it is clear that $\varrho^t(A, B)\neq S$ (if $S=\varrho^t(A, B)$, then $S\subset\varrho^t(A, B)$, which is a contradiction, because $\varrho^t(A, B)\subset S$ and $\varrho^t(A, B)\neq S$). But then the pair $(S, \varrho^t(S, B))$ is non-zero and $(S, \varrho^t(S, B))\in\mathcal{S}(\varrho^t)$. Thus, there exists a right ideal D of R, for which $S\supset D\supset\varrho^t(S, B)\supset B$, $S\neq D$ and $(S, D)\in\mathcal{C}_1$. By (viii) there exists a right ideal T of R, such that $B\subset T\subset A$, T does not $\supset S$ and $(A, T)\in\mathcal{C}_1$. Thus there is $\alpha\in J$ such that $T=T_\alpha\supset S$, a contradiction. The proposition is proved.

3. Supplementary radical closures. Definition 6. *The pair $(A, B)\in\mathfrak{U}(R)$ will be called irreducible, when the intersection $H=\bigcap Q_\alpha$ of all right ideals Q_α of R for which $B\subset Q_\alpha\subset A$, $B\neq Q_\alpha$ is different from B, i. e. when $H\supset B$, $H\neq B$. In this case, the pair (H, B) will be called the heart of (A, B). The pair (A, B) will be called simple, when there are no right ideals of R between B and A.*

It is clear, that the heart of any irreducible pair is a simple pair.

It is obvious, that the following lemma holds:

Lemma 2. *For every pair $(A, B)\in\mathfrak{U}(R)$, there exists a set $\{T_\alpha|\alpha\in J\}$ of right ideals of R, such that $B\subset T_\alpha\subset A$, $B=\bigcap T_\alpha$ and (A, T_α) $(\alpha\in J)$ are irreducible pairs.*

Proposition 5. *If the pair* $(A, B)\in\mathfrak{U}(R)$ *is irreducible with a heart* (H, B) *and* C *is a right ideal of* R, *such that* $A\cap C\supset B\cap C$, $A\cap C\neq B\cap C$, *then the pair* $(A\cap C, B\cap C)$ *is irreducible with heart* $(H\cap C, B\cap C)$.

Proof. Let $\{Q_\alpha \mid \alpha\in J\}$ be the set of all right ideals of R, which the condition $B\cap C\subset Q_\alpha\subset A\cap C$, $B\cap C\neq Q_\alpha$ holds and let $H_1=\bigcap Q_\alpha$. We have $B\cap C\subset H_1\subset A\cap C$ and our aim is to prove that $B\cap C\subset H_1$, $B\cap C\neq H_1$ and $H_1=H\cap C$. For every $\alpha\in J$ we have $Q_\alpha+B\supset B$, $Q_\alpha+B\neq B$ and therefore $\bigcap(Q_\alpha+B)\supset B$, $\bigcap(Q_\alpha+B)\neq B$ $(\alpha\in J)$ and $\bigcap(Q_\alpha+B)\supset H$. On the other hand, it is not difficult to see that $\bigcap(Q_\alpha+B)=\bigcap Q_\alpha+B=H_1+B$ and therefore $H_1+B\supset H$. It follows that $(H_1+B)\cap C\supset H\cap C\supset B\cap C$ or $H_1\supset H\cap C\supset B\cap C$, because if $(H_1+B)\cap C=A\cap C+H_1=H$. To conclude the proof it is enough to show that $H\cap C\neq B\cap C$.

Suppose $H\cap C=B\cap C$. But as $H\supset B$, $H\neq B$, there exists an element with $h\in H$ and $h\notin B$. However by $H_1\subset H_1+B\subset A\cap C+B\subset C+B$ it follows that $h\in C+B$, or $h=c+b$, where $c\in C$ and $b\in B$. Then $c=h-b\in H+B=H$ and therefore $c\in H\cap C=B\cap C\subset B$, i. e. we get the contradiction $h\in B$. The proposition is proved.

Irreducible pairs are of great importance in the construction of the so-called supplementary radical closures.

