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THE EFFECT OF NUMERICAL INTEGRATION IN THE FINITE ELEMENT APPROXIMATION OF HYPERBOLIC PROBLEMS

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The effect of numerical integration in the finite element is analyzed for solving hyperbolic problems. A discrete Galerkin procedure is introduced and a priori error bounds are derived.

Introduction. A finite element method approximates the general problem $Lu=f$ by a matrix problem of the $\Sigma_{i=0}^{s} B_i D_t^i \beta = F$. The elements of the matrices B_i and the vector F involve integrals of basis functions and coefficients of the operator L. Since these integrals generally cannot be evaluated exactly, the integration is usually done by a numerical scheme. The goal of this paper is to analyze the size of the error in the finite element approximation of hyperbolic problems introduced by the estimation of these integrals with numerical quadrature methods. The effect of numerical integration in finite element methods for solving elliptic problems has been analyzed by G. Strang [7], G. Strang and G. Fix [8], P. Ciarlet and P. Raviart [2]. The case of parabolic problems has been investigated by P. Raviart [6] and G. Fix [4]. The results in this paper are from the author's thesis [5].

1. Hyperbolic problems. In this section we discuss the use of a Galerkin type procedure "to discretize" the space variables in initial boundary value problems for linear hyperbolic problems with time dependent coefficients. In particular we consider the problem

$$D_t^2 u - \sum_{i,j=1}^{n} D_{x_j}(a_{ij}(x,t)) D_{x_i} u) = f \text{ in } \Omega\times[0,T), \tag{1.1}$$

$$u=0 \text{ on } \Gamma\times[0,T), \tag{1.2}$$

$$u(x,0)=u_0(x)\in L^2(\Omega),\quad D_t u(x,0)=u_1(x)\in L^2(\Omega),$$

where Ω is a bounded polyhedral domain of R^n with boundary Γ and a_i are functions continuous over $\Omega\times[0,T]$. Also, we assume that the second order differential operator $L(t)=-\Sigma_{i,j=1}^{n} D_{x_j}(a_{ij}(x,t)D_{x_i})$ satisfies the usual ellipticity property, i. e. there exists a positive constant K such that $\Sigma_{i,j=1}^{n} a_{ij}(x,t)\xi_i\xi_j \geq K\Sigma_{i=1}^{n}\xi_i^2$ for all $(x,t)\in\Omega\times[0,T]$ and $\xi\in R^m$.

Let us define

$$a(t;u,v)=\sum_{i,j=1}^{n}\int_{\Omega} a_{ij}(x,t) D_{x_i}u(x) D_{x_j}v(x)dx$$

for any $u,v\in H^1(\Omega)$ where we recall that $H^{p,q}$ is the collection of all real-valued functions $v(x)\in L^q(\Omega)$ with $D_x^{\alpha} v\in L^q(\Omega)$ for all $|\alpha|\leq p$.

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We use the notation $|\alpha| = \alpha_1 + \alpha_2 + \cdots + \alpha_n$, $D_x^\alpha = D_{x_1}^{\alpha_1} D_{x_2}^{\alpha_2} \cdots D_{x_n}^{\alpha_n}$ and $H^p = H^{p,2}$.

We say that u is a generalized solution of (1.1) iff $u(x, t) \in L^2(0, T; H_0^1(\Omega))$, $u(x, 0) = u_0(x)$, $D_t u(x, 0) = u_1(x)$ and

(1.3) $$(D_t^2 u, v) + a(t; u, v) = (f, v), \quad 0 \le t \le T \text{ for all } v \in H_0^1(\Omega).$$

Integrating by parts and using the Gronwall inequality we can prove the following result:

Theorem 1. *If u is a classical solution of* (1.1), (1.2) *then it is a generalized solution.*

Throughout we will assume that the generalized solution exists to (1.1) and (1.2).

In order to define a "semi-discrete Galerkin" approximation to the generalized solution u of (1.1) and (1.2), we construct a triangulation $\mathcal{T}_h$ of the domain Ω with finite elements K having diameters $\le h$. With this triangulation we associate a finite dimensional subspace S_h of $H_0^1(\Omega) \cap C[\Omega]$ which is spanned by the basis functions $\{B_i(x)\}_1^N$. Then the semi-discrete problem associating with the space S_h consists of finding an approximation $u_h(x, t)$ of the form $u_h(x, t) = \Sigma_{i=1}^N \beta_i(t) B_i(x)$. The coefficients $\{\beta_i(t)\}_1^N$ are determined by the following system of ordinary differential equations

(1.4) $$(D_t^2 u_h, B_i) + a(t; u_h, B_i) = (f, B_i),$$

$1 \le i \le N$, for all $t \in (0, T]$ and $(u_h(0), B_i) = (u_0, B_i)$, $(D_t u_h(0), B_i) = (u_1, B_i)$, $1 \le i \le n$.

