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ON DUAL SPACES OF LOCALLY CONVEX SPACES DEFINED BY IDEALS OF OPERATORS

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Let $\mathcal{A}$ be an ideal of linear operators between Banach spaces according to A. Pietsch. A locally convex space E is called an $\mathcal{A}$-space if it admits a projective canonical spectrum containing only operators of the ideal $\mathcal{A}$. In this paper it will be proved that if $\mathcal{A}$ is an injective, symmetric and complete metric space ideal then the strong dual space of each locally convex metrizable $\mathcal{A}$-space is an $\mathcal{A}$-space again. This statement generalizes the well-known relevant results for nuclear, semi-Schwartz and Schwartz spaces.

1. Many facts about nuclear, strongly nuclear and Schwartz spaces may be proved without using special properties of nuclear or compact operators, but only that of being an ideal. The concept of $\mathcal{A}$-spaces was introduced by A. Pietsch [8], where $\mathcal{A}$ denotes an ideal of operators between Banach spaces (see definition 2.1). This approach enables a simultaneous treatment of many classes of locally convex spaces (l. c. s.).

A very important theorem of the theory of nuclear spaces asserts that the strong dual space of any nuclear F- or DF-space is also nuclear. That is why Pietsch [9] raised the question under what assumptions on the ideal $\mathcal{A}$ the strong dual of an $\mathcal{A}$-space is of the respective type again. We will call this the *dual space problem* and discuss it in this paper.

It turns out that it is useful to introduce two further classes of l. c. s., called co-$\mathcal{A}$-spaces and mix-$\mathcal{A}$-spaces, because the dual space problem can be treated only by using the interrelationship between these three classes (see definition 2.1). These initiating considerations will be done in section 2. Martineau pointed out in 1964 that the strong dual of each nuclear F-space is even strongly nuclear. These peculiarities will be described generally in section 3 and will be important for the following sections. In section 4, in the main part of this paper, the dual space problem will be solved for F-spaces (see Theorem 4.1 and Corollary 4. 2). Section 5 deals with applications to several classical ideals. Furthermore, there will be shown that the dual space problem in the case of DF-spaces is not solvable for numerous ideals $\mathcal{A}$. For the class of all l. c. s. the dual space problem is not solvable if the ideal $\mathcal{A}$ is different from the class $\mathfrak{L}$ of all linear bounded operators between Banach spaces. This can be proved as follows. Let B be any Banach space. According to [3] there is a quasicomplete (of course not metrizable) nuclear space E with $E'_b=B$. Accordingly to Mackey we have $(E_\sigma)'_b=E'_b=B$. Since E_σ surely is an $\mathcal{A}$-space, B would be an $\mathcal{A}$-space, too. But this is possible only for $1_B\in\mathcal{A}$. This shows $\mathcal{A}=\mathfrak{L}$.

2. Classes of locally convex spaces defined by operator ideals. At first some concepts of the theory of operator ideals in the sense of Pietsch [8, 9] which are necessary for the following description shall be given (see also [1, 4]).

A subclass $\mathcal{A}$ of the class $\mathcal{L}$ of all linear bounded operators between Banach spaces will be called an *operator ideal*, if the components $\mathcal{A}(E, F) = \mathcal{A} \cap \mathcal{L}(E, F)$ satisfy the following three conditions:

(I 1) $\mathcal{A}(E, F)$ contains the class $\mathcal{F}(E, F)$ of all finite dimensional operators;

(I 2) $\mathcal{A}(E, F)$ is a linear subspace of $\mathcal{L}(E, F)$;

(I 3) If $T \in \mathcal{L}(E_0, E)$, $S \in \mathcal{A}(E, F)$ and $R \in \mathcal{L}(E, F_0)$ then $RST \in \mathcal{A}(E_0, F_0)$.

The ideal $\mathcal{A}$ will be called *injective*, if any operator $T \in \mathcal{L}(E, F)$ belongs to $\mathcal{A}(E, F)$, as soon as there exists an isomorphic embedding $J: F \to F_0$ such that $JT \in \mathcal{A}(E, F_0)$.

For each operator ideal $\mathcal{A}$ there exists the *dual operator ideal* $\mathcal{A}^d$ with components $\mathcal{A}^d(E, F) = \{S \in \mathcal{L}(E, F) : S' \in \mathcal{A}(F', E')\}$.

