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A DIAGONAL THEOREM, UNIFORM BOUNDEDNESS AND EQUICONTINUITY THEOREMS FOR TRIANGLE SET FUNCTIONS

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A nonnegative set function μ is triangle iff $\mu(A \cup B) \leqq \mu(A) + \mu(B)$ and $\mu(A) \leqq \mu(A \cup B) + \mu(B$ for any sets A, B from a given collection of sets. A set function is continuous if it vanishes on a disjoint sequence. It is proved that any pointwise bounded family of triangle and continuous set functions is uniformly bounded and any pointwise convergent to null sequence of such functions is uniformly convergent on each disjoint sequence of sets.

1. By N we denote the set of all positive integers and by $\mathfrak{N}$ the collection of all nonempty subsets of N. R_∞ will denote the extended nonnegative real number system with usual addition and convergence.

A set function μ from $\mathfrak{N}$ to R_∞ is said to be continuous iff $\lim_{n\to\infty} \mu(E_n) = 0$ for any disjoint sequence $\{E_n\}$ in $\mathfrak{N}$. It is subadditive iff $\mu(A \cup B) \leqq \mu(A) + \mu(B)$ for any disjoint sets A, B in $\mathfrak{N}$.

Instead of writing $\mu(\{j\})$, $\mu(\{i_1, \ldots i_k\})$, $\mu(\{i_1, i_2, \ldots\})$ we shall write $\mu(j)$, $\mu(i_1, \ldots, i_k)$ and $\mu(i_1, i_2, \ldots)$, respectively.

Diagonal Theorem (for set functions). *Let μ_i $(i \in N)$ be a subadditive and continuous set functions on $\mathfrak{N}$ such that* $\lim_{i\to\infty} \mu_i(j) = 0$ $(j \in N)$ *and* $\lim_{i\to\infty} \mu_i(i) = 0$. *Then there exists an infinite set I, $I \subset N$, such that* $\lim \mu_i(I) = 0$, $i \in I$, $i \to \infty$.

We show that this Diagonal Theorem can successfully be used in proofs of theorems on uniform boundedness and equicontinuity of families and sequences of triangle and continuous set functions. In turn these theorems are proved for additive and continuous set functions on a σ-ring $\mathfrak{R}$ with values in triangle-normed groupoids or quasi-normed groups. The Diagonal Theorem for set functions replaces more sophisticated auxiliary diagonal type or other theorems [1; 2; 5; 6; 7]. The Diagonal Theorem for set functions and the diagonal type theorems [1] are independent in the sense that the knowledge of one of them does not help in proving the other. At the end of the paper it is shown that Phillip's Lemma [8] follows from the Diagonal Theorem.

2. At first let us prove the following

Lemma 1. *If μ is a continuous set function from $\mathfrak{N}$ to R_∞, then for each positive number ε and for each infinite set A of positive integers there exists an infinite subset B such that for each $C \subset B$ we have $\mu(C) < \varepsilon$.*

Proof. Let B_n $(n \in N)$ be infinite pairwise disjoint subsets of A. We assert that there exists an index n_0 such that for each $C \subset B_{n_0}$ we have $\mu(C) < \varepsilon$. In fact if this is not true then for each $n \in N$ there exists a set C_n with $C_n \subset B$ and $\mu(C_n) \geqq \varepsilon$. Since $\{B_n\}$ is a disjoint sequence, so is $\{C_n\}$. Consequently μ is not continuous. This implies our assertion.

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Proof of the Diagonal Theorem (for set functions). We put $i_0=0$, $I_0=N$ and select positive integers i_n and infinite sets I_n such that for each $n \in N$

(i) $i_{n-1}<i_n$,

(ii) $I_{n-1} \supset I_n$,

(iii) $i_n<I_n$,

(iv) $\mu_{i_n}(i_1, \ldots, i_n \cup A)<1/n$ for each $A \subset I_{n+1}$,

(v) $i_{n+1} \in I$,

(vi) $\mu_{i_{n+1}}(i_1, \ldots, i_n)<1/3(n+1)$,

(vii) $\mu_{i_{n+1}}(i_{n+1})<1/3(n+1)$.