Definition 7. *If* ϱ *is a radical closure, then the radical closure* ϱ^* *will be called supplementary to* ϱ, *if*

(a) *For any pair* $(A, B)\in\mathfrak{U}(R)$, *we have* $\varrho(A, B)\cap\varrho^*(A, B)=B$.

(b) *For every radical closure* η *for which* $\eta(A, B)\cap\varrho(A, B)=B$ *we have* $\eta\leq\varrho^*$.

The radical closure ϱ *will be called a dual one if the radical closures* ϱ^* *and* $\varrho^{**}=(\varrho^*)^*$ *exist and* $\varrho=\varrho^{**}$.

We shall construct a radical closure, which is supplementary to a hereditary radical closure. However let us previously prove

Lemma 3. *If* ϱ *is a hereditary radical closure and* (A, B) *is a irreducible pair with heart* (H, B), *then the pair* (A, B) *is* ϱ*-semi-simple iff the pair* (H, B) *is* ϱ*-semi-simple.*

Proof. It is clear that if $\varrho(A, B)=B$, then $\varrho(H, B)=B$. Conversely let $\varrho(H, B)=B$. If we suppose that $\varrho(A, B)\neq B$, then $\varrho(A, B)\supset B, \varrho(A, B)\neq B$ and therefore $\varrho(A, B)\supset H$. But it follows that $\varrho(H, B)=\varrho(A, B)\cap H=H$, which is a contradiction.

Let ϱ be a radical closure. Denote by $\mathscr{C}(\varrho)$ the class of all irreducible pairs with a ϱ-radical heart. Then the class

$$\mathscr{C}=\{(A\cap C, B\cap C)\mid (A, B)\in\mathscr{C}(\varrho),\ C\text{— a right ideal of } R\}$$

consists of only irreducible pairs and satisfies (iii). Therefore $\mathscr{C}$ determines an upper radical closure η^t. If ξ is a radical closure, for which $\mathscr{C}(\varrho)\subset\mathscr{S}(\xi)$, then also $\mathscr{C}\subset\mathscr{S}(\xi)$, or $\xi\leq\varrho^t$. This justifies the name upper radical closure determined by the class of all irreducible pairs with a ϱ = radical heart.

Definition 8. *Let* ϱ *be a radical closure. The pair* (A, B) *will be called strongly* ϱ*-semi-simple if for every right ideal* C *of* R *we have* $\varrho(A+C, B+C)=B+C$.

Theorem 1. *If* ϱ *is a hereditary radical closure,* η^t *is an upper radical closure, determined by the class* $\mathscr{C}(\varrho)$ *of all irreducible pairs with a* ϱ*-radical heart and* $(A, B)\in\mathfrak{U}(R)$, *then the following conditions are equivalent:*

(a) $\eta^t(A, B)=A$.

(b) *For every right ideal D of R there exists a family of right ideals* $\{Q_\alpha \mid \alpha \in J\}$ *of* R, *such that* $B+D=\bigcap Q_\alpha$, *the pairs* $(A+D, Q_\alpha)$ *are irreducible ones and* $\varrho(A+D, Q_\alpha)=Q_\alpha$.

(c) (A, B) *is a strongly* ϱ*-semi-simple pair.*

(d) *For every right ideal D of R and every subpair* $(C, B+D)$ *of* $(A+D, B+D)$, *there exists a family of right ideals* $\{K_\alpha \mid \alpha \in I\}$, *such that* $C=\bigcap K_\alpha (\alpha \in J)$, $(A+D, K_\alpha)$ *are irreducible pairs and* $\varrho(A+D, K_\alpha)=K_\alpha$.

Proof. (a) $\Rightarrow$ (b). Let D be a right ideal of R. By lemma 2 there exists a family of right ideals $\{Q_\alpha \mid \alpha \in J\}$ of R, such that $B+D=\bigcap Q_\alpha (\alpha \in J)$ and $(A+D, Q_\alpha)$ are irreducible pairs. Let (H_α, Q_α) be the heart of $(A+D, Q_\alpha)$. However by proposition 1 we have $\eta^t(A+D, B+D)=A+D$ and $\eta^t(A+D, Q_\alpha)\supset\eta^t(A+D, B+D)=A+D$. Therefore $(A+D, Q_\alpha)\notin\mathcal{S}(\eta^t)$ and $(A+D, Q_\alpha)\in\mathfrak{T}(\varrho)$. This shows that $\varrho(H_\alpha, Q_\alpha)=Q_\alpha$ and by lemma 3 : $\varrho(A+D, Q_\alpha)=Q_\alpha$.

