Serdica

Bulgariacae mathematicae publicationes

Сердика

Българско математическо списание

BOUNDED FAMILIES OF HOLOMORPHIC MAPPINGS BETWEEN COMPLEX MANIFOLDS

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For complex manifolds M and N, denote by $H(M, N)$ the space of all holomorphic mappings form M into N with the compact-open topology. We define the concept of a bounded subset of $H(M, N)$ in a natural manner. We call a complex manifold N a Montel manifold if for each complex manifold M every bounded and closed subset of $H(M, N)$ is compact. The Montel manifolds are described by hyperbolic complex manifolds introduced by S. Kobayashi. Riemann's continuation theorem and Osgood's lemma are extended for mappings into Montel manifolds.

1. Preliminaries. The convention in force troughout this paper is that all complex manifolds are Hausdorff connected and second countable.

For two complex manifolds M and N, we shall denote by $H(M, N)$ the space of all holomorphic mappings from M into N with the compact-open topology. The space $H(M, N)$ is Hausdorff and second countable [1].

On every complex manifold N one can introduce the Kobayashi pseudodistance k_N [2]. The most important property of the Kobayashi pseudodistance is the following: If M and N are complex manifolds and if $f: M \to N$ is a holomorphic mapping, then $k_N(f(x), f(y)) \leq k_M(x, y)$; $x, y \in M$. It follows that $H(M, N)$ is an equicontinuous family with respect to k_M and k_N. A complex manifold N is called hyperbolic if the Kobayashi pseudodistance k_N is a distance. In this case k_N induces the topology of N [3].

Let N be a complex manifold and d a distance on N inducing its topology. The pair (N, d) is called a tight manifold if for every complex manifold M the family $H(M, N)$ is equicontinuous with respect to d [4]. A complex manifold N is hyperbolic if and only if it is tight for some distance d [5].

2. Montel manifolds.

Definition 1. *Let M and N be complex manifolds and let F be a subset of $H(M, N)$. We shall say that F is uniformly bounded on compact subsets of M (shortly "F is bounded") if for every compact subset K of M there is a compact subset Q of N such that $f(K) \subset Q$ for every $f \in F$.*

Obviously, this definition coincides with the usual one when $N = \mathbb{C}^n$. Note that if F is bounded and evenly continuous, then F is equicontinuous with respect to each distance on N inducing its topology. Therefore, if F is bounded and equicontinuous with respect to some distance on N inducing its topology, then F is equicontinuous with respect to every such distance [1].

Proposition 1 [7]. *Let M and N be complex manifolds. Then every compact subset F of $H(M, N)$ is bounded and closed.*

Proof. The set F is closed since $H(M, N)$ is Hausdorff. Let K be a compact subset of M. For $f \in F$, $f(K)$ is compact in N and we can find an open and relatively compact neighbourhood $U(f)$ of $f(K)$. If we put $W(f)$

$=\{\varphi\in H(M, N)\colon \varphi(K)\subset U(f)\}$, then there are finite numbers $f_s\in F$, $s=1, \ldots, k$, such that $F\subset \bigcup_{s=1}^{k} W(f_s)$. We set $Q=\bigcup_{s=1}^{k}\overline{U(f_s)}$. The set Q is compact in N and $f(K)\subset Q$ for every $f\in F$.

The converse is not true as shows the following

Example 1. Let Δ be the unit disk in $\mathbb{C}$. Let $\pi\colon \mathbb{C}^2-\{0\}\to\mathbb{P}^1(\mathbb{C})$ be the natural projection. We set $f_n(z)=\pi(1, n^2z-n)$ for $z\in\Delta$, $n=1, 2, \ldots$. We denote by F the closure of the set $\{f_n\colon n=1, 2, \ldots\}$ in $H(\Delta, \mathbb{P}^1(\mathbb{C}))$. The family F is bounded and closed but it is not compact. In fact, if we assume that F is compact, then there is a subsequence $\{f_{n_k}\}$ of $\{f_n\}$, which converges in $H(\Delta, \mathbb{P}^1(\mathbb{C}))$ to a holomorphic mapping f. We have $f_{n_k}(1/n_k)=\pi(1, 0)$, hence $f(0)=\lim_{k\to\infty}f_{n_k}(1/n_k)=\pi(1, 0)$. On the other hand, $f_{n_k}(0)=\pi(1, -n_k)=\pi(-1/n_k, 1)$, hence $f(0)=\lim_{k\to\infty}f_{n_k}(0)=\pi(0, 1)\neq\pi(1, 0)=f(0)$.