The ideal $\mathcal{A}$ will be called *symmetric*, if $\mathcal{A} \subseteq \mathcal{A}^d$. For instance the ideals $\mathcal{F}$ (resp. $\mathcal{C}, \mathcal{V}, \mathcal{E}, \mathcal{S}_0$) of all finite dimensional (resp. compact, weakly compact, nuclear, strongly nuclear) operators are symmetric.

A functional α defined on all ideal components $\mathcal{A}(E, F)$ will be called a *quasinorm* on $\mathcal{A}$ if the following conditions are satisfied:

If $1_{\mathbf{R}}$ denotes the identical mapping of the reals, then $\alpha(1_{\mathbf{R}}) = 1$.

There exists a constant $\varkappa \geqq 1$ such, that $\alpha(S_1 + S_2) \leqq \varkappa(\alpha(S_1) + \alpha(S_2))$ holds for all $S_1, S_2 \in \mathcal{A}(E, F)$ and all Banach spaces E, F.

$T \in \mathcal{L}(E_0, E)$, $S \in \mathcal{A}(E, F)$ and $R \in \mathcal{L}(F, F_0)$ implies $\alpha(RST) \leqq \|R\| \alpha(S) \|T\|$.

It is easy to prove that $\|S\| \leqq \alpha(S)$ for each $S \in \mathcal{A}$. Therefore, a quasinorm α on $\mathcal{A}$ defines on each component $\mathcal{A}(E, F)$ of $\mathcal{A}$ a separated uniform topology. An operator ideal $\mathcal{A}$ is called a *complete metric ideal*, if it is topologized by a countable system of quasinorms such that all components $\mathcal{A}(E, F)$ are complete with respect to this topology.

Now we shall introduce the operator-ideal-defined classes of locally convex spaces. Let E be any locally convex space (l. c. s.) and let $\mathfrak{U}(E)$ be a neighbourhood basis of the origin and $\mathfrak{B}(E)$ be a basis of the bounded subsets of E. We will assume that all elements of both $\mathfrak{U}(E)$ and $\mathfrak{B}(E)$ are absolutely convex and closed. Let the gauge functionals of $U \in \mathfrak{U}(E)$ and $B \in \mathfrak{B}(E)$ be denoted by p_U and p_B, respectively, and let us define the linear spaces

$$E(B) = \bigcup_{n=1}^{\infty} nB \quad \text{and} \quad E/U = E/p_U^{-1}(0).$$

By $C_B: E(B) \to E$ and $C_U: E \to E/U$ we denote the corresponding canonical linear mappings. Now, the spaces $E(B)$ and E/U can be normed by $\|x\|_B = p_B(x)$ and $\|C_U(x)\|_U = p_U(x)$, respectively. Let their completions be denoted by E_B and E_U. The mapping $C_{BU} = C_U C_B$ can be extended to a linear bounded mapping $C_{BU}: E_B \to E_U$. For $A, B \in \mathfrak{B}(E)$ with $A \subseteq \varrho B$ and $U, V \in \mathfrak{U}(E)$ with $U \subseteq \varrho V$ for some $\varrho > 0$ there exist canonical mappings $C_{AB}: E_A \to E_B$ and $C_{UV}: E_U \to E_V$. These mappings are determined by $C_B C_{AB} = C_A$ and $C_{VU} C_V = C_U$.

Definition 2.1. *Let $\mathcal{A}$ be an operator ideal. A locally convex space E is called:*

an $\mathcal{A}$-space (denoted by $E \in \operatorname{sp}\mathcal{A}$), if for any $V \in \mathfrak{U}(E)$ there is a $U \in \mathfrak{U}(E)$ with $U \subseteq \varrho V$ and $C_{UV} \in \mathcal{A}(E_U, E_V)$;

a co-$\mathcal{A}$-space (denoted by $E \in \operatorname{co}\mathcal{A}$) if for any $A \in \mathfrak{B}(E)$ there is a $B \in \mathfrak{B}(E)$ with $A \subseteq \varrho B$ and $C_{AB} \in \mathcal{A}(E_A, E_B)$;

a mix-$\mathcal{A}$-space (denoted by $E \in \operatorname{mix} \mathcal{A}$) if for any $A \in \mathfrak{B}(E)$ and for any $U \in \mathfrak{U}(E)$ the mapping C_{AU} belongs to $\mathcal{A}(E_A, E_U)$.