In order to compute the solution of (1.4) we must calculate the integrals which appear in (1.4) and this is usually done by numerical integration scheme. We denote by $\Sigma_{l=1}^K \omega_{l,K} f(\xi_{l,K})$ the quadrature sum over K that approximates $\int_K f(x)dx$ for some specified points $\xi_{l,K}$ and weights $\omega_{l,K} \in R$, $1 \le l \le k$.

Moreover, we define

(1.5) $$(\varphi, \psi)_h = \sum_{K \in \mathcal{T}_h} \sum_{l=1}^{k} \omega_{l,K} \varphi(\xi_{l,K}) \psi(\xi_{l,K}).$$

Let u_h denote the solution of (1.4) when the problem is perturbed by numerical integration, i. e. u_h is the solution of the following Galerkin type problem

(1.6) $$(D_t^2 u_h, v)_h + a_h(t; u_h, v) = (f, v)_h$$

for $v \in S_h$ and $0 \le t \le T$. With initial conditions $u_h(0) = u_{h,0} \in S_h$, $D_t u_h(0) = u_{h,1} \in S_h$ and u_h, $D_t^2 u_h \in L^2(0, T; S_h)$, where $L^2(0, T; S_h)$ denotes the space of functions $t \to v(t)$ which are L^2 on $[0, T]$ and $\|v(\cdot, t)\|_{S_h}$ is finite. We now proceed to examine the order of magnitude of the error $\|u - u_h\|$.

2. Error estimates. In this section we derive a priori error bounds on the error $\|u - u_h\|$ for a specific choice of the subspace S_h and the quadrature schemes (1.5). The subspace S_h is defined as follows:

1) we assume that for any function $v \in S_h$ and any (closed) finite element $K \in \mathcal{T}_h$ we have $v|_K \in C^{k+1}(K)$ for some integer $k \ge 1$;

2) we assume that for any integer s with $2 \leq s \leq k+1$ and any real number q with $2 \leq q \leq +\infty$, there exists a linear operator $\pi_h \in \mathcal{L}(H^{s,q}(\Omega) \cap H_0^{1,q}(\Omega);\ S_h)$ such that

$$\Big(\sum_{K \in \mathcal{F}_h} \| \pi_h v - v \|^q_{H^{m,q}(K)}\Big)^{1/q} \leq C h^{s-m} \| v \|_{H^{s,q}(\Omega)},\quad 0 \leq m \leq s,$$

for all $v \in H^{s,q}(\Omega) \cap H_0^{1,q}(\Omega)$, where the constant C is independent of h.

We present an example of a subspace S_h whose abstract formulation and its approximate properties have been studied by Raviart and Ciarlet in [1; 2]. Let S_h be a finite dimensional space of real functions defined in $\Omega \subset \mathbb{R}^n$ and spanned by $\varphi_1^h, \ldots, \varphi_N^h$, where the basis functions (or shape functions in engineering terminology) are determined so that to each φ_j^h there is associated a node z_j and $\varphi_j^h(z_i) = \delta_{ij}$, $i = 1, \ldots, N$. Assume that the basis functions φ_j^h are uniform to order q, that is there exists a constant C_s such that for all h, i and j:

$$\max_{x \in K,\, |\alpha| = s} | D^\alpha \varphi_j^h(x) | \leq C_s h^{-s}$$

for all $s \leq q$.

Define $\pi_h v = \sum_{j=1}^N v(z_j) \varphi_j^h$ and suppose S_h contains the set of polynomials in $x_1, \ldots, x_n$ of total degree less than k. Then the following theorem has been proved (see Strang and Fix [8]).

Theorem 2. *Suppose $u(x_1, \ldots, x_n)$ has k derivatives in the mean-square sense and any derivative D^α of order $|\alpha| = s \leq q$. Suppose also that $k > n/2$. Then*

$$\int_K | D^\alpha u(x) - D^\alpha \pi_h u(x) |^2 dx \leq C_s^2 h^{2(k-s)} \| u \|^2_{H^k(K)}$$

and

$$\Big(\sum_{K \in \mathcal{F}_h} \| u - \pi_h u \|^2_{H^s(K)}\Big)^{1/2} \leq C_s h^{k-s} \| u \|_{H^k(\Omega)}.$$

