Provided for non-commercial research and educational use.
Not for reproduction, distribution or commercial use.

Serdica

Bulgariacae mathematicae publicationes

Сердика

Българско математическо списание

The attached copy is furnished for non-commercial research and education use only.
Authors are permitted to post this version of the article to their personal websites or institutional repositories and to share with other researchers in the form of electronic reprints.
Other uses, including reproduction and distribution, or selling or licensing copies, or posting to third party websites are prohibited.

For further information on
Serdica Bulgaricae Mathematicae Publicationes
and its new series Serdica Mathematical Journal
visit the website of the journal http://www.math.bas.bg/~serdica
or contact: Editorial Office
Serdica Mathematical Journal
Institute of Mathematics and Informatics
Bulgarian Academy of Sciences
Telephone: (+359-2)9792818, FAX:(+359-2)971-36-49
e-mail: serdica@math.bas.bg

A SET OF POLYNOMIALS RELEVANT TO PROBLEMS IN QUANTUM MECHANICS

T. R. PRABHAKAR, SUDHA JAIN

Recently Cohen (1968) introduced a polynomial $\Theta_n(x)$, which is relevant to problems of quantum mechanics. In this paper a generalized Rodrigues formula and three double integrals are obtained for the polynomials $R_n^\mu(x)$, which contain as special cases a number of known polynomials including $\Theta_\lambda(x)$ and also squares and products of several standard polynomials.

1. Introduction. In many problems of quantum mechanics and related fields one encounters not only various polynomials such as those of Gegenbauer, Legendre, Hermite but also the squares and products of these polynomials (see for instance [1]). Here we consider the polynomials $R_n^\mu(x)$ defined by

$$(1)\qquad (1-t)^{-c}{}_pF_q\left[\begin{matrix}a_1,\dots,a_p\\ b_1,\dots,b_q\end{matrix};\frac{-\lambda xt}{(1-t)^\mu}\right]=\sum_{n=0}^{\infty} t^n\frac{(c)_n}{n!}R_n^\mu(x),$$

where ${}_pF_q$ is the generalized hypergeometric function. They include not only the mentioned standard polynomials but also those of Laguerre, Sister Celine and Jacobi (see [6]) and the polynomials recently introduced by Konhauser [4] and Cohen [1], as also the squares and products of many of these polynomials.

In this paper we obtain a generalized Rodrigues formula and evaluate three double integrals for the polynomial $R_n^\mu(x)$. A specialization of parameters yields a large number of formulae for various polynomials, as well as for their squares and products. In [8] we obtained integrals for $R_n^\mu(x)$ and in [9] we derived some generating functions for this polynomial.

From (1), one gets the following hypergeometric representation for $R_n^\mu(x)$

$$R_n^\mu(x)=R_n^\mu(c;\ a_1,\dots,a_p;\ \ b_1,\dots,b_q;\ \delta x)$$

$$={}_{p+\mu}F_{q+\mu}\left[\begin{matrix}-n,\ a_1,\dots,a_p,\ \ \Delta(\mu-1,\ c+n)\\ a_1,\dots,a_p,\ \ \Delta(\mu,\ c)\end{matrix};\ \delta x\right],$$

where

$$\delta=\begin{cases}(1-1/\mu)^{\mu-1}\lambda/\mu, & \text{when}\quad \mu>1,\\ \lambda\qquad\qquad\qquad, & \text{when}\quad \mu=1.\end{cases}$$

Specializing the parameters in $R_n^\mu(x)$ one gets a number of particular cases, most of which have been listed in [8].

2. A contour integral representation and a generalized Rodrigues formula for $R_n^\mu(x)$. Denoting the function in the left of (1) by $f(t)$ and using arguments from the theory of functions, we obtain the following contour integral representation for $R_n^\mu(x)$

$$(2)\qquad (c)_n R_n^\mu(x) = \frac{n!}{2\pi i}\int_C t^{-n-1}(1-t)^{-c}{}_pF_q\left[\begin{matrix}a_1,\ldots,a_p\\ b_1,\ldots,b_q\end{matrix};\ \frac{-\lambda xt}{(1-t)^\mu}\right]dt,$$

where C is a circle $|t|=r$ and $r<1$. Putting $u=x/(1-t)$ in (2), we get another integral representation

$$(3)\qquad R_n^\mu(x) = \frac{x^{1-c}n!}{(c)_n 2\pi i}\int_{C'}\frac{u^{n+c-1}}{(u-x)^{n+1}}\,{}_pF_q\left[\begin{matrix}a_1,\ldots,a_p\\ b_1,\ldots,b_q\end{matrix};\ \frac{\lambda u^{\mu-1}(x-u)}{x^{\mu-1}}\right]du,$$

where C' may again be taken as a circle $|u-x|=\varrho$ of small radius.

