Serdica

Bulgariacae mathematicae publicationes

Сердика

Българско математическо списание

MAPPINGS FROM CONVERGENCE GROUPS INTO QUASI-NORMED GROUPS

PIOTR ANTOSIK

An abelian group endowed with a set G of sequences is called a convergence group. I is proved that pointwise limits of sequence of mappings vanishing on sequences from G are vanishing on quasi-unconditionally G-convergent sequences.

1. In [3] is proved the following

Diagonal Theorem (for nonnegative matrices). *Let* $\{x_{ij}\}$ $(i, j \in N)$ *be a matrix of nonnegative numbers such that*

$$\lim_{j\to\infty} x_{ij}=0,\ (i\in N),\ \lim_{i\to\infty} x_{ij}=0,\ (j\in N),\ \textit{and}\ \lim_{i\to\infty} x_{ii}=0.$$

Then there exists an infinite set I, $I\subset N$, *such that* $\Sigma_{i,j\in I}\, x_{ij}<\infty$.

Hence, the elements of I *can be arranged into an increasing sequence* $\{p_i\}$ *such that*

$$\lim_{i\to\infty}\sum_{j=1}^{\infty} x_{p_i p_j}=0,\ \lim_{j\to\infty}\sum_{i=1}^{\infty} x_{p_i p_j}=0.$$

We show that this theorem can successfully be used in proofs of difficult theorems in Functional Analysis and Measure Theory. It replaces more sophisticated auxiliary diagonal type or other theorems and lemmas [1, 3, 5, 6, 7]. The Diagonal Theorems [1] and the lemmas on double sequences [7] are independent in the sense that knowledge of one of them does not help in proving the other.

2. Let $\mathfrak{X}$ be an abelian group and let G be a set of sequences $\{x_n\}$ of elements from $\mathfrak{X}$. Instead of writing $\{x_n\}\in G$ we shall equivalently write $x_n\to o(G)$.

An additive real (quasi-normed group valued) function on $\mathfrak{X}$ is called G-continuous, iff $x_n\to o(G)$ implies $f(x_n)\to 0$. By TG we denote the set of all additive real and G-continuous functions on $\mathfrak{X}$, and by $\overline{TG}$ the set of all pointwise limits of sequences $\{f_n\}$ of elements from TG. Thus $f\in \overline{TG}$ iff there exists a sequence $\{f_n\}$, $f_n\in TG$, such that $f(x)=\lim_{n\to\infty} f_n(x)$ for each $x\in\mathfrak{X}$. We note that each element of $\overline{TG}$ is an additive real function on $\mathfrak{X}$. Moreover, we have $TG\subset \overline{TG}$. A question arises when the converse inclusion holds, i. e., when $TG\supset\overline{TG}$.

We say that a sequence $\{x_n\}$ of elements x_n from $\mathfrak{X}$ is quasi-unconditionally G-convergent iff from each its subsequence $\{y_n\}$ we can select a subsequence $\{z_n\}$ such that $z-\Sigma_{m=1}^{n} z_m\to o(G)$ for some $z\in\mathfrak{X}$. Instead of $z-\Sigma_{m=1}^{n} z_m \to o(G)$ we shall write $\Sigma z_m=z(G)$.

We note that each subsequence of a quasi-unconditionally G-convergent soquence is a quasi-unconditionally G-convergent sequence. If $\Sigma z_m=z(G)$ and

$f \in TG$, then $\Sigma_{m=1}^{\infty} f(z_m) = f(z)$, and $f(z_m) \to 0$, as $m \to \infty$. Thus, from each subsequence $\{y_n\}$ of a quasiunconditionally G-convergent sequence $\{x_n\}$ we can select a subsequence $\{z_n\}$ such that $f(z_m) \to 0$ for each $f \in TG$. This implies that if $\{x_n\}$ is a quasi-unconditionally G-convergent sequence, then $f(x_n) \to 0$ for each $f \in TG$. In other words $\{x_n\} \in LTG$ if LTG denotes the set of all sequences $\{x_n\}$ such that $f(x_n) \to 0$ for each $f \in TG$ [4].

Theorem 1. *If $\{f_n\}$ is a sequence of elements from TG such that*

$$\lim_{n \to \infty} f_n(x) = 0 \tag{1}$$

for each $x \in \mathfrak{X}$, then we have

$$\lim_{m \to \infty} \sup_n |f_n(x_m)| = 0, \quad \lim_{n \to \infty} \sup_m |f_n(x_m)| = 0 \tag{2}$$

for each quasi-unconditionally G-convergent sequence $\{x_m\}$.