Because $\mathcal{A}$ is an ideal, this definition does not depend on the chosen bases $\mathfrak{B}(E)$ and $\mathfrak{U}(E)$. If $\mathcal{A}$ is the ideal of the nuclear (compact) mappings, one will get the nuclear spaces (Schwartz spaces) and the conuclear spaces (co-Schwartz spaces) as $\mathcal{A}$-spaces and co-$\mathcal{A}$-spaces, respectively. The $\mathfrak{V}$-spaces will be called infra-Schwartz-spaces.

A straightforward computation proves Proposition 2.2:

Proposition 2.2 (Permanence Properties).

The class $\operatorname{sp} \mathcal{A}$ is closed with respect to the formation of arbitrary products (and finite direct sums).

The class $\operatorname{co} \mathcal{A}$ is closed with respect to arbitrary locally convex direct sums (and finite products).

The class $\operatorname{mix} \mathcal{A}$ is closed with respect to both arbitrary locally convex sums and products.

There hold the obvious relations $\operatorname{sp} \mathcal{A} \subseteq \operatorname{mix} \mathcal{A}$ and $\operatorname{co} \mathcal{A} \subseteq \operatorname{mix} \mathcal{A}$.

Proposition 2.3. *Let $\mathcal{A}$ be an injective operator ideal and let E be any l. c. s and E'_b its strong dual space. Then:*

$E \in \operatorname{co} \mathcal{A}^d$ implies $E'_b \in \operatorname{sp} \mathcal{A}$;

$E'_b \in \operatorname{sp} \mathcal{A}^d$ implies $E \in \operatorname{co} \mathcal{A}$. (Particularly, if $\mathcal{A}$ is also symmetric then $E \in \operatorname{co} \mathcal{A}$ and $E'_b \in \operatorname{sp} \mathcal{A}$ are equivalent [1]);

If E is quasibarelled, then $E \in \operatorname{sp} \mathcal{A}^d$ implies $E'_b \in \operatorname{co} \mathcal{A}$, and vice versa;

If E is quasibarelled and $E'_b \in \operatorname{co} \mathcal{A}^d$ then $E \in \operatorname{sp} \mathcal{A}$;

If E is quasibarelled and $E \in \operatorname{mix} \mathcal{A}^d$ then $E'_b \in \operatorname{mix} \mathcal{A}$, and vice versa;

$E'_b \in \operatorname{mix} \mathcal{A}^d$ implies $E \in \operatorname{mix} \mathcal{A}$.

Proof. Let U^0 and A^0 denote the polars of the sets $U \in \mathfrak{U}(E)$ and $A \in \mathfrak{B}(E)$, respectively. The Proposition is proved by the following statements:

(i) $(E_U)' = E'_{U^0}$;

(ii) There is an isometric embedding of E'_{A^0} into $(E_A)'$;

(iii) There is an isometric embedding of E_A into $(E'_{A^0})'$.

In view of the above proposition the dual space problem can be reformulated to the question under what assumptions is a mix-$\mathcal{A}$-space an $\mathcal{A}$- or co-$\mathcal{A}$-space. This will be answered in section 4.

3. DF-spaces belonging to $\operatorname{sp} \mathcal{A}$. In this section properties of DF-$\mathcal{A}$-spaces will be described. To do this we start from a statement given in an earlier paper [5].

Theorem 3.1 [5, Theorem 4.2]. *Let $\mathcal{A}$ be an injective operator ideal. A mapping $S: B \to B_1$ (B, B_1 Banach spaces) is said belonging to $\mathcal{A}^{sp}(B, B_1)$ if S can be linearly and continuously factored through an $\mathcal{A}$-space E. Then:*

$\mathcal{A}^{sp}$ is an injective oparator ideal contained in all powers $\mathcal{A}^n$ of $\mathcal{A}$;

$\mathcal{A}^{sp} = \mathcal{A}$ iff $\mathcal{A}$ is idempotent, i. e. $\mathcal{A} = \mathcal{A}^2$.