For the quadrature schemes we assume that if r is an integer with $0 \leq r \leq k+1$ and q a real number with $2 \leq q \leq \infty$, $r - 1 - N/q > 0$ (that is $H^{r-1,q}(\Omega) \subset C(\Omega)$) we have, for all $u, v \in S_h$, the following inequalities

$$|(u, v) - (u, v)_h| \leq C h^{r-l} \Big(\sum_{K \in \mathcal{F}_h} \| u \|^q_{H^{r-l,q}(K)}\Big)^{1/q} \Big(\sum_{K \in \mathcal{F}_h} \| v \|^2_{H^{l+1}(K)}\Big)^{1/2},\ l = 0, 1,$$

$$| a(t;\, u, v) - a_h(t;\, u, v) | \leq C h \max_{1 \leq i, j \leq N} \| a_{ij}(.\,, t) \|_{H^{1,\infty}(\Omega)} \| u \|_{H^1(\Omega)} \| v \|_{H^1(\Omega)},$$

if $a_{ij}(.\,, t) \in H^{1,\infty}(\Omega)$, $1 \leq i, j \leq N$;

$$| a(t;\, u, v) - a_h(t;\, u, v) |$$

$$\leq C h^{r-l} \max_{1 \leq i, j \leq N} | a_{ij}(.\,, t) \|_{H^{r-l,\infty}(\Omega)} \Big(\sum_{K \in \mathcal{F}_h} \| u \|^q_{H^{r,q}(\Omega)}\Big)^{1/q} \Big(\sum_{K \in \mathcal{F}_h} \| v \|^2_{H^{l+1}(K)}\Big)^{1/2}$$

if $a_{ij}(.\,, t) \in H^{r-l,\infty}(\Omega)$, $1 \leq i, j \leq N$, $l = 0,1$;

$$(f, v)-(f, v_h)| \leq Ch^{r-l} \| f(\cdot, t) \|_{H^{r-l,q}(\Omega)} \Big(\sum_{K\in\mathcal{F}_h} \| v \|^2_{H^{l+1}(K)} \Big)^{1/2}$$

if $f(\cdot, t) \in H^{r-l,q}(\Omega)$, $l=0,1$, where C is used as a generic constant independent of h.

Lemma 2.1. *Assume that there exists a constant $\gamma>0$ independent of h such that*

$$(2.1) \qquad a_h(t; v, v) \geq \gamma \| v \|^2_{H^1(\Omega)}$$

for all $v \in S_h$ and all $t \in [0, T]$ and $u \in L^2(H^{r,q}(\Omega))$, f, $D_t^2 u \in L^2(H^{r-1,q}(\Omega))$, $a_{ij} \in L^\infty(H^{r-1,\infty}(\Omega))$, $1 \leq i, j \leq N$. Then the problem

$$(2.2) \qquad a_h(t;\ w_h, v) = (f - \pi_h D_t^2 u, v)_h,\ t \in (0, T],\ v \in S_h$$

has a unique solution $w_h \in L^2(S_h)$ such that

$$(2.3) \qquad \| w_h - u \|_{L^2(H^1(\Omega))} \leq Ch^{r-1} \{ \| u \|_{L^2(H^{r,q}(\Omega))} + \| D_t^2 u \|_{L^2(H^{r-1,q}(\Omega))} + \| f \|_{L^2(H^{r-1,q}(\Omega))} \},$$

where C is independent of u, h, f.

Proof. First, we observe that the existence and uniqueness of the solution w_h is a consequence of the assumption (2.1). Using the fact that u is a solution (1.1), we get that

$$a_h(t;\ w_h - \pi_h u, v) = a(t;\ u - \pi_h u, v) + (D_t^2 u - \pi_h D_t^2 u, v) + a(t;\ \pi_h u, v) - a_h(t;\ \pi_h u, v)$$
$$+ (\pi_h D_t^2 u, v) - (\pi_h D_t^2 u, v)_h - (f, v) - (f, v)_h$$

for all $v \in S_h$.

We choose $v = w_h - \pi_h u$ and use assumption (2.1) to obtain, after integration with respect to t,

$$\| w_h - \pi_h u \|_{L^2(H^1(\Omega))} \leq C\{ \| u - \pi_h u \|_{L^2(H^1(\Omega))} + \| D_t^2 u - \pi_h D_t^2 u \|_{L^2(L^2(\Omega))} \}$$
$$+ \sup_{v\in L^2(S_h)} \| v \|^{-1}_{L^2(H^1(\Omega))} \{ \Big| \int_0^T [a(t;\ \pi_h u, v) - a_h(t;\ \pi_h u, v)] dt \Big|$$
$$+ \Big| \int_0^T [(\pi_h D_t^2 u, v) - (\pi_h D_t^2 u, v)_h] dt \Big| + \Big| \int_0^T [(f, v) - (f, v)_h] dt \Big| \}.$$