Proof. Let $\{x_m\}$ be a quasi-unconditionally G-convergent sequence. Then $\lim f_n(x_m) = 0$ for each $n \in N$. Hence and from (1) it follows that if (2) does not hold then there exist a positive number ε and two increasing sequences $\{k_n\}$ and $\{l_n\}$ of positive integers such that $|f_{k_n}(x_{l_n})| > \varepsilon$, $n \in N$. Let $g_n = f_{k_n}$ and $y_n = x_{l_n}$. Then we can write

$$\lim_{n \to \infty} g_n(x) = 0 \tag{3}$$

for each $x \in \mathfrak{X}$ and

$$|g_n(y_n)| > \varepsilon \tag{4}$$

for each $n \in N$. Let $x_{ij} = |g_i(y_j)|$ for $i \neq j$ and $x_{ii} = 0$. Since $\{x_n\}$ is quasi-unconditionally G-convergent, we have $f(x_n) \to 0$ for each $f \in TG$. Hence we have $f(y_n) \to 0$ for each $f \in TG$. This implies that $\lim_{j \to \infty} x_{ij} = 0$. From (3) it follows that $\lim_{i \to \infty} x_{ij} = 0$. Moreover, we have $\lim_{i \to \infty} x_{ii} = 0$. Thus, by the Diagonal Theorem for nonnegative matrices, there exists an increasing sequence of positive integers p_n such that

$$\lim_{i \to \infty} \sum_{j=1}^{\infty} x_{p_i p_j} = 0. \tag{5}$$

Let $g_{p_j} = h_j$ and $y_{p_j} = z_j$. Then, by (3), (4) and (5) we can write, respectively,

$$\lim_{j \to \infty} h_j(x) = 0, \tag{6}$$

$$|h_j(z_j)| > \varepsilon, \tag{7}$$

$$\lim_{i \to \infty} \sum_{j=1,\, j \neq i}^{\infty} h_i(z_j) = 0. \tag{8}$$

We note that $\{z_j\}$ is a subsequence of a quasi-unconditionally G-convergent sequence $\{x_j\}$. Thus, there exists a subsequence $\{z_{p_j}\}$ of $\{z_j\}$ and an element $z \in \mathfrak{X}$ such that $\Sigma z_{p_j} = z(G)$. Since $h_{p_i} \in TG$, we can write $\Sigma_{j=1}^{\infty} h_{p_i}(z_{p_j}) = h_{p_i}(z)$ for each $i \in N$. Since

$$\left| \sum_{j=1}^{\infty} h_{p_i}(z_{p_j}) \right| \geq |h_{p_i}(z_{p_i})| - \left| \sum_{j=1,\, j \neq i}^{\infty} h_{p_i}(z_{p_j}) \right|$$

and

$$\left|\sum_{j=1,\, j\neq i}^{\infty} h_{p_i}(z_{p_j})\right| \leq \sum_{j=1,\, j\neq i}^{\infty} |h_{p_i}(z_{p_j})|,$$

we have

$$|h_{p_i}(z)|+\sum_{j=1,\, j\neq i} |h_{p_i}(z_{p_j})| \geq |h_{p_i}(z_{p_i})|.$$

Hence, by (6) and (8) we have $\lim_{i\to\infty} h_{p_i}(z_{p_i})=0$ which leads to contradiction with (7). This contradiction implies our assertion.

3. We say that G is a quasi-unconditional convergence iff each element of G is a quasi-unconditionally G-convergent sequence. Thus if $x_n \to o(G)$, then for each subsequence $\{y_n\}$ of $\{x_n\}$ there exists a subsequence $\{z_n\}$ and an element $z \in \mathfrak{X}$ such that $\Sigma z_n = z(G)$.

Theorem 2. *If G is a quasi-unconditional convergence, then* $\overline{TG} = TG$.

Proof. Let $\{f_n\}$ be a sequence of elements f_n from TG and let f be a function such that

$$f(x)=\lim_{n\to\infty} f_n(x) \tag{9}$$

for each $x \in \mathfrak{X}$. Let $\{x_n\}$ be a sequence such that $x_n \to O(G)$. We have to show that

$$\lim_{n\to\infty} f(x_n)=0. \tag{10}$$

At first we shall show that

$$\lim_{m\to\infty} \sup_n f_n(x_m)=0. \tag{11}$$

If (11) does not hold, then there exist a positive number ε, an increasing sequence $\{k_n\}$ and a sequence $\{l_n\}$ such that $|f_{l_n}(x_{k_n})| > \varepsilon$ for each $n \in N$. Since $f_n \in TG$ and $x_n \to o(G)$, we see that $\lim_{m\to\infty} f_{l_n}(x_{k_m})=0$ for each $n \in N$. Thus we may assume that $\{l_n\}$ is an increasing sequence. Let $g_n = f_{l_n}$ and $y_n = x_{k_n}$. Then by (9) we have