(b) $\Rightarrow$ (c). Let D be a right ideal of R. Then $B+D=\bigcap Q_\alpha (\alpha \in J)$, $\varrho(A+D, Q_\alpha)=Q_\alpha$ and $\varrho(A+D, B+D)=\varrho(A+D, \bigcap Q_\alpha)\subset\varrho(A+D, Q_\alpha)=Q_\alpha$ for any $\alpha \in J$. Therefore $\varrho(A+D, B+D)\subset\bigcap Q_\alpha=B+D$ and $\varrho(A+D, B+D)=B+D$.

(c) $\Rightarrow$ (d). Let D be a right ideal of R and $(C, B+D)$ be a subpair of $(A+D, B+D)$. Then $\varrho(A+D, C)=\varrho(A+D, B+D+C)=\varrho(A+(D+C), B+(D+C))=B+D+C=C$. On the other hand, by lemma 2 there exists a family of right ideals $\{K_\alpha \mid \alpha \in J\}$ of R, such that $C=\bigcap K_\alpha$ and $(A+D, K_\alpha)$ are irreducible pairs. It follows from $A+D\supset K_\alpha\supset C\supset B+D$ that $\varrho(A+D, K_\alpha)=\varrho(A+(D+K_\alpha), B+(D+K_\alpha))=B+D+K_\alpha=K_\alpha$.

(d) $\Rightarrow$ (a). Suppose $\eta^t(A, B)\neq B$ and denote $D=\eta^t(A, B)$. Then $A=A+D\neq B+D=D$ and $\eta^t(A+D, B+D)=\eta^t(A, D)=\eta^t(A, B)=D=B+D$. We get that $(A+D, B+D)\in\mathcal{S}(\eta^t)$ and therefore there exists a right ideal C of R, such that $(A+D, C)=(U\cap K, V\cap K)$, where $(U, V)\in\mathfrak{T}(\varrho)$ and K is a right ideal of R. Let (X, V) be a heart of the pair (U, V). Then $\varrho(X, V)=X$ and $(X\cap K, V\cap K)$ is a heart of $(A+D, C)$. Also $(X\cap K)+V=X$ and $(V\cap K)+V=V$, from which it follows that the pair $(U, V)=((U\cap K)+V, (V\cap K)+V)$ is an irreducible one with heart (X, V). However $V=C+V$ and $(C+V, B+(D+V))$ is a subpair of $(A+(D+V), B+(D+V))$. Therefore there exists a family of right ideals $\{K_\alpha \mid \alpha \in J\}$ of R, such that $C+V=\bigcap K_\alpha$, the pairs $(A+D+V, K_\alpha)$ are irreducible ones and $\varrho(A+D+V, K_\alpha)=K_\alpha$. But from the irreducibility of $(U, V)=(A+D+V, C+V)$ it follows that for some $\alpha \in J$: $K_\alpha=V$ and $\varrho(U, V)=V$, which is a contradiction.

The theorem is proved.

Theorem 2. *If ϱ is a hereditary radical closure, then the upper radical closure η^t, determined by the class of all irreducible pairs with a ϱ-radical heart, is supplementary to ϱ.*

Proof. Let $(A, B)\in\mathfrak{U}(R)$ and denote $Q=\varrho(A, B)\cap\eta^t(A, B)$. By theorem 1 it follows from $\eta^t(\eta^t(A, B), B)=\eta^t(A, B)$ that $\varrho(\eta^t(A, B), B)=B$ and therefore $Q=\varrho(A, B)\cap Q=\varrho(Q, B)\subset B$, which shows that $B=Q$.