We use the properties of the space S_h and error bounds for the quadrature schemes to obtain

$$\| w_h - u \|_{L^2(H^1(\Omega))} \leq \| w_h - \pi_h u \|_{L^2(H^1(\Omega))} + \| \pi_h u - u \|_{L^2(H^1(\Omega))}$$
$$\leq Ch^{r-1} \{ \| u \|_{L^2(H^r(\Omega))} + \| D_t^2 u \|_{L^2(H^{r-1}(\Omega))} + \max_{1\leq i, j\leq N} \| a_{ij} \|_{L^\infty(H^{r-1,\infty}(\Omega))} \| u \|_{L^2(H^{r,q}(\Omega))}$$
$$+ \| D_t^2 u \|_{L^2(H^{r-1,q}(\Omega))} + \| f \|_{L^2(H^{r-1,q}(\Omega))} \}.$$

Since

$$\Big[\int_0^T \Big(\sum_{K\in\tau} \| \pi_h u \|^q_{H^{r,q}(\Omega)} \Big)^{2/q} dt \Big]^{1/2} \leq C \| u \|_{L^2(H^{r,q}(\Omega))}$$

and

$$[\int_0^T (\sum_{K\in \mathcal{C}_h} \|\pi_h D_t^2 u\|^q_{H^{r-1,q}(\Omega)})^{2/q} dt]^{1/2} \leq C \|D_t^2 u\|_{L^2(H^{r-1,q}(\Omega))},$$

as it follows easily from the properties of the S_h space.

Now, in order to find a priori bounds for the $\|w_h - u\|_{L^2(L^2(\Omega))}$ we assume that the adjoint operator

$$L^* = -\sum_{i,j=1}^{N} D_{x_i}(a_{ij}(x, t)D_{x_j})$$

satisfies the following regularity property

$$\|v\|_{H^2(\Omega)} \leq C \|L^* v\|_{L^2(\Omega)} \tag{2.4}$$

for all $v \in H^2(\Omega) \cap H_0^1(\Omega)$, $t \in [0, T]$. Notice that property (2.4) is satisfied if $a_{ij} \in L^\infty(H^{1,\infty}(\Omega))$ for $1 \leq i, j \leq N$.

Lemma 2.2. *Consider $f, u, D_t^2 u \in L^2(H^{r,q}(\Omega))$, $a_{ij} \in L^\infty(H^{r,\infty}(\Omega))$ $1 \leq i, j \leq N$ where q is some real number with $2 \leq q \leq +\infty$, $r-1-N/q>0$. Assume that hypotheses (2.1) and (2.4) hold.*

Then the solution w_h of the equation (2.2) satisfies

$$\|w_h - u\|_{L^2(L^2(\Omega))} \leq Ch^r \{\|u\|_{L^2(H^{r,q}(\Omega))} + \|D_t^2 u\|_{L^2(H^{r,q}(\Omega))} + \|f\|_{L^2(H^{r,q}(\Omega))}\}, \tag{2.5}$$

where the constant C is independent of h, u and f.

Proof. To prove (2.5) we use a generalization of the Aubin-Nitsche duality argument. We have

$$\|w_h - u\|_{L^2(L^2(\Omega))} = \sup_{\varphi \in L^2(L^2(\Omega))} \{|\int_0^T (w_h - u, \varphi)dt| / \|\varphi\|_{L^2(L^2(\Omega))}\}. \tag{2.6}$$

Given $\varphi \in L^2(L^2(\Omega))$ we consider the problem of finding $\Psi(x, t)$ such that $L^*\Psi = \varphi$ in Ω, $\Psi = 0$ on Γ. Since L^* satisfies property (2.4) we have $\Psi \in L^2(H^2(\Omega) \cap H_0^1(\Omega))$. Then $(w_h - u, \varphi) = a(t; w_h - u, \Psi)$. On the other hand, for any function $v \in L^2(S_h)$ we use equation (2.2) to get

$$a(t; w_h - u, v) = a(t; w_h, v) - a_h(t; w_h, v) - (D_t^2 u, v) + (\pi_h D_t^2 u, v)_h + (f, v) - (f, v)_h.$$

Therefore

$$(w_h - u, \varphi) = a(t; w_h - u, \Psi) = a(t; w_h - u, \Psi - v) + a(t; w_h - u, v) = a(t; w_h - u, \Psi - v) + a(t; w_h, v) - a_h(t; w_h, v) - (D_t^2 u, v) + (\pi_h D_t^2 u, v)_h + (f, v) - (f, v)_h.$$

We choose $v = \pi_h \Psi$ to obtain

$$|\int_0^T (w_h - u, \varphi)dt| \leq C\{h\|w_h - u\|_{L^2(H^1(\Omega))} \|\Psi\|_{L^2(H^2(\Omega))} \tag{2.7}$$

$$+ \|D_t^2 u - \pi_h D_t^2 u\|_{L^2(L^2(\Omega))} \|\Psi\|_{L^2(H^2(\Omega))} + |\int_0^T [a(t; w_h - \pi_h u, \pi_h \Psi) - a_h(t; w_h - \pi_h u, \pi_h \Psi)]dt|$$

$$+\left|\int_0^T [(f, \pi_h \Psi)-(f, \pi_h \Psi)]dt\right|+\left|\int_0^T [a(t;\ \pi_h u, \pi_h \Psi)-a_h(t;\ \pi_h u, \pi_h \Psi)]dt\right|\}.$$