Evaluating the integral (3) by the Cauchy residue theorem, we obtain a generalized Rodrigues formula given by

$$(4)\qquad R_n^\mu(x) = \frac{x^{1-c}}{(c)_n}\frac{\partial^n}{\partial u^n}\left(u^{n+c-1}\,{}_pF_q\left[\begin{matrix}a_1,\ldots,a_p\\ b_1,\ldots,b_q\end{matrix};\ \frac{\lambda u^{\mu-1}(x-u)}{x^{\mu-1}}\right]\right)_{u=x}.$$

Some special cases

(i) When $\mu=2$, $\lambda=4$, $c=2\nu$ and $a=\nu$ we get the following generalized Rodrigues formula for the polynomial Θ_n of Cohen [1]:

$$\Theta_n(\nu;\ a_2,\ldots,a_p;\ b_1,\ldots,b_q;\ x)$$
$$=\frac{x^{1-c}}{n!}\frac{\partial^n}{\partial u^n}\left(u^{n+2\nu-1}{}_pF_q\left[\begin{matrix}\nu,\ a_2,\ldots,a_p\\ b_1,\ldots,b_q\end{matrix};\ \frac{4u(x-u)}{x}\right]\right)_{u=x}.$$

(ii) When $p=1$, $q=2$, $\mu=\lambda=1$, $a_1=1+\alpha+n$, $b_1=(1+\alpha)/2$, $b_2=1+\alpha/2$ and $c=1+\alpha$ we have

$$L_n^{(\alpha)}(x)L_n^{(\alpha)}(-x)=\frac{(1+\alpha)_n}{(n!)^2}\left(\frac{x}{2}\right)^{2-2c}\times\frac{\partial^n}{\partial u^n}\left(u^{n+\alpha}{}_1F_2\left[\begin{matrix}1+\alpha+n\\ (1+\alpha)/2,\ 1+\alpha/2\end{matrix};\ \frac{x^2}{4}-u\right]\right)_{u=x^2/4}$$

where $L_n^{(\alpha)}(x)$ is the generalized Laguerre polynomial [6].

(iii) Taking $p=2$, $q=1$, $\mu=2$, $\lambda=4$, $a_1=a_2=\nu$, $b_1=2\nu=c$, we get an interesting representation for the square of the Gegenbauer polynomials [6]:

$$\{C_n^\nu(x)\}^2=\frac{(2\nu)_n(1-x^2)^{1-c}}{(n!)^2}\times\frac{\partial^n}{\partial u^n}\left(u^{n+2\nu-1}{}_2F_1\left[\begin{matrix}\nu,\ \nu\\ 2\nu\end{matrix},\ \frac{4u(1-x^2-u)}{1-x^2}\right]\right)_{u=-x^2+1}.$$

(iv) Letting $\mu=2$, $\lambda=4$, $c=2\nu$, $p=2$, $q=1$, $a_1=a_2=\nu$ and $b_1=2\nu$ we get

$$\{p_{n+\nu-1/2}^{1/2-\nu}(x)\}^2=\{[\Gamma(1/2+\nu)]^2(2\nu_n\}^{-1}(-x/4)^{\nu-1/2}(1-x^2)^{1-c}$$
$$\times\frac{\partial^n}{\partial u^n}\left(u^{n+2\nu-1}{}_2F_1\left[\begin{matrix}\nu,\ \nu\\ 2\nu\end{matrix};\ \frac{4u(1-x^2-u)}{1-x^2}\right]\right)_{u=(1-x^2)},$$

fwhere $p_n^\lambda(x)$ is the associated Legendre function [3].