Now we give the following

Definition 2. *Let N be a complex manifold. We shall say that N is a Montel manifold if for each complex manifold M every bounded and closed subset of $H(M, N)$ is compact.*

Example 2. Every domain in $\mathbb{C}^n$ is a Montel manifold by Montel's theorem.

Example 3. Every hyperbolic manifold is a Montel manifold by Ascoli's theorem. Note that $\mathbb{C}^n$ is a Montel manifold but it is not hyperbolic ($k_{\mathbb{C}^n}\equiv 0$).

Let N be a complex manifold. A subset of N is called a (bounded) domain in N if it is (relatively compact) open and connected.

Proposition 2. *A complex manifold N is a Montel manifold iff every bounded domain in N is a hyperbolic manifold.*

Proof. Sufficiency. Let M be a complex manifold and let F be a bounded subset of $H(M, N)$. We want to show that F is relatively compact. Let ϱ be a distance on N inducing its topology. Let A be a bounded domain in M biholomorphically equivalent to a bounded domain in $\mathbb{C}^m$, $m=\dim M$. Every bounded domain in $\mathbb{C}^m$ is a hyperbolic manifold [2], hence so is A. Let Q be a compact subset of N such that $f(A)\subset Q$ for every $f\in F$. We can find a sequence $\{D_\nu\}$ of bounded domains in N such that $N=\bigcup_{\nu=1}^{\infty}D_\nu$ and $D_\nu\subset D_{\nu+1}$, $\nu=1, 2, \ldots$. Then Q is contained in some domain D_{ν_0}. The domain $D=D_{\nu_0}$ is a hyperbolic manifold by the assumption. The family $F_A=\{f\mid A\colon f\in F\}$ is a bounded subset of $H(A, D)$ and F_A is equicontinuous with respect to the Kobayashi distances k_A and k_D. Therefore, F_A is equicontinuous with respect to the distance $d=\varrho\mid(D\times D)$. This proves that F is equicontinuous with respect to ϱ. Hence, by Ascoli's theorem, F is relatively compact.

Necessity. Let D be a bounded domain in N and ϱ a distance on N inducing its topology. Then (D, d), where $d=\varrho\mid(D\times D)$ is a tight manifold, hence D is hyperbolic. In fact, let M be a compex manifold. Then $H(M, D)$ is a bounded subset of $H(M, N)$, hence $H(M, D)$ is relatively compact in $H(M, N)$. Therefore, by Ascoli's theorem, $H(M, D)$ is equicontinuous with respect to ϱ. Thus, $H(M, D)$ is equicontinuous with respect to d.

Corollary 1. *A compact complex manifold is a Montel manifold iff it is a hyperbolic manifold.*

Corollary 2. *If N_1 and N_2 are Montel manifolds, so is $N_1\times N_2$.*

Corollary 3. *A complex submanifold of a Montel manifold is a Montel manifold.*

A complex manifold $\tilde{N}$ is called a domain (or a spread) over a complex manifold N with projection π if π is a locally biholomorphic mapping from $\tilde{N}$ into N. Since a domain over a hyperbolic manifold is hyperbolic [2], we have:

Corollary 4. *A domain over a Montel manifold is a Montel manifold. In particular, every domain over* $\mathbb{C}^n$ *is a Montel manifold.*

Corollary 5. *If a complex manifold N' is holomorphically immersed in a Montel manifold N, then N' is also a Montel manifold.*

Proof. Let $\varphi: N' \to N$ be a holomorphic immersion. Let D' be a bounded domain in N'. Then $\varphi(D')$ is relatively compact in N, hence it is contained in a bounded domain D in N. The domain D is a hyperbolic manifold and D' is holomorphically immersed in D. Hence D' is hyperbolic [2].