Since $\lim_{i\to\infty} \mu_i(i)=0$, there exists an index r such that $\mu_i(i)<1/2$ for $i \geqq r$. We put $i_1=r$. By Lemma 1 there exists an infinite subset I_1 of I_0 such that $I_1 \subset I_0$, $i_1<I_0$ and $\mu_{i_1}(A)<1/2$ for each $A \subset I_1$. Then by the subadditivity of μ_i we have $\mu_{i_1}(i_1 \cup A) \leqq \mu_{i_1}(i_1)+\mu_{i_1}(A)<1$ for each $A \subset I_1$. Thus we see that (i)—(iv) hold for $n=1$. Since $\lim_{i\to\infty} \mu_i(i_1)=0$ and $\lim_{i\to\infty} \mu_i(i)=0$, there exists an index such that $\mu_i(i_1)<1/3.2$ and $\mu_i(i)<1/3.2$ for $i \geqq r$. For I_1 is an infinite set, here exists an index i_2 such that $i_2 \in I_1$ and $i_2 \geqq r$. Therefore also (v)—(vii) told for $n=1$. Assume that we have already found $i_1, \ldots, i_{p+1}$ and $I_1, \ldots, I_{p+1}$ huch that (i)—(vii) hold for $n=p$. By Lemma 1 there exists an infinite set s_{p+2} such that $I_{p+2} \subset I_{p+1}$, $i_{p+1}<I_{p+2}$ and $\mu_{i_{p+1}}(A)<1/2(p+1)$ for each $A \subset I_{p+2}$. Thus we see that (i)—(iii) hold for $n=p+1$. Moreover, we have

$$\mu_{i_{p+1}}(i_1, \ldots, i_{p+1} \cup A) \leqq \mu_{i_{p+1}}(i_1, \ldots, i_p)+\mu_{i_{p+1}}(i_{p+1})+\mu_{i_{p+1}}(A)<1/(p+1).$$

Therefore also (iv) holds for $n=p+1$. Since $\lim_{i\to\infty} \mu_i(j)=0$, $j \in N$ and μ_i is subadditive, there exists an index r such that $\mu_i(i_1, \ldots, i_{p+1})<1/3(p+2)$ and $\mu_i(i)<1/3(p+2)$ for $i \geqq r$. For I_{p+1} is an infinite set, there is i_{p+2} such that $i^p{}_{+2} \in I_{p+1}$ and $i_{p+2} \geqq r$. Then we see that also (v)—(vii) hold for $n=p+1$. By induction the existence of i_n and I_n such that (i)—(vii) hold follows. Let $I=\{i_1, i_2, \ldots\}$. By (i) we see that I is an infinite set of positive integers. By (ii) and (v) we have $\{i_{n+1}, i_{n+2}, \ldots\}=A$, $A \subset I_n$ and by (iv) we can write $\mu_{i_n}(I)=\mu_{i_n}(i_1, \ldots, i_n \cup A)<1/n$. Hence our assertion follows.

The Diagonal Theorem for set functions implies the following

Diagonal Theorem (for nonnegative matrices). *Let* x_{ij} $(i, j \in N)$ *be a matrix of nonnegative numbers such that* $\lim_{j\to\infty} x_{ij}=0$, $(i \in N)$, $\lim_{i\to\infty} x_{ij}=0$, $(j \in N)$ *and* $\lim_{i\to\infty} x_{ii}=0$. *Then there exists an infinite set* I, $I \subset N$, *such that* $\Sigma_{i,j\in I}\, x_{ij}<\infty$.

To prove this theorem it is enough to put $\mu_n(E)=\Sigma_{j \in E}\, x_{ij}$ for each $E \in \mathfrak{N}$ and to apply the Diagonal Theorem for set functions.