(v) If $\mu=2$, $\lambda=4$, $c=1+\alpha+\beta$, $a_1=(\alpha+\beta+1)/2$, $a_2=(\alpha+\beta+2)/2$, $b_1=1+\alpha$, and $b_2=1/2$, then we have the following formula for the generalized Sister Celine's polynomial [10]

$$f_n^{(\alpha,\beta)}\left[\begin{matrix} a_3, \ldots, a_p; \\ b_3, \ldots, b_q \end{matrix}\; x\right]=\frac{(1+\alpha)_n}{(1+\alpha+\beta)_n n!}x^{1-c}$$

$$\times\frac{\partial^n}{\partial u^n}\left(u^{n+\alpha+\beta}{}_pF_q\left[\begin{matrix}(\alpha+\beta+1)/2,\ (\alpha+\beta+2)/2,\ a_3, \ldots, a_p; \\ 1+\alpha,\ 1/2,\ b_3, \ldots, b_q\end{matrix}\ \frac{4u(x-u)}{x}\right]\right)_{u=x}.$$

For $\alpha=\beta=0$, this reduces to a formula for Sister Celine's polynomial [6].

(vi) When $\mu=\lambda=1$, $p=1$, $q=k$, $a_1=c$, $b_i=(\alpha+i)/k$, $i=1, \ldots, k$, this reduces to the generalized Rodrigues formula [7, (4.5)]

$$Z_n^{\alpha}(x;k)=\Gamma(kn+\alpha+1)x^{k(1-c)}[n!(c)_n]$$

$$\times\frac{\partial^n}{\partial u^n}\left(u^{c+n-1}{}_1F_k\left[\begin{matrix}c \\ (\alpha+1)/k, \ldots, (\alpha+k)/k\end{matrix};\ \frac{x^k-u}{x^k}\right]\right)_{u=x^k}$$

or the polynomial $Z_n^{\alpha}(x;k)$ studied recently by Konhauser [4] and Prabhakar [7].

On further specialization of parameters (4) reduces to the well-known Rodrigues formulae for the classical orthogonal polynomials. Thus for instance in (4), put $\mu=\lambda=1$, $c=1+\alpha$, $p=q=0$ for $L_n^{(\alpha)}(x)$ and $\mu=2$, $\lambda=4$, $c=1$, $a_1=1/2$, $p=1$, $q=0$ for $P_n(x)$.

3. Double integral representations for $R_n^{\mu}(x)$

(i) We first obtain the following double integral representation

$$R_n^{\mu}(x)=[B(l,s)(l+s+r)]^{-1}\int_0^{\infty}\int_0^{\infty}e^{-(x+y)}(x+y)^{\gamma}x^{l-1}y^{s-1}\tag{5}$$

$$\times R_n^{\mu}(c;\,a_1, \ldots, a_p,\ l+s;\ b_1, \ldots, b_q,\ l,\ l+s+r;\ \delta tx)dxdy.$$

Puting in equality [2]

$$\int_0^{\infty}\int_0^{\infty}\varphi(x+y)x^{l-1}y^{s-1}dxdy=B(l,s)\int_0^{\infty}\varphi(u)u^{l+s+1}du\tag{6}$$

$\varphi=u^re^{-u}$, we get

$$\int_0^{\infty}\int_0^{\infty}e^{-(x+y)}(x+y)^ry^{s-1}x^{l-1}$$

$$\times R_n^{\mu}(c;\,a_1, \ldots, a_p,\ l+s;\ b_1, \ldots, b_q,\ l,\ l+s+r;\ \delta tx)dxdy$$

$$=\sum_{k=0}^{\infty}\frac{(l+s)_k(a_1)_k\ldots(a_p)_k(-n)_k\Delta_k(\mu-1,c+n)}{(l)_k(l+s+r)_k(b_1)_k\ldots(b_q)_kk!}\delta^kt^kB(l+k,s)\times\int_0^{\infty}e^{-u}u^{l+k+s+r-1}du,$$

which through [6, 15(1) and 19(3)] leads to (5).