Proposition 3. *Let N be a complex manifold satisfying the following condition:*

(*) *for every point $p \in N$, there exists a holomorphic mapping $z_p: N \to \mathbb{C}^n$, $n = \dim N$, which is a local coordinate system of N in a neighbourhood of p.*

Then N is a Montel manifold.

To prove this proposition we need the following

Lemma 1 [7]. *Let M and N be complex manifolds,* $\dim M = m$, $\dim N = n$. *Let V be an open and relatively compact subset of M with a local coordinate system $\zeta = (\zeta^1, \ldots, \zeta^m)$. Let A be a bounded domain in V. Let d_A be the distance on A yielded by the metric $ds_A^2 = \sum_{r=1}^m |d\zeta^r|^2$. Assume that N satisfies the condition* (*).

If F is a bounded subset of $H(M, N)$, then there is a distance d_N on N inducing its topology such that

$$d_N(f(x), f(y)) \leq d_A(x, y) \tag{1}$$

for each $f \in F$; $x, y \in A$.

Proof. There is a compact subset Q of N such that $f(V) \subset Q$ for each $f \in F$. We can find an open cover $\{\{U_s\}_{s=1}^k, \{U_q\}_{q \in N-Q}\}$ of N such that: (a) $\bar{U}_s$, $1 \leq s \leq k$, is compact in N; (b) for each s, $1 \leq s \leq k$, there is a holomorphic mapping $z_s = (z_s^1, \ldots, z_s^n): N \to \mathbb{C}^n$, which is a local coordinate system in a neighbourhood of $\bar{U}_s$; (c) U_q, $q \in N - Q$, is a coordinate neighbourhood with a local coordinate system $z_q = (z_q^1, \ldots, z_q^n)$; (d) $Q \subset \bigcup_{s=1}^k U_s$ and $U_q \subset N - Q$ for $q \in N - Q$. Let $\{\{\varphi_s\}_{s=1}^k, \{\varphi_q\}_{q \in N-Q}\}$ be a partition of the unity such that $\operatorname{supp} \varphi_s \subset U_s$, $\operatorname{supp} \varphi_q \subset U_q$. We set:

$$\theta_s = \begin{cases} \varphi_s \sum_{l=1}^n |dz_s^l|^2 & \text{in } U_s \\ 0 & \text{in } N - U_s \end{cases}; \quad \theta_q = \begin{cases} \varphi_q \sum_{l=1}^n |dz_q^l|^2 & \text{in } U_q \\ 0 & \text{in } N - U_q \end{cases}.$$

The metric $ds_N^2 = \sum_{s=1}^k \theta_s + \sum_{q \in N-Q} \theta_q$ yields a distance ϱ_N on N, which induces the topology of N.

The mapping $z_s \circ f$ is holomorphic in V and maps V in the bounded set $\bigcup_{j=1}^k z_j(U_j) \subset \mathbb{C}^n$ for every $f \in F$ and s, $1 \leq s \leq k$. The set $\bar{A}$ is compact in V hence, by Cauchy's inequality, there is a constant $C > 0$ such that

$$|\partial(z_s^l \circ f)/\partial\zeta^r| \leq C \text{ in } A \text{ for every } f \in F,\ 1 \leq s \leq k,\ 1 \leq l \leq n,\ 1 \leq r \leq m. \tag{2}$$

Obviously, we have

$$(3)\qquad (f^*ds_N^2)(a) \leq \sum_{s=1}^{k} (\varphi_s \circ f)(a) \sum_{l=1}^{n} \left| \sum_{r=1}^{m} (\partial(z_s^l \circ f)/\partial\zeta^r)(a)d\zeta^r(a) \right|^2$$

for $f \in F$ and $a \in A$. By means of the Cauchy-Schwartz inequality, the inequalities (2) and $0 \leq \varphi_s \leq 1 (1 \leq s \leq k)$ we obtain from (3)

$$(4)\qquad (f^*ds_N^2)(a) \leq C^2mnk \sum_{r=1}^{m} |d\zeta^r(a)|^2 = C^2mnk\, ds_A^2(a);\ f \in F,\ a \in A$$

If we set $d_N = (C^2mnk)^{-1/2}\varrho_N$, then (1) follows from (4).