3. A set function μ from a collection $\mathfrak{R}$ of sets to R_∞ is said to be triangle iff μ satisfied the conditions

(T1) $\mu(B \cup B) \leqq \mu(A)+\mu(B)$,

(T2) $\mu(A) \leqq \mu(A \cup B)+\mu(B)$

for any disjoint sets A, B in $\mathfrak{R}$ such that $A \cup B \in \mathfrak{R}$.

If μ is a triangle set function, then we have $\mu(A) \leqq \mu(B)+\mu(B \setminus A)$.

In the sequel we shall write $\sup_E$, $\sup_m$, $\sup_\mu$ instead of writing $\sup_{E\in\mathfrak{R}}$, $\sup_{m\in N}$, $\sup_{\mu\in\mathfrak{M}}$, respectively, where $\mathfrak{M}$ denotes the family of triangle and continuous set functions.

Lemma 2. *Let $\mathfrak{R}$ be an algebra (σ-ring) of subsets of a set S and let μ_n $(n \in N)$ be triangle set functions on $\mathfrak{R}$. Then we have*

$$\lim_{n\to\infty} \sup_E \mu_n(E) = 0, \tag{*}$$

iff

$$\lim_{n\to\infty} \sup_m \mu_n(E_m) = 0 \tag{**}$$

for each disjoint sequence $\{E_m\}$ in $\mathfrak{R}$.

Proof. Evidently $(*)$ implies $(**)$. Suppose that $(*)$ does not hold. If there exists a set $E \in \mathfrak{R}$ such that $\lim_{n\to\infty} \mu_n(E) \geqq \varepsilon > 0$, then putting $E_1 = E$ and $E_n = \varnothing$ for $n = 2, 3, \ldots$ we see that also $(**)$ does not hold. Therefore we may additionaly assume that

$$\lim_{n\to\infty} \mu_n(E) = 0 \tag{1}$$

for each $E \in \mathfrak{R}$. Under this additional assumption we select a positive number ε, positive integers k_n and sets A_n such that for each $n \in N$

(i) $k_{n-1} < k_n$,

(ii) $A_{n-1} \supset A_n$,

(iii) $\mu_{k_n}(A_{n-1}) < \varepsilon$ and $\mu_{k_n}(A_n) > 2\varepsilon$

and

(iv) $\lim_{m\to\infty} \sup_E \mu_m(A_n \cap E) \geqq 2\varepsilon$.

Since $(*)$ does not hold, there exists a positive number ε such that

$$\lim_{n\to\infty} \sup_E \mu_n(E) > 2\varepsilon. \tag{2}$$

We put $k_0 = 0$ and $A_0 = S$. By the additional assumption (1) there exists an integer r such that $\mu_n(A_0) < \varepsilon$ for $n \geqq r$. Then there exist sets $E_1, E_2, \ldots$ such that

$$\lim_{n\to\infty} \sup_m \mu_n(E_m) > 2\varepsilon.$$

Let $A = \bigcup_{m=1}^{\infty} E_m$. Then we have

$$\lim_{n\to\infty} \sup_E \mu_n(E \cap A) > 2\varepsilon. \tag{3}$$

Thus there exist a set E_1 and an index $k_1 \geqq r$ such that $\mu_{k_1}(E_1 \cap A) > 2\varepsilon$. Assuming $A_1 = E_1 \cap A$ we see that (i)—(iv) hold for $n = 1$. Suppose that we have already found $k_1, \ldots, k_p$, $A_1 \ldots, A_p$ such that (i)—(iv) hold for $n = p$. By the additional assumption there exists an $r > k_p$ such that $\mu_n(A_{k_p}) < \varepsilon$ for each $n \geqq r$. Since (iv) holds, there exist sets $E_1, E_2, \ldots$ such that $\lim_{n\to\infty} \sup_m \mu_n (A_{k_p} \cap E_m) > 2\varepsilon$. Let $A = \bigcup_{m=1}^{\infty} A_{k_p} \cap E_m$. Then we can write $\lim_{n\to\infty} \sup_E \mu_n(E \cap A) > 2\varepsilon$. Thus there exists a set E_0 and an index k_{p+1} such that $k_{p+1} > \max(k_p, r)$ and $\mu_{k_{p+1}}(E_0 \cap A) > \varepsilon$. Assuming $A_{p+1} = E_0 \cap A$ we see that (i)—(iv) hold for $n = p+1$. Therefore, the existence of k_n and A_n with (i)—(iv) hold for $n \in N$ follows by induction.