The ideals $\mathfrak{F}, \mathfrak{C}, \mathfrak{V}$ and $\mathcal{S}_0$ are both idempotent and injective. The ideal of the nuclear (or more exactly quasi-nuclear) operators $\mathcal{S}$ fulfils $\mathcal{S}^{sp} = \mathcal{S}_0$.

As mentioned in the introduction the strong dual of a nuclear F-space is already strongly nuclear. This is a special case of the following proposition.

Proposition 3.2. *Let $\mathcal{A}$ be an injective operator ideal and E any DF-space. Then E is an $\mathcal{A}$-space iff E is an $\mathcal{A}^{sp}$-space.*

P r o o f. Let $\{B_n\}$ be an increasing fundamental system of the closed absolutely convex bounded subsets of the $\mathcal{A}$-space E. For a given neighbourhood $V_0 \in \mathfrak{U}(E)$ we select a sequence $\{V_n\}$ of neighbourhoods of zero $V_n \in \mathfrak{U}(E)$ such that the canonical mappings $C_{V_{n+1}V_n}: E_{V_{n+1}} \to E_{V_n}$ belong to $\mathcal{A}$. Let $\{\varrho_{nm}\}_{n,m \in \mathbf{N}}$ be a system of positive real numbers with $B_m \subseteq \varrho_{mn} V_n$ and let $V = \cap\{\varrho_{nn} V_n | 0 \leq n < \infty\}$. Since the chain $B_m \subseteq B_n \subseteq \varrho_{nn} V_n$ is valid for $n \geq m$, we get

$$B_m \cap V = B_m \cap \bigcap_{n=0}^{m-1} \varrho_{nn} V_n.$$

Because of $\cap\{\varrho_{nn} V_n | 0 \leq n \leq m-1\} \in \mathfrak{U}(E)$ the set V is a neighbourhood of zero [12, IV 6. 7]. The space $E_0 = \operatorname{proj} \lim E_{V_n}$ is obviously an $\mathcal{A}$-space. Because of $V \subseteq \varrho_{nn} V_n$ the mappings $C_{VV_n}: E_V \to E_{V_n}$ are continuous and consistent with the mappings of the projective spectrum $\{E_{V_n}\}$. Therefore the mapping C_{VV_0} is factored as $E_V \to E_0 \to E_{V_0}$. This proves that C_{VV_0} belongs to $\mathcal{A}^{sp}$. By means of this proposition we will show in Section 5 that there are numerous operator ideals such that the strong dual space of a DF-$\mathcal{A}$-space is in general not an $\mathcal{A}$-space.

4. Strong duals of metrizable $\mathcal{A}$-spaces. Now we state the important theorem of the theory of $\mathcal{A}$- and co-$\mathcal{A}$-spaces.

T h e o r e m 4.1. *Let $\mathcal{A}$ be an injective and complete metric ideal of operators, the topology of which is given by an increasing system $\{\alpha_n\}_{n \in \mathbf{N}}$ of quasinorms α_n. Then each metrizable mix-$\mathcal{A}$-space E is also a co-$\mathcal{A}$-space.*

Particularly each metrizable $\mathcal{A}$-space E is a co-$\mathcal{A}$-space.

P r o o f. Let $\{U_n\}$ be a countable basis of neighbourhoods of zero of E and let $A \in \mathfrak{B}(E)$ be given. The abbreviation E_n will be used to denote the Banach spaces E_{U_n}. By assumption the mappings $C_{A,n}: E_A \to E_n$ belong to $\mathcal{A}(E_A, E_n)$. Let $\varkappa_j$ be the constant corresponding to the quasinorm α_j. We choose numbers $0 < c_n \leq 1$ which satisfy

$$c_n\Big(\prod_{j=1}^{n} \varkappa_j\Big)\alpha_n(C_{A,n}) \leq 1 \quad (n \in \mathbf{N}). \tag{1}$$