Let now ξ be a radical closure, such that $\xi(A, B)\cap\varrho(A, B)=B$ for any pair $(A, B)\in\mathfrak{U}(R)$. Choose a pair $(A, B)\in\mathfrak{T}(\varrho)$. Then (A, B) is an irreducible one with heart (H, B) and $\varrho(H, B)=H$. If we suppose that $\xi(A, B)\neq B$, then $\xi(A, B)\supset H$ and $\varrho(A, B)\cap\xi(A, B)\supset H$ which is not true. Therefore $\xi(A, B)=B$, $(A, B)\in\mathcal{S}(\xi)$ and $\mathfrak{T}(\varrho)\subset\mathcal{S}(\xi)$. Finally we receive that $\xi\leq\eta^t$. The theorem is proved.

Theorem 3. *If ϱ is a hereditary radical closure, then its supplementary radical closure ϱ^* is also a hereditary one.*

Proof. Let $(A, B)\in\mathfrak{U}(R)$ and $\varrho^*(A, B)=A$. Suppose a subpair (C, B) of (A, B) exists, such that $\varrho^*(C, B)\neq C$. Then $(C, \varrho^*(C, B))\in\mathcal{S}(\varrho^*)$ and by theorems 1 and 2 there exists a right ideal D of R, such that $C\supset D\supset\varrho^*(C, B)$, $C\neq D$ and $(C, D)=(U\cap K, V\cap K)$, where (U, V) is an irreducible pair with a ϱ-radical heart (X, V) and K is a right ideal of R. It follows from here that the pair (C, D) is also an irreducible one with a heart $(X\cap K, V\cap K)$. But $(X\cap K)+V=X$, $(V\cap K)+V=V$ and then the pair $(U_1, V)=((U\cap K)+V, (V\cap K)+V)$ is irreducible one with a heart (X, V). On the other hand, $\varrho^*(A, D)\supset\varrho^*(A, B)=A$ and therefore $\varrho(A+V, D+V)=D+V$ and $\varrho(C+V, D+V)=\varrho(A+V, D+V)\cap(C+V)=D+V$. Thus $\varrho(X, V)=V$, which is a contradiction. The theorem is proved.

Corollary 1. *If ϱ is a hereditary radical closure, then there exists the radical closure $\varrho^{**}=(\varrho^*)^*$ and $\varrho\leq\varrho^{**}$.*

Corollary 2. *If ϱ is a hereditary radical closure, then its supplementary radical closure ϱ^* is a dual one.*

Proof. It is clear that $\varrho^*\leq\varrho^{***}=(\varrho^{**})^*$. Let now $(A, B)\in\mathcal{C}(\varrho)$. Then (A, B) is an irreducible pair and if (H, B) is its heart, then $\varrho(H, B)=H$. However it follows from $\varrho\leq\varrho^{**}$ that $\varrho(H, B)=H$, i. e. $(A, B)\in\mathcal{C}(\varrho^{**})$. Therefore $\mathcal{C}(\varrho)\subset\mathcal{C}(\varrho^{**})\subset\mathcal{S}(\varrho^{***})$ or $\bar{\varrho}^{**}\leq\varrho^*$, which shows finally that $\varrho^*=\varrho^{***}$.

4. Supernilpotent and underidempotent radical closures. Definition 9. *The pair (A, B) will be called a weakly regular one, if for every element $a\in A$, there exists an element $a'\in(a)$*[1]*, such that $a-aa'\in B$.*

Definition 10. *The pair (A, B) will be called a nilpotent pair if there exists a positive integer n such that $A^n\subset B$.*

It is easy to prove the following

Proposition 6. *If $(A, B)\in\mathfrak{U}(R)$, then the following conditions are equivalent:*

(a) *(A, B) is a weakly regular pair.*

(b) *For every $a\in A: a\in a(a)+B$.*

(c) *For every $a\in A: a\in |a)^2+B$*[2]*.*

(d) *For every $a\in A: |a)\subset |a)^2+B$.*

(e) *For every right ideal T of R, such that $T\subset A$, it holds $T\subset T^2+B$.*

Definition 11. *The radical closure ϱ will be called a supernilpotent radical closure if it is hereditary and if every nilpotent pair is a ϱ-radical pair. The radical closure ϱ will be called an underidempotent radical closure if it is hereditary and if every ϱ-radical pair is a weakly regular pair.*

We shall examine the relation between underidempotent and supernilpotent radical closures.