Proof of Proposition 3. Let M be a complex manifold and let F be a bounded subset of $H(M, N)$. Then, by Lemma 1, F is evenly continuous. Hence, by Ascoli's theorem, F is relatively compact.

Corollary 6. *Every Stein manifold is a Montel manifold.*

This corollary also follows from the imbedding theorem and corollary 3.

The following lemma can be proved similary to Lemma 1.

Lemma 2 [7]. *Let (N, π) be a domain over $\mathbb{C}^n$ such that $\pi(N)$ is bounded in $\mathbb{C}^n$. Denote by d_N the distance on N induced by the metric $ds_N^2 = \Sigma_{k=1}^n |d\pi^k|^2$, where $(\pi^1, \ldots, \pi^n) = \pi$. Let $M, V, \zeta, A, ds_A^2, d_A$ be as in Lemma 1.*

Then there is a constant $C > 0$ such that

$$(5)\qquad d_N(f(x), f(y)) \leq Cd_A(x, y)$$

or every $f \in H(M, N)$ and $x, y \in A$.

Proof. In view of the boundedness of $\pi(N)$ and Cauchy's inequality, there is a constant $C_1 > 0$ such that $|\partial(\pi^k \circ f)/\partial\zeta^r| \leq C_1$ in A for every $f \in H(M, N)$ and $1 \leq k \leq n$, $1 \leq r \leq m (m = \dim M)$. We have

$$(f^*ds_N^2)(a) = \sum_{k=1}^{n} \left| \sum_{r=1}^{m} (\partial(\pi^k \circ f)/\partial\zeta^r)(a)d\zeta^r(a) \right|^2 \leq C_1^2mn\, ds_A^2(a)$$

for every $f \in H(M, N)$ and $a \in A$. This inequality implies (5).

Corollary 7. *Let (N, π) be a domain over $\mathbb{C}^n$ such that $\pi(N)$ is bounded in $\mathbb{C}^n$. Then (N, d_N) is a tight manifold, where d_N is the distance from Lemma 2 (hence N is hyperbolic).*

The last result was proved by S. Kobayashi [2] by a different method and in more general situation.

3. Locally bounded mappings into Montel manifolds.

Definition 3. *Let M and N be complex manifolds, and let A be a subset of M. We shall say that a mapping $f: M - A \to N$ is locally bounded at a point $x \in M$ if there is a neighbourhood U of x such that $f(U - A)$ is relatively compact in N. We shall say that f is locally bounded on M if f is locally bounded at every point of M.*

Obviously, this definition coincides with the usual one when $N = \mathbb{C}^n$. If $A = \emptyset$, then f is loccally bounded on M iff $\{f\}$ is bounded in the sence of Definition 1.

From Proposition 2 and the theorem of Mrs. M. Kwack [2, Theorem 3.1 p. 83], we obtain the following lemma.

Lemma 3. *Denote $\Delta = \{z \in \mathbb{C}: |z| < 1\}$, $\Delta^* = \{z \in \mathbb{C}: 0 < |z| < 1\}$. Let N be a Montel manifold. Let $f: \Delta^* \to N$ be a locally bounded at the origin* *holomorphic mapping. Then f extends to a holomorphic mapping $\tilde{f}\colon \Delta \to N$.*

Proposition 4. *Let M and N be complex manifolds. Let A be an analytic subset of M of a dimension $\leq \dim M - 1$. Assume that N is a Montel manifold. Then every locally bounded on M holomorphic mapping $f\colon M - A \to N$ can be extended to an unique holomorphic mapping $\tilde{f}\colon M \to N$.*

Proof. Proceeding by induction on the dimension of A, we may assume that A is nonsingular (cf. [2, Theorem 4.1, p. 86]). For every point $a \in A$, we want to find a neighbourhood U of a such that the restriction $f|(U-A)$ can be extended to a holomorphic mapping from U into N. Therefore, we may assume that: (a) M is the unit polydisk $\Delta^m = \{(z, t^1, \dots, t^{m-1}) \in \mathbb{C}^m : |z| < 1, |t^1| < 1, \dots, |t^{m-1}| < 1\}$, $m = \dim M$; (b) $a = 0$; (c) A lies in the subset of Δ^m defined by $z = 0$; (d) $f(\Delta^m - A)$ is relatively compact in N. The set $\overline{f(\Delta^m - A)}$ is contained in a bounded domain D in N. The domain D is a hyperbolic manifold, by Proposition 2, and we end the proof as in [2, Theorem 4.1, p. 86] using Lemma 3.