Let $Z = l^1(\{2^{-n}\}, \{E_n\})$ be the Banach space containing all sequences $\{\xi_n\}_{n \in \mathbf{N}}$ with $\xi_n \in E_n$ and $\|(\xi_n)\|_Z = \Sigma_{n=1}^{\infty} 2^{-n} \|\xi_n\|_n < \infty$. By $J_n(\xi_n) = (\delta_{in} c_n \xi_n)_{i \in \mathbf{N}}$, where δ_{in} denotes the Kronecker delta, an operator $J_n: E_n \to Z$ with $\|J_n\| = 2^{-n} c_n$ is defined. Let $T_n = J_n C_{A,n} \in \mathcal{A}(E_A, Z)$. Then

$$\alpha_i(T_i) = \alpha_i(J_i C_{A,i}) \leq \|J_i\| \alpha_i(C_{A,i}) = 2^{-i} c_i \alpha_i(C_{A,i}).$$

We claim that the sequence of operators $S_n = \Sigma_{i=1}^{n} T_i \in \mathcal{A}(E_A, Z)$ is a Cauchy sequence with respect to any quasinorm α_k. In fact, for any $n \geq k$ and each $p \in \mathbf{N}$ the estimation

$$\alpha_k(S_{n+p} - S_n) \leq \alpha_{n+1}\Big(\sum_{i=n+1}^{n+p} T_i\Big) \leq \sum_{i=n+1}^{n+p}\Big(\prod_{j=n+1}^{i} \varkappa_j\Big)\alpha_i(T_i) \leq \sum_{i=n+1}^{n+p}\Big(\prod_{j=1}^{i} \varkappa_j\Big) 2^{-i} c_i \alpha_i(C_{A,i})$$

is valid. Thus we obtain $\alpha_k(S_{n+p}-S_n)\leq\Sigma_{i=n+1}^{n+p}2^{-i}$ using (1). Since $\mathcal{A}$ is complete, the sequence S_n converges to an operator $S\in\mathcal{A}(E_A, Z)$. Because of $\|S-S_n\|\leq\alpha_i(S-S_n)$ it also converges with respect to the uniform operator norm. Therefore we have $Sx=\lim_{n\to\infty}S_nx$ for each $x\in E_A$. Because of

$$S_nx=\sum_{i=1}^{n}T_ix=(c_1C_{A,1}x,\ c_2C_{A,2}x,\dots,\ c_nC_{A,n}x,0,\dots)$$

we obtain

$$(2)\qquad Sx=(c_iC_{A,i}x)_{i\in\mathbf{N}}.$$

Let p_i denote the gauge functional of the neighbourhood U_i. Then the set $B=\{x\in E:\ p_B(x)=\Sigma_{i=1}^{\infty}2^{-i}c_ip_i(x)\leq 1\}$ is absolutely convex, closed and bounded, i. e. $B\in\mathfrak{B}(E)$. The set $M=\{(c_iC_{B,i}x)_{i\in\mathbf{N}}:\ x\in E(B)\}$ is a subspace of Z because of

$$(3)\qquad \|(c_iC_{B,i}x)_{i\in\mathbf{N}}\|_Z=\sum_{i=1}^{\infty}2^{-i}c_i\|C_{B,i}x\|_i=\sum_{i=1}^{\infty}2^{-i}c_ip_i(x)=p_B(x).$$

A linear mapping R from M into $E(B)$ will be defined by $R((c_iC_{B,i}x)_{i\in\mathbf{N}})=x$, because $C_{B,i}x=C_{B,i}x'$ for all $i\in\mathbf{N}$ implies $p_i(x-x')=0$, and consequently $x=x'$. Because of (3) R is isometrical. From (1) it follows $c_i\|C_{A,i}\|\leq 1$ for all $i\in\mathbf{N}$. Thus for all $x\in E(A)$ we obtain the inequality

$$p_B(x)=\sum_{i=1}^{\infty}2^{-i}c_ip_i(x)=\sum_{i=1}^{\infty}2^{-i}c_i\|C_{A,i}x\|_i\leq\sum_{i=1}^{\infty}2^{-i}c_i\|C_{A,i}\|p_A(x)\leq p_A(x).$$

Therefore $A\subseteq B$ and the canonical mapping $C_{A,B}$ exists.