Lemma 4. *If ϱ is a supernilpotent radical closure and $\varrho(A, B)=B$, then for every right ideal C of R, for which $C\subset A$ and $C^2\subset B$, it holds $C\subset B$.*

Proof. Suppose that C does not $\subset B$. Then $C\supset B\cap C$, $C\neq B\cap C$ and $C^2\subset B\cap C$. Therefore $\varrho(C, B\cap C)=C$. On the other hand, it follows from

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[1] We shall denote (a), the two-sided ideal of R, generated by a, i. e. $(a)=\{\alpha a+ar_1+r_2a+s_1as_2 \mid \alpha\in\Phi,\ r_1, r_2, s_1, s_2\in R\}$.

[2] We shall denote $|a)$, the right ideal of R, generated by a, i. e. $|a)=\{\alpha a+ar \mid \alpha\in\Phi,\ r\in R\}$.

$\varrho(A, B)=B$, that $\varrho(A\cap C, B\cap C)=\varrho(C, B\cap C)=B\cap C$ which is a contradiction.

Theorem 4. *If ϱ is a supernilpotent radical closure, then ϱ^* is a dual underidempotent radical closure, ϱ^{**} is a dual supernilpotent radical closure and ϱ^{**} is a minimal dual supernilpotent radical closure, such that $\varrho\leqq\varrho^{**}$.*

Proof. It follows by theorem 3 and corollary 2 that ϱ^* is a dual hereditary radical closure. Let now $(A, B)\in\mathfrak{U}(R)$ and $\varrho^*(A, B)=A$. We shall show, that (A, B) is a weakly regular pair. Let T be a right ideal of R, such that $T\subset A$. Then $\varrho^*(A+T^2, B+T^2)=\varrho^*(A, B+T^2)=A$ and therefore $\varrho(A, B+T^2)=B+T^2$. But $T^2\subset B+T^2$ and it follows by lemma 4, that $T\subset B+T^2$ which shows that (A, B) is a weakly regular pair, or ϱ^* is an underidempotent radical closure.

By corollary 2 we have $\varrho\leqq\varrho^{**}$, which shows that ϱ^{**} is a dual supernilpotent radical closure. Finally let η be a dual supernilpotent radical closure and $\varrho\leqq\eta$. Then $\varrho^*\geqq\eta^*$ and $\varrho^{**}\leqq\eta^{**}=\eta$. The theorem is proved.

5. Prime pairs. Let $A\supset B$ be right ideals of R. We denote by $(B:A)$ the quotient $(B:A)=\{x\in R\mid Ax\subset B\}$. It is clear that $(B:A)$ is a two-sided ideal of R.

Definition 12. *The non-zero pair $(A, B)\in\mathfrak{U}(R)$ will be called a prime pair, if the following conditions hold:*

(ix) AR does not $\subset B$.

(x) If $x\in A$, *I is a two-sided ideal of R and $xI\subset B$ then $x\in B$ or $I\subset(B:A)$.*

(xi) $A\cap(B:A)\subset B$.

The conditions (ix) and (x) show that A/B is a prime R-module [2]. It is not difficult to see that the two-sided ideal P of R is a prime one if and only if the pair (R, P) is a prime pair.

Proposition 7. *If (A, B) is a prime pair and C is a right ideal of R such that $A\cap C\supset B\cap C$, $A\cap C\neq B\cap C$, then the pair $(A\cap C, B\cap C)$ is a prime one.*

Proof. Suppose $(A\cap C)R\subset B\cap C$. Since $A\cap C\neq B\cap C$ then there is an element $x\in A\cap C$, such that $x\notin B\cap C$. Then $x\in A$ and $x\notin B$. But $xR\subset (A\cap C)R\subset(B\cap C)\subset B$ and by (x) we recieve that $R\subset(B:A)$, i. e. $AR\subset B$, a contradiction. Therefore the condition (ix) is true for the pair $(A\cap C, B\cap C)$.