Let $E_m = A_m \setminus A_{m+1}$, $m \in N$. Then by (ii) $\{E_m\}$ is a disjoint sequence in $\mathfrak{R}$. Moreover, since μ_n $(n \in N)$ are triangle set functions, we can write $\mu_{k_n}(E_n) \geqq \mu_{k_n}(A_{n+1}) - \mu_{k_n}(A_n) > \varepsilon$ by (iii). By (i) we see that $k_n \to \infty$. This implies that

$\lim_{n\to\infty}\sup_m \mu_n(E_m)>\varepsilon$. Thus we have proved that if (*) does not hold then also (**) does not hold. This implies our assertion in the case when $\mathfrak{R}$ is an algebra of sets. The case when $\mathfrak{R}$ is a σ-ring can be easily reduced to what have been proved. In fact, if (*) does not hold, then there exists a sequence of sets E_m in $\mathfrak{R}$ such that

$$\lim_{n\to\infty}\sup_m \mu_n(E_m)\neq 0.$$

Assuming $S=\bigcup_{m=1}^{\infty} E_m$ we see that S is an algebra and (*) does not hold with respect to S. Hence by what have been proved our assertion follows.

A set function μ_n from a collection $\mathfrak{R}$ of sets is continuous, iff $\lim_{n\to\infty} \mu(E_n)=0$ for each disjoint sequence $\{E_n\}$ in $\mathfrak{R}$.

Theorem 1. *Let $\mu_n (n\in N)$ be triangle and continuous set functions on a σ-ring $\mathfrak{R}$ such that*

$$\lim_{n\to\infty}\mu_n(E)=0 \tag{4}$$

for each $E\in\mathfrak{R}$. Then we have

$$\lim_{n\to\infty}\sup_E \mu_n(E)=0. \tag{5}$$

Proof. If (5) does not hold, then by Lemma 2 there exists a disjoint sequence $\{E_m\}$ in $\mathfrak{R}$ and $\varepsilon>0$ such that

$$\lim_{n\to\infty}\sup_m \mu_n(E_m)\geqq\varepsilon. \tag{6}$$

Thus there are two increasing sequences of positive integers k_n and l_n such that

$$\mu_{k_n}(E_{l_n})>\varepsilon. \tag{7}$$

Let $\nu_i(A)=|\mu_{k_i}(\bigcup_{j\in A} E_{kj})|$ for $A\urcorner\supset\{i\}$ and $\nu_i(A)=0$ if $A\supset\{i\}$. It is easy to verify that $\nu_n (n\in N)$ are triangle and continuous set functions on $\mathfrak{R}$ and $\lim_{i\to\infty}\nu_i(j)=0$, $(j\in N)$. Hence by the Diagonal Theorem for set functions there exists an infinite set I, $I\subset N$ such that

$$\lim_{i\to\infty,\, i\in I}\nu_i(I)=0. \tag{8}$$

Since $\mu_{k_n}(n\in N)$ are triangle, we have

$$\mu_{k_i}(F)+\mu_{k_i}(F_i)\geqq\mu_{k_i}(E_{k_i}), \tag{9}$$

where $F=\bigcup_{j\in I}E_{k_j}$ and $F_i=\bigcup\{F_{k_j}\,|\,j\in I,\ j\neq i\}$. For $\nu_i(I)=\mu_{k_i}(F_i)$ we may write

$$\lim_{i\to\infty}\mu_{k_i}(E_{k_i})=0 \tag{10}$$

by (8), (4) and (9). But (10) contradicts to (6). This contradiction implies our assertion.