The condition that N is a Montel manifold cannot be omitted in Proposition 4.

Example 4. The mapping $z \to e^{1/z}$ of $\mathbb{C} - \{0\}$ into $P^1(\mathbb{C})$ is holomorphic and locally bounded on $\mathbb{C}$ but it cannot bo extended to a continuous mapping at the origin.

It is well-known that if $N = \mathbb{C}^n$ and $\operatorname{codim} A \geq 2$, every holomorphic mapping $f\colon M - A \to N$ is locally bounded on M. But, in general, this is not true as shows the following

Example 5. Denote $\Delta^m = \{(z^1, \dots, z^m) \in \mathbb{C}^m : |z^1| < 1, \dots, |z^m| < 1\}$, where $m \geq 2$, $M = \Delta^m$, $A = \{0\}$, $N = \Delta^m - \{0\}$, $f = \mathrm{id}_{\Delta^m - \{0\}}$. The set $f(U - \{0\}) = U - \{0\}$ is not relatively compact in N for every neighbourhood U of 0 in M.

Let $M_1, \dots, M_k, N$ be complex manifolds. Let G be an open subset of $M_1 \times \dots \times M_k$. Let $a = (a_1, \dots, a_k) \in G$. The set $G_{s,a} = \{m_s \in M_s : (a_1, \dots, a_{s-1}, m_s, a_{s+1}, \dots, a_k) \in G\}$ is open in M_s, $1 \leq s \leq k$. A mapping $f\colon G \to N$ is called separately holomorphic if for every point $a \in G$ and every natural number s, $1 \leq s \leq k$, the mapping $f_{s,a}\colon G_{s,a} \to N$, defined by $f_{s,a}(m_s) = f(a_1, \dots, a_{s-1}, m_s, a_{s+1}, \dots, a_k)$, $m_s \in G_{s,a}$, is holomorphic. If $f\colon G \to \mathbb{C}^n$ is a locally bounded separately holomorphic mapping, then f is a holomorphic mapping by a classical result of Osgood, well-known as Osgood's lemma. This assertion is not true if we replace $\mathbb{C}^n$ by an arbitrary complex manifold as shows the following example due to T. Barth [6]:

Example 6. Let $\pi\colon \mathbb{C}^2 - \{0\} \to P^1(\mathbb{C})$ be the natural projection. Let $g\colon \mathbb{C}^2 \to \mathbb{C}^2 - \{0\}$ be the mapping defined by

$$g(z, w) = \begin{cases} (1,\ (w+z)^2/(w-z)), & w \neq z; \\ (0,\ 1), & w = z \neq 0; \\ (1,\ 0), & w = z = 0. \end{cases}$$

Then the mapping $f = \pi \circ g$ is locally bounded and separately holomorphic but it is not continuous at the origin.

It is easy to show that the conclusion of Osgood's lemma holds if we replace $\mathbb{C}^n$ by a Montel manifold.

Proposition 5. *Let $M_1, \dots, M_k, N$ be complex manifolds and let G be an open subset of the complex manifold $M_1 \times \dots \times M_k$. Assume that N*

is a Montel manifold. Then every locally bounded on G separately holomorphic mapping $f: G \to N$ is holomorphic.

P r o o f. Since the conclusion is local, we may assume that G is the unit polydisk Δ^m in $\mathbb{C}^m$, $m = \Sigma_{s=1}^{k} \dim M_s$, and that $f(\Delta^m)$ is relatively compact in N. Then $f(\Delta^m)$ is contained in a bounded domain D in N. The domain D is a hyperbolic manifold, by Proposition 2, and $f: \Delta^m \to D$ is separately holomorphic. It follows that f is holomorphic [6].

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Received 13. 1. 1978