Let now $x\in A\cap C$, I be a two-sided ideal of R and $xI\subset B\cap C$. If $x\notin B\cap C$, then $x\notin B$ and $xI\subset B$, which shows that $I\subset(B:A)$, i. e. $AI\subset B$. It follows that $(A\cap C)I\subset B\cap C$ or $I\subset(B\cap C:A\cap C)$. Condition (x) holds for $(A\cap C, B\cap C)$.

Finally let $x\in A\cap C\cap(B\cap C:A\cap C)$. Then $(A\cap C)x\subset B\cap C$. Take an element $y\in A\cap C$ with $y\notin B\cap C$. Then $y\in A$, $y\notin B$ $y(x)\subset B\cap C\subset B$ and $(x)\subset(B:A)$. Therefore $x\in(B:A)\cap A\subset B$. We receive that $x\in B\cap C$ and $A\cap C\cap(B\cap C:A\cap C)\subset B\cap C$. The proposition is proved.

Corollary 3. $(B:A)=(B\cap C:A\cap C)$

Proof. If $x\in(B:A)$, then $Ax\subset B$ and therefore $(A\cap C)x\subset Ax\subset B$ and $(A\cap C)x\subset Cx\subset C$, i. e. $x\in(B\cap C:A\cap C)$. Conversely, let $x\in(B\cap C:A\cap C)$. Then $(x)\subset(B\cap C:A\cap C)$. If y is an element for which $y\in A\cap C$, $y\notin B\cap C$, then $y\in A$, $y\in B$ and $y(x)\subset(A\cap C)(x)\subset(B\cap C)\subset B$. It follows that $(x)\subset(B:A)$ and $x\in(B:A)$.

Proposition 8. *If (A, B) is a prime pair, then there exists a right ideal P of R, such that (R, P) is a prime pair and $P \cap A = B$.*

Proof. To examine the right ideal $B+(B:A)$. We have $[B+(B:A)] \cap (A \diagup B) = \varnothing$. If we suppose that $x \in (B+(B:A)) \cap (A \setminus B)$ then $x \in A$, $x \notin B$ and $x = b_1 + b_2$, where $b_1 \in B$, $b_2 \in (B:A)$. Therefore $b_2 = x - b_1 \in (B:A) \cap A \subset B$ and $x \in B$, a contradiction. It follows that the set $\mathfrak{M}$ of all right ideals T of R such that $B+(B:A) \subset T$ and $T \cap (A \setminus B) = \varnothing$ is not empty. By Zorn's Lemma there are maximal elements of $\mathfrak{M}$ and let P be such an element. It is clear that $P \cap A = B$. Now we shall prove that (R, P) is a prime pair. Suppose $R^2 \subset P$. Then $AR \subset R^2 \subset P$ or $AR \subset P \cap A = B$, a contradiction. Therefore (R, P) fulfils (ix).

Let $x \in (P:R)$. Then $Rx \subset P$ and $Ax \subset P \cap A = B$, i. e. $x \in (B:A) \subset P$. It follows that $(P:R) \subset P$. The condition (xi) is true for (R, P). If I is a two-sided ideal of R and $I \subset P$, then $I \subset (P:R)$.

Finally let $x \in R$, I be two-sided ideal of R, $xI \subset P$ and $x \notin P$. Then $(x)+P \neq P$ and there exists an element $y \in A \setminus B$ such that $y \in (x)+P$, i. e. $y = p + \alpha x + xr$ where $p \in P$, $r \in R$, $\alpha \in \Phi$. However $yI \subset P \cap A = B$ and $I \subset (B:A) \subset P$. Therefore $I \subset (P:R)$ and the condition (x) is true for the pair (R, P). The proposition is proved.

6. Special radical closures. Definition 13. *The class of pairs S will be called a special class if the following conditions hold:*

S. 1. *If $(A, B) \in S$, then (A, B) is a prime pair.*

S. 2. *If $(A, B) \in S$ and C is a right ideal of R, then $(A \cap C, B \cap C) \in S$.*

S. 3. *If (C, B) is a non-zero subpair of (A, B) and $(C, B) \in S$ then there exists a right ideal T of R, such that $B \subset T \subset A$, C does not $\subset T$ and $(A, T) \in S$.*

It follows from conditions S. 1., S. 2., S. 3. that if S is a special class of pairs, then it determines an upper radical closure ϱ^s, which is a hereditary one and for any pair $(A, B) \in \mathfrak{U}(R)$ it holds $\varrho^s(A, B) = \bigcap T_\alpha$, where $\{T_\alpha\}$ is the set of all right ideals of R such that $B \subset T_\alpha \subset A$ and $(A, T_\alpha) \in S$.