Theorem 2. *Let $\mathfrak{M}$ be a family of triangle and continuous set functions on a σ-ring $\mathfrak{R}$ such that*

$$\sup_\mu \mu(E)<\infty$$

for each $E\in\mathfrak{R}$. Then we have

$$\sup_E \sup_\mu \mu(E) < \infty.$$

P r o o f. If the theorem is false, then there exist a sequence $\{E_n\}$ in $\mathfrak{R}$ and a sequence $\{k_n\}$ of positive integers such that

$$\lim_{n\to\infty} \mu_{k_n}(E_n) = \infty.$$

Let $\nu_n(E) = \mu_{k_n}^{-1}(E_{k_n})\mu_{k_n}(E)$. Evidently ν_n $(n \in N)$ are triangle and continuous set functions on $\mathfrak{R}$ and $\lim_{n\to\infty} \nu_n(E) = 0$. Therefore, by Theorem 1 we have

$$\lim_{n\to\infty} \sup_E \nu_n(E) = 0.$$

On the other hand, $\nu_n(E_{k_n}) = 1$ for $n \in N$. Hence we have

$$\lim_{n\to\infty} \sup_E \nu_n(E) \geqq 1.$$

This contradiction implies our assertion.

4. This section is concerned with additive and continuous set functions from a σ-ring $\mathfrak{R}$ to a triangle normed groupoid X.

A triangle normed groupoid is a set X (called groupoid) with a binary operation $(x, y) \to x+y$ and with a function (called a triangle norm) $|x|$ on X with values in R_∞ such that

(N1) $|x+y| \leq |x| + |y|$,

(N2) $|x| \leqq |x+y| + |y|$

for any x, y in X.

We see that if μ is an additive set function from $\mathfrak{R}$ to a triangle normed groupold X and $\nu(E) = |\mu(E)|$, then ν is a triangle function on $\mathfrak{R}$. Morover, if μ is continuous, i. e., $\lim_{n\to\infty} \mu(E_n) = 0$ for each disjoint sequence $\{E_n\}$ in $\mathfrak{R}$, then ν is continuous.

T h e o r e m 3. *Let μ_n $(n \in N)$ be additive and continuous set functions on a σ-ring $\mathfrak{R}$ with values in a triangle normed groupoid X such that*

$$\lim |\mu_n(E)| = 0$$

for each $E \in \mathfrak{R}$. Then we have

$$\lim_{n\to\infty} \sup_E |\mu_n(E)| = 0.$$

P r o o f. To prove the theorem it is enough to consider the family of functions ν such that $\nu(E) = |\mu(E)|$, $\mu \in \mathfrak{M}$, and apply Theorem 1.

T h e o r e m 4. *Let $\mathfrak{M}$ be a family of additive and continuous set functions on a σ-ring $\mathfrak{R}$ with values in a triangle normed groupoid X such that*

$$\sup_\mu |\mu(E)| < \infty$$

for each $E \in \mathfrak{R}$. Then we have

$$\sup_E \sup_\mu |\mu(E)| < \infty.$$

P r o o f. To prove the theorem it is enough to consider the family of unctions ν such that $\nu(E) = |\mu(E)|$, $\mu \in \mathfrak{M}$, and to apply Theorem 2.

When X is a normed group, Theorem 4 reduces to the case proved in [5]. When X is a quasi-normed group and members of μ are countable additive set functions the theorem reduces to the case proved in [7].

5. We recall that a quasi-normed group is an abelian group X with a function (called a quasi-norm) $|x|$ on X such that $|0|=0$, $|-x|=|x|$ and $|x+y|\leqq |x|+|y|$. We write $x_n \to x$, iff $|x_n-x|\to 0$. A sequence $\{x_n\}$ is said to be fundamental, iff for any increasing sequence of positive integers p_n we have $|x_{p_{n+1}}-x_{p_n}|\to 0$.