From the convergence conditions of the quadrature schemes (1.4) we have the following inequalities:

$$\left|\int_0^T [a(t;\ w_h-\pi_h u, \pi_h \Psi)-a_h(t;\ w_h-\pi_h u, \pi_h \Psi)]dt\right|$$

$$\leq Ch \max_{1\leq i, j\leq N} \|a_{ij}\|_{L^\infty(H^{1,\infty}(\Omega))} \|w_h-\pi_h u\|_{L^2(H^1(\Omega))} \|\Psi\|_{L^2(H^2(\Omega))},$$

$$\left|\int_0^T [a(t;\ \pi_h u, \pi_h \psi)-a_h(t;\ \pi_h u, \pi_h \psi)]dt\right|$$

$$\leq Ch^r \max_{1\leq i, j\leq N} \|a_{ij}\|_{L^\infty(H^{r,\infty}(\Omega))} \|u\|_{L^2(H^{r,q}(\Omega))} \|\psi\|_{L^2(H^2(\Omega))},$$

$$\left|\int_0^T [(\pi_h D_t^2 u, \pi_h \psi)-(\pi_h D_t^2 u, \pi_h \psi)_h]dt\right| \leq Ch^r \|D_t^2 u\|_{L^2(H^{r,q}(\Omega))} \|\psi\|_{L^2(H^2(\Omega))}$$

and

$$\left|\int_0^T [(f, \pi_h \psi)-(f, \pi_h \psi)_h]dt\right| \leq Ch^r \|f\|_{L^2(H^{r,q}(\Omega))} \|\psi\|_{L^2(H^2(\Omega))},$$

where C is a generic constant independent of u, f, h. The proof is then completed by observing that (2.5) follows from (2.6) and (2.7) along with these inequalities.

Lemma 2,3. *Consider u, $D_t^2 u \in L^2(H^{r,q}(\Omega))$, $D_t^3 u$, f, $D_t f \in L^2(H^{r-1,q}(\Omega))$ and a_{ij}, $D_t a_{ij} \in L^2(H^{r-1,\infty}(\Omega))$, $1\leq i, j\leq N$, where q is some real number with $2\leq q\leq +\infty$, $-1-N/q>0$. Assume that (2.1) holds. Then the solution of equation (2.2) w_h satisfies $D_t w_h \in L^2(S_h)$ and*

$$\|D_t w_h-D_t u\|_{L^2(H^1(\Omega))} \leq Ch^{r-1}\{\|u\|_{L^2(H^{r,q}(\Omega))}+\|D_t u\|_{L^2(H^{r,q}(\Omega))} \tag{2.8}$$
$$+\|D_t^2 u\|_{L^2(H^{r,q}(\Omega))}+\|D_t^2\|_{L^2(H^{r-1,q}(\Omega))}+\|f\|_{L^2(H^{r-1,q}(\Omega))}+\|D_t f\|_{L^2(H^{r-1,q}(\Omega))}\},$$

where the constant C is independent of u, f, h.

Proof. We define, analogous to our previous definition of $a(t;\ u, v)$ and $a_h(t;\ u, v)$,

$$a'(t;\ u, v)=\sum_{i,j=1}^N \int_\Omega D_t a_{ij}(x, t) D_{x_i} u(x, t) D_{x_j} v(x, t) dx,\ u, v\in H^1(\Omega),$$

$$a_h'(t;\ u, v)=\sum_{K\in \mathcal{T}_h} \sum_{m=1}^M \omega_{m,k}\left(\sum_{i,j=1}^N D_t a_{ij}(.\,, t) D_{x_i} u D_{x_j} v\right)(b_{m,k}),\ u, v\in S_h.$$

Clearly $D_t w_h \in L^2(S_h)$. After differentiation of the equation (2.2) with respect to t we obtain $a_h(t;\ D_t w_h, v)=(D_t f-D_t^3 u, v)_h-a_h'(t;\ w_h, v)$ for all $v\in S_h$. Therefore we can write

$$a_h(t;\, D_t w_h - \pi_h D_t u,\, v) = a(t,\, D_t u - \pi_h D_t u,\, v) + a'(t;\, u - w_h,\, v)$$
$$+ (D_t^3 u - \pi_h D_t^3 u,\, v) + a(t;\, \pi_h D_t u,\, v) - a_h(t;\, \pi_h D_t u,\, v) + a'(t;\, w_h - \pi_h u,\, v)$$
$$- a'_h(t;\, w_h - \pi_h u,\, v) + a'(t;\, \pi_h u,\, v) - a'_h(t;\, \pi_h u, v) + (\pi_h D_t^3 u,\, v)$$
$$-(\pi_h D_t^3 u,\, v)_h - (D_t f,\, v) + (D_t f,\, v)_h.$$