Definition 14. *The radical closure ϱ will be called a special radical closure if there exists a special class of pairs S such that $\varrho = \varrho^s$.*

We shall discuss now some elementary properties of the special radical closures.

Lemma 5. *If S be a special class of pairs and (A, B) — an irreducible pair, then $\varrho^s(A, B) = B$ iff $(A, B) \in S$.*

Proof. It is obvious that if $(A, B) \in S$, then $\varrho^s(A, B) = B$. Conversely, let $\varrho^s(A, B) = B$. Then $B = \bigcap T_\alpha$, where $\{T_\alpha\}$ is the set of all right ideals of R such that $B \subset T_\alpha \subset A$ and $(A, T_\alpha) \in S$. However it follows from the irreducibility of (A, B) that there exists some α such that $T_\alpha = B$. Therefore $(A, B) \in S$.

Corollary 4. *If (A, B) is a simple pair, then $\varrho^s(A, B) = A$ or $(A, B) \in S$.*

Lemma 6. *If $\{\varrho_\alpha \mid \alpha \in I\}$ is a family of special radical closures, then there exists a special radical closure ϱ such that $\varrho \leq \varrho_\alpha$ for every $\alpha \in I$ and if η is a radical closure for which $\eta \leq \varrho_\alpha$ for every $\alpha \in I$, then $\eta \leq \varrho$.*

Proof. Let ϱ_α be determined by a special class of pairs S_α $(\alpha \in I)$. It is clear that the class of pairs $S = \bigcup S_\alpha (\alpha \in I)$ is a special one. Let ϱ be a special radical closure determined by S. It follows by $S_\alpha \subset S \subset \mathcal{S}(\varrho)$ that

$\varrho \leqq \varrho_\alpha$ for every $\alpha \in I$. Let now η is a radical closure such that $\eta \leqq \varrho_\alpha$ for every $\alpha \in I$. Then $S_\alpha \subset \mathcal{S}(\eta)$ for every α. Therefore $S \subset \mathcal{S}(\eta)$ and $\eta \leqq \varrho$.

We shall denote with $\varrho = \Lambda \varrho_\alpha$ such determined radical closure and shall examine now the link between special and supernilpotent radical closures.

Proposition 9. *Every special radical closure is a supernilpotent one.*

Proof. Let ϱ be a special radical closure, determined by a special class of pairs S. In order to prove that ϱ is a supernilpotent radical closure it is enough to show that for every nilpotent pair (A, B) we have $\varrho(A, B) = A$. Suppose the contrary. Then there is a non-zero nilpotent pair (A, B) such that $\varrho(A, B) \neq A$. But $\varrho(A, B) = \bigcap T_\alpha$, where $B \subset T_\alpha \subset A$ and $(A, T_\alpha) \in S$. Therefore there exists a right ideal T or R such that (A, T) is a non-zero nilpotent pair and $(A, T) \in S$. Let n be a positive integer such that $A^n \subset T$ but A^{n-1} does not $\subset T$. However (A, T) is a prime pair and therefore from $A^{n-1}. A = A^n \subset T$ it follows that $A^{n-1} \subset A \bigcap (T : A) \subset T$, a contradiction.

Proposition 10. *If ϱ is a supernilpotent radical closure and (A, B) is an irreducible pair with a ϱ-semi-simple heart, then (A, B) is a prime pair.*

Proof. We shall consider two cases:

(a) Let (A, B) be a simple pair. Then it follows from $\varrho(A, B) = B$ that A^2 does not $\subset B$ and AR does not $\subset B$ (condition (ix)). Let now $x \in A$, I be a two-sided ideal of R, $xI \subset B$ and $x \notin B$. On the other hand, (A, B) is a simple pair and therefore $|x) + B = A$. We get from here that $AI = (|x) + B)I \subset B$, i. e. $I \subset (B : A)$ (condition (x)). Finally let $x \in (B : A) \bigcap A$ and suppose $x \notin B$. Then the pair $(|x), B \bigcap |x))$ is a non-zero one and $|x)^2 \subset (B \bigcap |x))$, i. e. $\varrho(|x), B \bigcap |x)) = |x)$. However it follows from $\varrho(A, B) = B$ that $\varrho(|x), B \bigcap |x)) = \varrho(A \bigcap |x), B \bigcap |x)) = B \bigcap |x)$, a contradiction. Therefore (A, B) is a prime pair.