(ii) We now show that

$$R_n^\mu(x)=\frac{2^{-(l+s-2)}\Gamma(l+s+v)}{\Gamma(l)\Gamma(s)\Gamma(v)}\int\int x^{2l-1}y^{2s-1}\left[1-\frac{x^2\quad y^2}{2}\right]^{v-1}$$

$$\times R_n^\mu\left(c;\ a_1,\ldots,a_p,\ l+s+v;\ b_1,\ldots,b_q,\ l;\frac{\delta x^2}{2}t\right)dxdy,$$

for $x\geqq 0$, $y\geqq 0$ and $x^2+y^2\leqq 2$, where $\mathrm{Re}(l)>0$, Re $s>0$ and Re $v>0$.

By taking $f(ax^2+by^2)=[1-(x^2+y^2)/2]^{v-1}$ in (6) we obtain

$$\int\int x^{2l-1}y^{2s-1}f(ax^2+by^2)dxdy=\frac{1}{4a\,b^s}B(l,s)\int_0^{C^2}z^{l+s-1}f(x)dz,$$

where the double integral on the left is taken over the domain defined by $x\geqq 0$, $y\geqq 0$ and $ax^2+by^2\leqq c^2$ and proceeding on the lines of (i), we get the desired result.

(iii) If we take $f(z)=z^\gamma(1-z)^\varrho$, then

$$\int_0^1\int_0^1 f(uv)(1-u)^{\alpha-1}v^\alpha(1-v)^{\beta-1}dudv=B[\alpha,b]\int_0^1 f(z)(1-z)^{\alpha+\beta+1}dz,$$

where $\mathrm{Re}(\alpha)>0$, $\mathrm{Re}(\beta)>0$, then we are led to the following interesting integral representation for $R_n^\mu(x)$

$$R_n^\mu(x)=[B(\alpha,\beta)B(\gamma+1,\ \alpha+\beta+\varrho)]^{-1}\int_0^1\int_0^1(uv)^\gamma(1-uv)^\varrho(1-u)^{\alpha-1}v^\alpha(1-v)^{\beta-1}$$

$$\times R_n^\mu[c;\ a_1,\ldots,a_p,\ \alpha+\beta,\ \alpha+\beta+\varrho+\gamma+1;\ b_1,\ldots,b_q,\ \beta,\ \alpha+\beta+\varrho;\ \delta x(1-v)]dudv,$$

where Re $\gamma>-1$, $\mathrm{Re}(\alpha+\beta+\varrho)>0$, $\mathrm{Re}(\alpha)>0$, $\mathrm{Re}(\beta)>0$.

As in 2, by specialization of the parameters, one can obtain double integral representations for several known polynomials as well as for products and squares of a number of them.

REFERENCES

1. M. E. Cohen. A new class of polynomials relevant to electron collision problems. *Math. Z.*, **108**, 1968—69, 121—128.
2. J. Edward. A treatise on integral calculus. New York, 1922.
3. A. Erdelyi et. al. Higher transcendental functions, Vol. 1. New York, 1953.
4. J. D. E. Konhauser, Biorthogonal polynomials suggested by the Laguerre polynomials. *Pacif. J. Math.*, **21**, 1967, 303—314.
5. P. R. Khandekar. On a generalization of Rice's polynomial-I. *Proc. Mat. Acad. Sci. India Sec.*, A, **24**, Part II, 1964, 157—162.
6. E. D. Rainville. Special functions. New York, 1960.
7. T. R. Prabhakar. On a set of polynomials suggested by the Laguerre polynomial. *Pacif. J. Math.*, **35**, 1970, 213—219.
8. T. R. Prabhakar, S. Jain. On a general class of polynomials relevant to quantum mechanics. *Proc. Nat. Acad. Sci. India*, **44**, 1978, 103—106.
9. T. R. Prabhakar, S. Jain. Some bilateral generating functions for a general class of polynomials relevant to quantum mechanics. Vijnana Parishad Anusandhan Patrika, **22**, 1979, 49—56.
10. Manilal Shah. Certain integrals involving the products of two hypergeometric polynomials. *Proc. Nat. Acad. Sci. India, Sec. A.*, **37**, Part I, 1967, 79—96.

Department of Mathematics
University of Delhi, Delhi-110007

Received 13. 6. 1977