We choose $v = D_t w_h - \pi_h D_t u$ and using hypothesis (2.1) and the inequality

$$\| D_t w_h - D_t u \|_{L^2(H^1(\Omega))} \leq \| D_t w_h - \pi_h D_t u \|_{L^2(H^1(\Omega))} + \| D_t u - \pi_h D_t u \|_{L^2(H^1(\Omega))},$$

we obtain

$$\| D_t w_h - D_t u \|_{L^2(H^1(\Omega))} \leq C \{ \| u - w_h \|_{L^2(H^1(\Omega))} + \| D_t u - \pi_h D_t u \|_{L^2(H^1(\Omega))}$$
$$+ \| D_t^3 u - \pi_h D_t^3 u \|_{L^2(L^2(\Omega))} + \sup_{v \in L^2(S_h)} \| v \|^{-1}_{L^2(H^1(\Omega))} [\, | \int_0^T [a(t;\, \pi_h D_t u,\, v) - a_h(t;\, \pi_h D_t u,\, v)]\, dt |$$
$$+ | \int_0^T [a'(t;\, w_h - \pi_h u,\, v) - a'_h(t;\, w_h - \pi_h u,\, v)] dt | + | \int_0^T [a'(t;\, \pi_h u,\, v) - a'_h(t;\, \pi_h u,\, v)] dt |$$
$$+ | \int_0^T [(\pi_h D_t^3 u,\, v) - (\pi_h D_t^3 u,\, v)_h] dt | + | \int_0^T [(D_t f,\, v) - (D_t f,\, v)_h] dt |]\},$$

where the constant C is independent of u, f, h.

By applying the hypotheses about the space S_h and the quadrature formulas we get the inequality (2.8) and complete the proof.

Lemma 2.4. *Consider u, $D_t u$, $D_t^2 u \in L^2(H^{r,q}(\Omega))$; $D_t^4 u$, $D_t^2 f$, f, $D_t^3 u$, $D_t f \in L^2(H^{r-1,q}(\Omega))$ and a_{ij}, $D_t a_{ij}$, $D_t^2 a_{ij} \in L^2(H^{r-1,\infty}(\Omega))$, $1 \leq i, j \leq N$, where q is some real number with $2 \leq q \leq +\infty$, $r - 1 - N/q > 0$. Assume that (2.1) holds. Then the solution of equation (2.2) w_h satisfies*

$$\| D_t^2 w_h - D_t^2 u \|_{L^2(H^1(\Omega))} \leq C h^{r-1} \{ \| u \|_{L^2(H^{r,q}(\Omega))} + \| D_t u \|_{L^2(H^{r,q}(\Omega))} \tag{2.9}$$
$$+ \| D_t^2 u \|_{L^2(H^{r,q}(\Omega))} + \| D_t^3 u \|_{L^2(H^{r-1,q}(\Omega))} + \| D_t^4 u \|_{L^2(H^{r-1,q}(\Omega))} + \| f \|_{L^2(H^{r-1,q}(\Omega))}$$
$$+ \| D_t f \|_{L^2(H^{r-1,q}(\Omega))} + \| D_t^2 f \|_{L^2(H^{r-1,q}(\Omega))}\},$$

where the constant C is independent of u, f, h.

Proof. We define, again analogous to previous definitions,

$$a''(t;\, u,\, v) = \sum_{i,j=1}^{N} \int_\Omega D_t^2 a_{ij}(x, t) D_{x_i} u D_{x_j} v dx, \quad u,\, v \in H'(\Omega),$$

$$a''_h(t;\, u,\, v) = \sum_{K \in \mathcal{T}_h} \sum_{m=1}^{M} \omega_{m,K} (\sum_{i,j=1}^{N} D_t^2 a_i(.\,, t) D_{x_i} u D_{x_j} v)(D_{m,K}), \quad u,\, v \in S_h.$$

Clearly $D_t^2 w_h \in L^2(S_h)$. After twice differentiating the equation (2.2) with respect to t we obtain

$$a_h(t;\ D_t^2 w_n, v) = (D_t^2 f - \pi_h D_t^4 u, v)_h - 2a_h'(t;\ D_t w_h, v) - a_h''(t;\ w_h, v)$$

for all $v \in S_h$. Hence, we can write

$$a_h(t;\ D_t^2 w_h - \pi_h D_t^2 u, v) = (D_t^2 f, v) - (D_t^2 f, v)_h + (D_t^4 u - \pi_h D_t^4 u, v) + (\pi_h D_t^4 u, v)$$
$$- (\pi_h D_t^4 u, v)_h + a(t;\ D_t^2 u - \pi_h D_t^2 u, v) + a(t;\ \pi_h D_t^2 u, v) - a_h(t;\ \pi_h D_t^2 u, v)$$
$$+ a''(t;\ u - w_h, v) + a''(t;\ w_h, v) - a_h''(t;\ w_h, v) + 2a'(t;\ D_t u - D_t w_h, v)$$
$$+ 2a'(t;\ D_t w_h, v) - 2a_h'(t;\ D_t w_h, v).$$