(b) Let (A, B) be an irreducible pair with a heart (H, B). Then (H, B) is a simple pair and $\varrho(H, B) = B$. Therefore (H, B) is a prime pair. By proposition 7 there exists a right ideal P or R such that (R, P) is a prime pair and $P \bigcap H = B$. Then the pair $(A, P \bigcap A)$ is a prime one. If $P \bigcap A \supset B$, $P \bigcap A \neq B$, then $P \supset P \bigcap A \supset H$ which is not true. Therefore the pair $(A, B) = (A, P \bigcap A)$ is a prime one.

We shall prove already the following main

Theorem 5. *Every dual supernilpotent radical closure ϱ is a dual special radical closure.*

Proof. Let us denote by $\mathcal{C}(\varrho^*)$ the class of all irreducible pairs with a ϱ^*-radical heart and let $\mathcal{C} = \{(A \bigcap C, B \bigcap C) \ (A, B) \in \mathcal{C}(\varrho^*),\ C$ — a right ideal of $R\}$.

Since ϱ is a dual radical closure then by theorem 2 we have $\varrho = \varrho^t$. We shall show that $\mathcal{C}$ is a special class of pairs which shall prove the theorem. It is clear that all pairs of $\mathcal{C}$ are irreducible. Let $(A, B) \in \mathcal{C}$ and (H, B) be the heart of (A, B). Then $\varrho(H, B) = B$ and by proposition 10 (A, B) is a prime pair (condition S. 1.). It is obvious that the condition S. 2. is also true for $\mathcal{C}$.

Finally let (C, B) be a non-zero subpair of (A, B) and $(C, B) \in \mathcal{C}$. Then (C, B) is an irreducible pair and let (H, B) be its heart. On the other hand, $(C, B) = (U \bigcap K, V \bigcap K)$, where $(U, V) \in \mathcal{C}(\varrho^*)$ and K is a right ideal of R. Denote by (X, V) the heart of (U, V). Then $(H, B) = (U \bigcap K, V \bigcap K)$. Let P be a right ideal of R which is maximal with respect to the conditions: $P \supset V + (V:U)$, $P \cap U = V$. (It exists by proposition 7.) Then (R, P) is a prime pair and it is clear that it is an irreducible one with heart $(X+P, P)$. If we suppose that $P = \varrho^*(X+P, P)$ then $\varrho^*((X+P) \cap X, P \cap X) = \varrho^*(X, V) = V$, a contradiction. We get from here that $(\varrho(X+P, P) = X+P$, $(R, P) \in \mathcal{C}(\varrho^*)$ and $(A, P \cap A) = (R \cap A, P \cap A) \in \mathcal{C}$. Further more $B \subset P \cap A \subset A$ and C does not $\subset P \cap A$. Therefore $\mathcal{C}$ has the condition s. 3. and $\mathcal{C}$ is a special class of pairs. The theorem is proved.

Combining theorem 5 with lemma 6 we have

Theorem 6. *If ϱ is a supernilpotent radical closure, then there exists a special radical closure ϱ', such that $\varrho \leq \varrho'$ and for every special radical closure η, for which $\varrho \leqq \eta$ holds $\varrho' \leq \eta$.*

Proof. Let $\{\varrho_\alpha) \alpha \in I\}$ be the set of all special radical closures such that $\varrho \leqq \varrho_\alpha$. Since $\varrho \leqq \varrho^{**}$ and by theorem 5 ϱ^{**} is a special radical closure, then this set is not empty. It is obvious that the special radical closure $\varrho' = \Lambda \varrho_\alpha$ satisfies the conditions of theorem. The theorem is proved.

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