Because of $C_{A,i}x=C_{B,i}C_{B,A}x$ for all $x\in E(A)$ the formulae (2) yield $Sx=(c_iC_{B,i}C_{A,B}x)_{i\in\mathbf{N}}$, where $C_{A,B}x\in E(B)$. This shows $S(E(A))\subseteq M$. Thus (2) also defines an operator $S':E_A\to\bar{M}$ with $S=J_{\bar{M}}S'$, where $\bar{M}$ denotes the closure of M in Z and $J_{\bar{M}}$ the inclusion $\bar{M}\to Z$. Since $\mathcal{A}$ is injective the operator S' belongs to $\mathcal{A}(E_A,\bar{M})$. From $RS'x=R((c_iC_{B,i}C_{B,A}x)_{i\in\mathbf{N}})=C_{A,B}x$ we conclude $C_{A,B}=RS'\in\mathcal{A}(E_A, E_B)$. This proves the theorem.

Corollary 4.2. *Let $\mathcal{A}$ be an injective, symmetric and complete metric ideal of operators. Then the strong dual space of each metrizable mix-$\mathcal{A}$-space E is an $\mathcal{A}^{sp}$-space. Particularly the strong dual of each metrizable $\mathcal{A}$-space is an $\mathcal{A}$-space.*

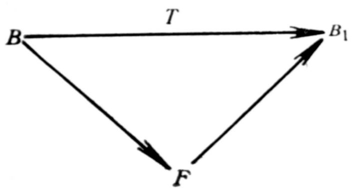
Proof. Theorem 4.1, Proposition 2.3 and Proposition 3.2 yield the assertion.

Corollary 4.3. *If $\mathcal{A}$ is an injective, symmetric, complete metric operator ideal then $\mathcal{A}^{sp}$ is symmetric and idempotent. Particularly $\mathcal{A}^{sp\,sp}=\mathcal{A}^{sp}$ is valid.*

To prove this Corollary we use the following Lemma.

Lemma 4.4. *If $\mathcal{A}$ is injective operator ideal then each operator $T\in\mathcal{A}^{sp}(B, B_1)$ can be factored through a metrizable $\mathcal{A}$-space E.*

Proof. According to the definition of $\mathcal{A}^{sp}$ there is an $\mathcal{A}$-space F such that the diagram commutes:



Therefore there is a sequence $\{U_n\}$ of neighbourhoods of zero such tha the canonical mappings $C_{n+1,n}: F_{n+1} \rightarrow F_n$ belorg to $\mathcal{A}$. Without loss of gene rality we can assume that the mapping $F \rightarrow B_1$ is factorable through F_1. Accord ing to [5, Proposition 1.7] the space $E = \text{proj lim } \{F_n\}$ is a metrizable $\mathcal{A}$-space Moreover, T factores through E.

Proof of the Corollary 4.3. Let T be any operator of $\mathcal{A}^{sp}$, factoring through the metrizable $\mathcal{A}$-space E. Its adjoint operator T' factores through the $\mathcal{A}^{sp}$-space E'_b by Corollary 4.2. This means $T' \in \mathcal{A}^{sp\, sp}$ or equivalently $T \in \mathcal{A}^{sp\, sp\, d}$. Therefore

$$\mathcal{A}^{sp} \subset \mathcal{A}^{sp\, sp\, d} \subset \mathcal{A}^{sp\, d}. \tag{4}$$

Now let $T \in \mathcal{A}^{sp\, d}$ or $T' \in \mathcal{A}^{sp}$. By (4) we obtain $T'' \in \mathcal{A}^{sp\, sp}$. But $\mathcal{A}^{sp\, sp}$ is injective, thus $T \in \mathcal{A}^{sp\, sp}$. This shows

$$\mathcal{A}^{sp\, d} \subset \mathcal{A}^{sp\, sp} \subset \mathcal{A}^{sp}. \tag{5}$$

Both (4) and (5) yield

$$\mathcal{A}^{sp} = \mathcal{A}^{sp\, sp} = \mathcal{A}^{sp\, d}.$$

Because of Theorem 3.1 this implies the idempotency of $\mathcal{A}^{sp}$.

If $\mathcal{A}$ is an injective, symmetric and complete metric operator ideal then by Theorem 3.1 and by Corollary 4.3 the ideal $\mathcal{A}^{sp}$ is the largest injective and idempotent ideal contained in $\mathcal{A}$. Theorefore $\mathcal{A}^{sp}$ should be called the *injective and idempotent kernel* of $\mathcal{A}$.