Theorem 5. *Let μ_n $(n\in N)$ be additive and continuous set functions on a σ-ring $\mathfrak{R}$ with values in a quasi-normed group X such that $\{\mu_n(E)\}$ is a fundamental sequence for each $E\in\mathfrak{R}$ and let $\{E_m\}$ be a sequence in $\mathfrak{R}$. Then we have*

(i) $$\lim_{m\to\infty} \mu_n(E_m)=0 \text{ uniformly on } N,$$

iff

(ii) $$\lim_{m\to\infty} \mu_n(E_m)=0 \text{ for each } n\in N.$$

Moreover if $\mu(E)=\lim_{n\to\infty}\mu_n(E)$, $E\in\mathfrak{R}$, then μ is an additive continuous set function and

(iii) $$\lim_{m\to\infty} \mu(E_m)=0.$$

Proof. Evidently (i) implies (ii). Suppose that (ii) holds. If (i) does not hold, then there exist two increasing sequences of positive integers $\{k_n\}$ and $\{l_n\}$ and a positive number ε such that $|\mu_{k_n}(E_{l_n})|>\varepsilon$ for each $n\in N$. We may assume that $k_n=n$ and $l_n=n$ for $n\in N$. Then we can write

(11) $$|\mu_n(E_n)|>\varepsilon.$$

Since (ii) holds, we can select an increasing sequence of positive integers p_n such that

(12) $$\lim_{n\to\infty} \mu_{p_n}(E_{p_{n+1}})=0.$$

Let now $\nu_n(E)=\mu_{p_{n+1}}(E)-\mu_{p_n}(E)$. Since $\{\mu_n(E)\}$ is a fundamental sequence for each $E\in\mathfrak{R}$, we see that $\lim_{n\to\infty}\nu_n(E)=0$ for each $E\in\mathfrak{R}$. Hence by Theorem 3

$$\lim_{n\to\infty}\sup_E |\mu_{p_{n+1}}(E)-\mu_{p_n}(E)|=0.$$

This implies that

$$\lim_{n\to\infty} |\mu_{p_{n+1}}(E_{p_{n+1}})-\mu_{p_n}(E_{p_{n+1}})|=0.$$

Hence by (12) it follows that

$$\lim_{n\to\infty} \mu_{p_n}(E_{p_{n+1}})=0,$$

which leads to contradiction with (11). Thus (ii) implies (i). To prove (iii) we note that

$$|\mu(E_m)|\leqq|\mu(E_m)-\mu_n(E_m)|+|\mu_n(E_m)|.$$

Hence and from (i) follows (iii). This completes the proof.

If in Theorem 4 $\{E_m\}$ is a disjoint sequence in $\mathfrak{R}$, then (ii) holds and consequently (i) and (iii) hold for disjoint sequences. In other words, any pointwise convergent sequence of additive and continuous set functions is uniformly continuous and the limit is continuous. This reduces to the case proved in [4] when X is a normed space.

Let μ be a countable additive set function on a σ-ring $\mathfrak{R}$, i. e., $\lim_{n\to\infty}\mu(E_n)=0$ for each sequence $\{E_n\}$ such that $E_n\supset E_{n+1}$ and $\bigcap_{n=1}^{\infty}E_n=\varnothing$. Then it is easy to verify that μ is an additive and continuous set function. Thus if in Theorem 4 μ_n $(n\in N)$ are countable additive set functions and $\{E_m\}$ is a nondecreasing to the empty set, then (ii) holds and consequently (ii) and (iii) hold for each nonincreasing to the empty set sequence. This means that the sequence $\{\mu_n\}$ is uniformly countable additive and that its pointwise limit is a countable additive set function. This is what Nikodym's type theorems on uniform countable additivity say. When X is a Banach space this reduces to the case proved in [6]. When X is a quasi-normed group this reduces to the case proved in [2].