If we choose $v = D_t^2 w_h - \pi_h D_t u$ and use hypothesis (2.1), and the inequality

$$\| D_t^2 w_h - D_t^2 u \|_{L^2(H^1(\Omega))} \leq \| D_t^2 w_h - \pi_h D_t^2 u \|_{L^2(H^1(\Omega))} + \| D_t^2 u - \pi_h D_t^2 u \|_{L^2(H^1(\Omega))}$$

we obtain the inequality

$$\| D_t^2 w_h - D_t^2 u \|_{L^2(H^1(\Omega))} \leq C \{ \| D_t u - D_t w_h \|_{L^2(H^1(\Omega))} + \| D_t^4 (u - \pi_h u) \|_{L^2(L^2(\Omega))}$$
$$+ \| D_t^2 (u - \pi_h u) \|_{L^2(H^1(\Omega))} + \| u - w_h \|_{L^2(H^1(\Omega))}$$
$$+ \sup_{v \in L^2(S_h)} \| v \|^{-1}_{L^2(H^1(\Omega))} [| \int_0^T [a''(t;\ w_h, v) - a_h''(t;\ w_h, v)] dt | + | \int_0^T [a'(t;\ w_h, v)$$
$$- a_h'(t;\ w_h, v)] dt | + | \int_0^T [(D_t^2 f, v) - (D_t^2 f, v)_h] dt | + | \int_0^T [(\pi_h D_t^4 u, v) - (\pi_h D_t^4 u, v)_h] dt |] \},$$

where C is a constant independent of u, h, f. By applying the properties of S_h as we have defined them and the hypotheses about quadrature formulas we get the inequality (2.9) and complete the proof. Notice that with similar arguments as in Lemma 2.2 we can find a priori bounds for

$$\| D_t w_h - D_t u \|_{L^2(L^2(\Omega))}, \quad \| D_t^2 (w_h - u) \|_{L^2(L^2(\Omega))}.$$

Theorem 3. *Assume that $\| v \|_h = (v, v)_h^{1/2}$ is a norm over S_h and there exists a constant μ independent of h such that*

$$\| v \|_h \leq \mu \| v \|_{L^2(\Omega)} \text{ for all } v \in S_h. \tag{2.10}$$

Moreover, we assume the hypotheses of Lemma 2.4.

Then the unique solution u_h of the problem (1.5) *satisfies*

$$\| D_t(u_h - u) \|_h + \| u_h - u \|_{L^2(H^1(\Omega))} \leq C \{ \| D_t(w_h - u_h)(0) \|_{L^2(\Omega)} + \| (w_h - u_h)(0) \|_{H^1(\Omega)} \tag{2.11}$$
$$+ h^{r-1} [\sum_{m=0}^{2} \| D_t^m u \|_{L^2(H^{r,q}(\Omega))} + \sum_{m=2}^{4} \| D_t^m u \|_{L^2(H^{r-1,q}(\Omega))} + \sum_{m=0}^{2} \| D_t^m f \|_{L^2(H^{r-1,q}(\Omega))}] \}.$$

Proof. Since $\| v \|_h^2$ is a norm over S_h the assumption (2.1) ensures that the semi discrete problem (1.5) has a unique solution u_h. Let $c_h = u_h - w_h$, where w_h is defined by (2.2); then we have

$$(D_t^2\zeta_h, D_t\zeta_h)_h + a_h(t; \zeta_h, D_t\zeta_h) = (\pi_h D_t^2 u - D_t^2 w_h, D_t\zeta_h)_h$$

or

$$\frac{1}{2} D_t \| D_t\zeta_h \|_h^2 + \frac{1}{2} D_t a_h(t; \zeta_h, \zeta_h) = \frac{1}{2} a_h'(t; \zeta_h, \zeta_h) + (\pi_h D_t^2 u - D_t^2 w_h, D_t\zeta_h)_h,$$

$$\frac{1}{2} D_t \{ \| D_t\zeta_h \|_h^2 + a_h(t; \zeta_h, \zeta_h)\} \leq C\{a_h(t; \zeta_h, \zeta_h) + \| \pi_h D_t^2 u - D_t^2 w_h \|_{L^2(\Omega)}^2 + \| D_t\zeta_h \|_h^2\}.$$