Problem. *Under which weaker assumptions on $\mathcal{A}$ the ideal $\mathcal{A}^{sp}$ is idempotent?*

Corollary 4.4. *Let $\mathcal{A}$ be an injective, symmetric and complete metric operator ideal.*

a. *For each metrizable l. c. s. E the following are equivalent:*

(i) $E \in \text{mix}\, \mathcal{A}$.

(ii) $E \in \text{co}\, \mathcal{A}$.

(iii) $E \in \text{co}\, \mathcal{A}^{sp}$.

b. *For reflexive DF-spaces E the following are equivalent:*

(iv) $E \in \text{mix}\, \mathcal{A}$.

(v) $E \in \text{sp}\, \mathcal{A}$.

(vi) $E \in \text{sp}\, \mathcal{A}^{sp}$.

If $\mathcal{A} \subset \mathcal{C}$ then each of the conditions (iv)—(vi) *imply the reflexivity for complete DF-spaces E.*

Proof. The equivalence of (i), (ii), (iii) and of (iv), (v), (vi) will be proved by the Corollaries 4.3, 4.2 and by Proposition 2.3.

If E is a DF-mix-$\mathcal{A}$-space then E'_b is a metrizable co-$\mathcal{A}$-space because of a. Since $\mathcal{A} \subseteq \mathcal{C}$ all bounded subsets of E'_b are separable. But E is σ-quasi-barrelled, so it must be quasi-barrelled. On the other hand, E is semireflexive (cf. section 5.3). This shows the reflexivity of E.

5. Applications and Counterexamples. 5.1. The results of the foregoing section are applicable immediately to the ideals $\mathcal{C}, \mathfrak{V}, \mathcal{S}_0$ and $\mathcal{E}$. In the case of the ideal $\mathcal{C}$ this yields the results contained in [13]. The resulting statement for the ideals $\mathfrak{V}$ and $\mathcal{S}_0$ are new as far as I know. Furthermore, theory is applicable to the φ-nuclear spaces according to Rosenberger [11].

A statement for DF-spaces which corresponds to Theorem 4.1 (true for nuclear DF-spaces) is missed. We shall show below that such assertion does not hold for a wide class of ideals.

5.2. An operator ideal $\mathcal{A}$ is called *l^2-factorable* if each operator $T \in \mathcal{A}(B, B_1)$ can be factored through the separable Hilbert space l^2. In this case the class sp$\mathcal{A}$ is determined only by the component $\mathcal{A}(l^2, l^2)$ and we obtain the following variation of the Theorem 4.1.

Proposition. *Let $\mathcal{A}$ be an l^2-factorable operator ideal such that the component $\mathcal{A}(l^2, l^2)$ is a complete metric space with respect to a countable system of seminorms. Then each metrizable $\mathcal{A}$-space E is a co-$\mathcal{A}$-space and its strong dual is an $\mathcal{A}^{\mathrm{sp}}$-space.*

Proof. We modify the proof of the Theorem 4.1 by an exclusive usage of separable Hilbert spaces. First of all we may assume that each E_n is isomorphic to l^2. For given $A \in \mathfrak{B}(E)$ we define:

$$Z = l^2(\{2^{-2n}\}, \{E_n\}) \text{ with } \|(\xi_n)\|_Z^2 = \sum_{n=1}^{\infty} 2^{-2n} \|\xi_n\|^2;$$

$J_n : E_n \to Z$ is defined by $J_n(\xi_n) = (\delta_{in} c_n \xi_n)_{i \in \mathbf{N}}$, where the c_i are chosen in conformity with (1) in the proof of the Theorem 4.1, maybe with $\|C_{A,i}\|$ instead of $\alpha_i(C_{A,i})$;

$$B = \{x \in E : p_B(x)^2 = \sum_{n=1}^{\infty} 2^{-2n} c_n^2 p_n(x)^2 \leq 1\};$$

$$M = \{(c_i C_{B,i} x)_{i \in \mathbf{N}} : x \in E(B)\}.$$

Now it is easy to show that B is bounded and contains A and that E_B is isomorphic to l^2. A repetition of this construction with B instead of A produces a bounded set C such that $C_{BC} : E_B \to E_C$ belongs to $\mathcal{A}(l^2, l^2)$.

