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INTERSECTIONS OF A RANDOM PROCESS OF LINEAR SEGMENTS WITH A GIVEN STRAIGHT LINE

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We consider a random process of linear segments in R^2 which can be interpreted as a Boolean model in the sense of Matheron. It is shown that the random number of intersections of the random segments with a given segment with fixed length and direction has a Poisson distribution.

1. Introduction. We consider a random process of linear segments in R^2 which can be interpreted as a Boolean model in the sense of Matheron [3, p. 137]. The "primary grains" are in our case linear segments of random length and direction. It is shown that the random number of intersections of the random segments has a Poisson distribution. This is true not only if we consider segments in the full R^2 but also if we consider only segments which lie completely in an infinite strip of R^2.

The process which we consider is a model for clefts in rocks or for tectonic fractures. Knowledge of the number of intersections with a given straight line (= a blast hole) is of importance for enginering geology. We remark that a similar model has been studied for quite other purposes [1].

2. The Model. We denote by R^2 the set of all pairs (x, y) (x, y real numbers) and by S_a its subset $S_a=\{(x, y)\in R^2, 0\leq y\leq a\}$. The random process of linear segments is a Boolean model, i. e. we construct it as follows:

1. We consider a homogeneous Poisson process of intensity λ in R^2 (S_a). Its points (the "primary germs") are the middles of the segments of the process.

2. The segments have lengths with distribution function $F(x)$ with mean m.

3. The orientation of the segments has a distribution $G(\alpha)$ on $(-\pi/2, \pi/2)$· where the segments parallel to the y-axis have the direction $\alpha=0$ (see Fig. 1).

4. We assume full independence between lengths and directions. The given straight line (the "test line") is given by the four parameters l, d_1, d_2 and ψ (see Fig. 1).

3. Result. We consider the random number ν of intersections of the test line with the random process of linear segments.

The R^2 case is simpler than the S_a case. For example, the distribution of ν in the R^2 case does not depend on d_1, whereas we have a remarkable dependence on the position of the test line in the S_a case.

Theorem. *The number ν has a Poisson distribution with parameter μ,*

$$\mu=\lambda ml\int_{-\pi/2}^{\pi/2}\cos(\alpha+\psi)dG(\alpha)\qquad \textit{in } R^2 \textit{ case},$$

$$\mu=\lambda 1[\int_{-\pi/2}^{\pi/2}\{2\int_{0}^{l\sin\psi/\cos\alpha}(1-F(2x))\cos(\alpha+\psi)\frac{x\cos\alpha}{l\sin\psi}\,dx$$

$$+\cos(\alpha+\psi)\sum_{i=1}^{2}\int_{l\sin\psi/\cos\alpha}^{(l\sin\psi+d_i)/\cos\alpha}(1-F(2x))\,dx\}dG(\alpha)] \quad \text{in } S_a \text{ case.}$$

P r o o f. We give a proof only for the S_a case, $0\leq\psi<\pi/2$. The idea which is used here is also applicable in the R^2 case and the similar case $\psi=\pi/2$.

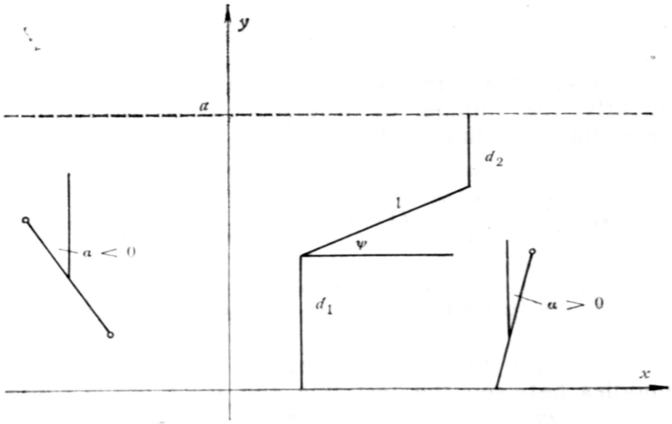

Fig. 1. Position of the test line and definition of angle α

Let us consider the sub-process of all linear segments having orientation α. (For brevity we assume $\alpha>0$.) The point process P_α of middles of these linear segments is a Poisson process with intensity $\lambda dG(\alpha)$.

Let $L_U(L_L)$ be the straight line with direction α which intersects the test line in its upper (lower) point $U(L)$. Only linear segments with middles in the strip between L_U and L_L can have intersections with the test line, if we consider the above mentioned sub-process. Now we consider only that part of this strip which lies between the test line and the x-axis. The projections parallel to the test line of the points of P_α into L_U form a Poisson process P_α^* on L_U with variable intensity $\mu(t)dG(\alpha)$, where t denotes the distance between U and the points of L_U. We have

$$\mu(t)=\begin{cases}\lambda l\cos(\alpha+\psi); & 0\leq t\leq d_1/\cos\alpha,\\ \lambda l\,\dfrac{d_1+l\sin\psi-t\cos\alpha}{l\sin\psi}\cos(\alpha+\psi); & \dfrac{d_1}{\cos\alpha}\leq t\leq\dfrac{d_1}{\cos\alpha}+\dfrac{l\sin\psi}{\cos\alpha},\\ 0; & t>\dfrac{d_1}{\cos\alpha}+\dfrac{l\sin\psi}{\cos\alpha}.\end{cases}$$

Let ν_α be the number of intersections of our sub-process with the test line. The distribution of ν_α can be found by a queueing theory argument. We interpret the points of P_α^* as arrival instants of customers and the upper half length of linear segments as service times. Then ν_α is the number of customers in the system. Our system is of type $M/GI/\infty$ with time-dependent arrival rate $\lambda(\tau)$ and service time d. f. $F(2x)$, where

$$\lambda(\tau)=\begin{cases}0; & \tau<0\\ \mu\left(\frac{d_1+l\sin\psi}{\cos\alpha}-\tau\right); & \tau\geqq 0.\end{cases}$$

Then the formulae for time-dependent $M/GI/\infty$ systems give that ν_α, which can be interpreted as the number of customers in the system at $\tau_0=(d_1+l\sin\psi)/\cos\alpha$, has a Poisson distribution [2] with mean μ_α,

$$\mu_\alpha=\int_0^{\tau_0}(1-F(2x))\lambda(x)dx.$$

As is well-known, the sum of independent Poisson r. v.'s is a Poisson r. v. Consequently, the number of intersections of all linear segments with middles between the test line and the x-axis has a Poisson distribution with mean $\mu^{(1)}$,

$$\mu^{(1)}=\lambda l\int_{-\pi/2}^{\pi/2}\int_0^{l\sin\psi/\cos\alpha}(1-F(2x))\cos(\alpha+\psi)\frac{x\cos\alpha}{l\sin\psi}dxdG(\alpha)$$

$$+\lambda l\int_{-\pi/2}^{\pi/2}\cos(\alpha+\psi)\int_{l\sin\psi/\cos\alpha}^{(l\sin\psi+d_1)/\cos\alpha}(1-F(2x))dxdG(\alpha)=m(d_1).$$

Similarly we obtain that the number of intersections of all linear segments with middles between the test line and $y=a$ has a Poisson distribution with mean $\mu^{(2)}$, $\mu^{(2)}=m(d_2)$. The total number of intersections has then a Poisson distribution with mean $\mu=\mu^{(1)}+\mu^{(2)}$.

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