Now apply Gronwall's lemma and integrate with respect to t to obtain

$$\| D_t\zeta_h \|_h^2 + a_h(t; \zeta_h, \zeta_h) \leq \| D_t\zeta_h(.\,, 0) \|_h^2 + a_h(0; \zeta_h, \zeta_h)$$
$$+ \| \pi_h D_t^2 u - D_t^2 u \|_{L^2(L^2(\Omega))} + \| D_t^2 u - D_t^2 w_h \|_{L^2(L^2(\Omega))}$$

and

$$(2.12) \qquad \| D_t\zeta_h \|_h^2 + \gamma \| \zeta_h \|_{H^1(\Omega)} \leq \| D_t\zeta_h(0) \|_{L^2(\Omega)}^2 + C \| \zeta_h(0) \|_{H^1(\Omega)}$$
$$+ \| \pi_h D_t^2 u - D_t^2 u \|_{L^2(L^2(\Omega))} + \| D_t^2 u - D_t^2 w_h \|_{L^2(L^2(\Omega))}.$$

We use the triangle inequality and assumption (2.10) to obtain

$$(2.13) \qquad \| D_t(u_h - u) \|_h + \| u_h - u \|_{L^2(H^1(\Omega))} \leq \| \zeta_h \|_{L^2(H^1(\Omega))} + \| D_t\zeta_h \|_h$$
$$+ \| D_t(u - w_h) \|_{L^2(H^1(\Omega))} + \| u - w_h \|_{L^2(H^1(\Omega))}$$

and, by the application of (2.12) and (2.13),

$$(2.14) \qquad \| D_t(u_h - u) \| + \| u_h - u \|_{L^2(H^1(\Omega))}$$
$$\leq C\{ \| D_t(w_h - u_h)(0) \|_{L^2(\Omega)} + \| (u_h - u)(0) \|_{H^1(\Omega)} + \| \pi_h D_t^2 u - D_t^2 u \|_{L^2(L^2(\Omega))}$$
$$+ \| D_t^2 u - D_t^2 w_h \|_{L^2(H^1(\Omega))} + \| D_t(u - w_h) \|_{L^2(H^1(\Omega))} + \| u - w_h \|_{L^2(H^1(\Omega))}.$$

Finally, the inequality (2.11) is a consequence of lemmas 2.4, 2.3, 2.1 and the approximate properties of the space S_h. This completes the proof of the theorem.

Notice that the H^1-optimal estimates that we have obtained in Theorem 3 using a perturbed Galerkin procedure are the same as those using a semi-discrete Galerkin method, under the same smoothness assumptions and the same subspace S_h. For H^1-estimates of the (1.1), (1.2) in Galerkin procedure see [3].

3. **Collocation on lines**. In this section we examine the relation between the numerical integration methods and the collocation on lines methods. First, we assume that the space S_h associated with the partition $\mathcal{T}_h$ of Ω with finite elements K satisfies the following properties:

First, we assume

(i) S_h is a finite dimensional subspace of $H^2(\Omega) \cap H_0^1(\Omega)$;

(ii) For all $v_h \in S_h$, $K \in \mathcal{T}_h$, $v_h|_K \in C^2(K)$.

Second, we choose the quadrature nodes $\xi_{l,K}$ so that

(iii) $\xi_{l,K} \in \text{int}\,(K)$, $1 \leq l \leq L$, for any $K \in \mathcal{T}_h$;

(iv) a function $v_h \in S_h$ is uniquely determined by its values at the points $\xi_{l,K}$, $1 \leq l \leq L$, $K \in \mathcal{T}_h$.

Third, we assume that $a_{ij}(t)\in C^1(\Omega)$, $1\leq i, j\leq n$ and choose for each u_h, $v_h\in S_h$,

$$a_h(t;\ u_h, v_h)=\sum_{K\in\mathcal{F}_h}\sum_{l=1}^{k}\omega_{l,K}(L(t)u_hv_h)(\xi_{l,K}).$$

The problem find $u_h\in S_h$ such that

$$(D_t^2u_h, v)_h+a_h(t;\ u_h, v)=(f, v)_h \text{ for } v\in S_h,\ 0\leq t\leq T, \tag{3.1}$$
$$u_h(0)=u_{h,0},\ D_tu_h(0)=u_{h,1}$$

can be stated equivalently as follows:

Find $u_h: [0, T]\to S_h$ such that

$$\{D_t^2u_h+L(t)u_h\}(\xi_{l,K})=f(\xi_{l,K}) \tag{3.2}$$
$$1\leq l\leq L,\quad K\in\mathcal{F}_h,\quad u_h(0)=u_{h,0},\quad D_tu_h(0)=u_{h,1}.$$

Thus, we obtain a collocation on lines method with collocation points the quadrature points $\xi_{l,K}$, $1\leq l\leq L$, $K\in\mathcal{F}_h$.

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