5.3. To provide some counterexamples we use the following proposition.

Proposition. *Let $\mathcal{A}$ be the ideal of the weakly compact operators and E a complete l. c. s. Then $E \in \operatorname{mix} \mathfrak{V}$ iff E is semireflexive.*

Proof. The l. c. s. E is semireflexive if and only if each set $A \in \mathfrak{B}(E)$ is weakly compact in E. Therefore, A is weakly compact in E_U too. This means $E \in \operatorname{mix} \mathfrak{V}$. Conversely if E is a mix-$\mathfrak{V}$-space then for given $A \in \mathfrak{B}(E)$ and any $U \in \mathfrak{U}(E)$ the weak closure of the image $C_U(A)$ of A is compact in $(E_U)_\sigma$. But $E_\sigma = \operatorname{proj} (E_U)_\sigma$ is closed in the product space $\Pi\{(E_U)_\sigma \mid U \in \mathfrak{U}(E)\}$. Therefore by Tichonov's Theorem A must be relatively weekly compact in E. Now A is absolutely convex and closed. Due to Mazur's Theorem it is weakly closed. This shows the weak compactness of A.

A similar consideration shows that any complete l. c. s. E is a mix-$\mathcal{C}$-space if and only if each closed and bounded subset of E is already compact.

Corollary. *If E is a complete metric l. c. s. then the following hold:*

$E \in \operatorname{mix} \mathfrak{V}$ iff E is semireflexive;

$E \in \operatorname{mix} \mathcal{C}$ iff E is a Montel space.

Let E denote Köthe's example of a Frechet-Montel-space having l^1 as a quotient space [6, § 31.5]. Thus E is not a $\mathfrak{V}$-space (cf. [9; 22.4.5] or [4; § 6(6)]). On the other hand, by the above Corollary E belongs to $\operatorname{mix} \mathcal{C}$. Therefore, if $\mathcal{A}$ is any operator ideal between $\mathcal{C}$ and $\mathfrak{V}$ then E is a mix-$\mathcal{A}$-space but not an $\mathcal{A}$-space. Because of Corollary 4.2 the dual space $F = E'_b$ belongs to $\operatorname{sp} \mathcal{C} \subseteq \operatorname{sp} \mathcal{A}$. Since E is reflexive we have $F'_b = E$. This proves:

Corollary. *If $\mathcal{A}$ is any operator ideal between $\mathcal{C}$ and $\mathfrak{V}$ then there is a complete DF-$\mathcal{A}$-space F such that its strong dual space is not an $\mathcal{A}$-space.*

5.4. Further counterexamples are obtained for some l^2-factorable ideals. Let $\mathcal{A}$ be an l^2-factorable ideal of operators such that $\mathcal{A}(l^2, l^2)$ is complete metric. If $\mathcal{A}(l^2, l^2)^2 \neq \mathcal{A}(l^2, l^2)$ then there is a DF-$\mathcal{A}^{sp}$-space, the strong dual of which is not an $\mathcal{A}^{sp}$-space. To prove this we choose an operator $T \in \mathcal{A}(l^2, l^2) \setminus \mathcal{A}(l^2, l^2)^2$. Without loss of generality we may assume that T is positive. Then $T^{2^k} \in \mathcal{A}(l^2, l^2)^{2^{k+1}}$ for all $k \in \mathbf{N}$. We define $E = \operatorname{proj} l^2(\{\lambda_n^{-2k}(T)\}_n, \{\mathbf{R}\}_n)$ where $\{\lambda_n(T)\}$ denotes the decreasing sequence of the eigenvalues of T. An easy computation confirms that E is an $\mathcal{A}(l^2, l^2)^k$-space for all $k \in \mathbf{N}$ but it is not an $\cap_{k=1}^{\infty} \mathcal{A}(l^2, l^2)^k$-space. Thus E is not an $\mathcal{A}$-^{sp}-space. Since E is reflexive the dual space E'_b provides us with the wanted counterexample.

Note added in proof. As I learned after completition of this paper E. Nelimarkka in his Doctor's thesis "On operator ideals and locally convex $\mathcal{A}$-spaces with applications to λ-nuclearity", Helsinki, August 1977, considered semi-ideals (i. e. sublpasses $\mathcal{A}$ of $\mathcal{L}$ satisfying only the conditions (1 1) and (1 3) of an ideal) to make a theorem like Corollary 4.2 for semi-ideals applicable to $\Lambda_N(\alpha)$-spaces.

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