$$\lim_{n\to\infty} g_n(x)=0 \tag{12}$$

for each $x \in \mathfrak{X}$ and

$$|g_n(y_n)| > \varepsilon. \tag{13}$$

Since $\lim_{m\to\infty} g_n(y_m)=0$, $n \in N$, we can select an increasing sequence $\{p_n\}$ such that

$$\lim_{n\to\infty} g_{p_n}(y_{p_{n+1}})=0. \tag{14}$$

Let $h_n(x) = g_{p_{n+1}}(x) - g_{p_n}(x)$. Then by (12) we can write $\lim_{n\to\infty} h_n(x)=0$ for each $x \in \mathfrak{X}$. Since $\{y_{p_{n+1}}\}$ is a subsequence of the quasi-unconditionally G-convergent sequence, the sequence $\{y_{p_{n+1}}\}$ is quasi-unconditionally G-convergent. Consequently, by Theorem 1, we can write

$$\lim_{m\to\infty} \sup_n |h_n(y_{p_{m+1}})| = 0.$$

Hence we have

$$\lim_{n\to\infty} h_n(y_{p_{n+1}})=0. \tag{15}$$

On the other hand, $|h_n(y_{p_{n+1}})|=|g_{p_{n+1}}(y_{p_{n+1}})-g_{p_n}(y_{p_{n+1}})|$. Hence by (15) and (14) we have $\lim_{n\to\infty} g_{p_{n+1}}(y_{p_{n+1}})=0$ which leads to contradiction with (13). This contradiction implies (11). To prove (10) we note that $|f(x_m)|\leq|f(x_m)-f_n(x_m)|+|f_n(x_m)|$. Hence from (9) and (11) we obtain (10). This proves that $f\in TG$ and consequently $\overline{TG}\subset TG$. Since $TG\subset \overline{TG}$, we have $TG=\overline{TG}$, which was to be proved.

In the case when $\mathfrak{X}$ is a linear space and G is a linear convergence such that each subsequence of a sequence in G contains an unconditionally convergent subsequence, the Theorem 2 reduces to [7, theorem VIII. 2].

When $\mathfrak{X}$ is a Banach space and G is the set of all sequences converging to 0, then the Theorem 2 is equivalent to the Banach Theorem.

If $\mathfrak{X}$ is a σ-ring of sets and G is the set of all disjoint sequences $\{E_n\}$ and nonincreasing sequences $\{E_n\}$ to the empty set, then the Theorem 2 is equivalent to the Nikodym theorem [6].

Final remark. One can easily see that all considerations in this paper make sense and remain valid if one replaces the additive real functions by additive functions on $\mathfrak{X}$ with values in a quasi-normed group [1].

REFERENCES

1. P. Antosik. On the Mikusinski Diagonal Theorem. *Bull. Acad. Polon. Sci., Sér. Math. Astronom. Phys.*, **19**, 1971, No. 4, 305—310.
2. P. Antosik. Mappings from *L*-groups into topological groups, II. *Bull. Acad. Polon. Sci., Sér. Math. Astronom. Phys.*, **21**, 1973, No. 2, 155—160.
3. P. Antosik. A diagonal theorem for nonnegative matrices and equicontinuous sequences of mappings. *Bull. Acad. Polon. Sci., Sér. Math. Astronom. Phys.*, **24**, 1976, No. 10, 855—860.
4. J. Mikusinski. Convergence and locally convex spaces. *Bull. Acad. Polon. Sci., Sér. Math. Astronom. Phys.*, **23**, 1975, No. 11, 1171—1174.
5. J. Mikusiński. A theorem on vector matrices and its applications in measure theory and functional analysis. *Bull. Acad. Polon. Sci., Sér. Math. Astronom. Phys.*, **18**, 1970, No. 4, 193—196.
6. O. M. Nikodym. Sur les suits convergents de fonctions parfaiment additives d'ensemble abstrait. *Monatshefte für Math. und Physik*, **40**, 1933, 427—432.
7. С. Л. Соболев. Введение в теорию кубатурных формул. Москва, 1974.

Mathematical Laboratory, Department of Complex Automation, Polish Academy of Sciences, Wieczorka 8, 40—013 Katowice

Received 14. 12. 1976