Let ν be a nonnegative measure on a σ-ring $\mathfrak{R}$ and let μ be a set function on $\mathfrak{R}$ with values in a quasi-normed group X. The set function μ is ν-continuous iff $\nu(E_n)\to 0$ as $n\to\infty$ implies $\mu(E_n)\to 0$ as $n\to\infty$. Let $\{E_m\}$ be a sequence in $\mathfrak{R}$ such that $\nu(E_n)\to 0$ as $n\to\infty$, let $E=\bigcup_{m=1}^{\infty}E_m$ and let $\mathfrak{R}'=\{F\cap E: F\in\mathfrak{R}\}$. Then $\mathfrak{R}'$ is a σ-ring and for any disjoint sequence $\{F_n\}$ in $\mathfrak{R}'$ we have $\nu(F_n)\to 0$ as $n\to\infty$. Consequently $\mu(F_n)\to 0$ as $n\to\infty$ for any disjoint sequence $\{F_n\}$ in $\mathfrak{R}'$. This shows that μ is a continuous function in $\mathfrak{R}'$. Thus if μ_n $(n\in N)$ are additive and ν-continuous set functions on a σ-ring $\mathfrak{R}$ and $\{E_n\}$ is a sequence in $\mathfrak{R}$ such that $\nu(E_n)\to 0$ as $n\to\infty$, then μ_n are additive continuous set functions on $\mathfrak{R}'$ and (ii) holds for the sequence $\{E_m\}$. Therefore, by Theorem 4 (ii) and (iii) hold for the sequence $\{E_m\}$. This is what Vitali-Hahn-Saks type theorems say. When X is the real number system this is the case proved by Vitali, Hahn and Saks. When X is a Banach space this is the case proved in [6].

Theorem 5. *Let μ_n $(n\in N)$ be additive and continuous set functions on a σ-ring $\mathfrak{R}$ with values in a quasi-normed group X such that*

$$\lim_{n\to\infty}\mu_n(E)=\mu(E) \tag{13}$$

for each $E\in\mathfrak{R}$. Then we have

$$\lim_{n\to\infty}\sup_{E}|\mu_n(E)-\mu(E)|=0. \tag{14}$$

Moreover, if X is a complete quasi-normed group and $\{E_m\}$ is a disjoint sequence in $\mathfrak{R}$, then we have

$$\lim_{n\to\infty}\sup_{M\in\mathfrak{N}}\Big|\sum_{m\in M}[\mu_n(E_m)-\mu(E_m)]\Big|=0. \tag{15}$$

Proof. By Theorem 4 μ is an additive and continuous set function. Therefore, the difference $\nu_n=\mu_n-\mu$ is an additive and continuous set function, moreover, $\lim_{n\to\infty}|\nu_n(E)|=0$ for each $E\in\mathfrak{R}$. Thus by Theorem 3 we have

$$\lim_{n\to\infty}\sup_{E}|\nu_n(E)|=0,$$

which is equivalent to (14). To prove (15) we take an arbitrary disjoint sequence $\{E_m\}$ in $\mathfrak{R}$. Since X is complete quasi-normed group and $\nu_n(E) = \mu_n(E) - \mu(E)$ is continuous, we infer that for each $M \subset N$ and $n \in N$, the series $\sum_{m \in M} \nu_n(E_m)$ is convergent. Let $\{M_n\}$ be an arbitrary sequence of sets in $\mathfrak{N}$ and let ε_n be a sequence of positive numbers tending to 0. Then for each $n \in N$, there exists a finite subset K_n of M_n such that

$$\left| \sum_{m \in M_n} \nu_n(E_m) \right| \leq \left| \sum_{m \in K_n} \nu_n(E_m) \right| + \varepsilon_n. \tag{16}$$

Since K_n $(n \in N)$ are finite and ν_n $(n \in N)$ are finitely additive, we can write

$$\sum_{m \in K_n} \nu_n(E_m) = \nu_n(F_n),$$

where $F_n = \bigcup_{m \in K_n} E_m$ for $n \in N$. Hence by (16) we have.

$$\lim_{n \to \infty} \sup_j \left| \sum_{m \in M_j} \nu_n(E_m) \right| = 0.$$

Since $\{M_n\}$ was an arbitrary sequence in $\mathfrak{R}$ this implies (15). Thus the proof of the theorem is complete.

In the case when X is the real number system the second part of the Theorem 5 is equivalent to Phillip's Lemma proved in [8].

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