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A PROOF OF TELYAKOVSKI — GOPENGAUZ THEOREM THROUGH INTERPOLATION

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This paper gives an elegant proof of Telyakovski's theorem for continuous functions defined on $[-1,1]$ by actually constructing interpolatory polynomials of degree not higher than $4n$ based on the nodes $x_k = \cos k\pi/n$, $k=\overline{0,n}$. This paper also includes a new proof of R. M. Trigub's inequality for the derivative of the polynomial.

1. Introduction. In their paper [2] O. Kis and P. Vertesi constructed the polynomials $P_n(f, x)$ of degree at most $4n$, which interpolate the given function $f(x) \in C[-1,1]$ at the points

$$x_{kn} = \cos 2k\pi/2n+1, \ k=\overline{0, n}, \tag{1.1}$$

where $k=\overline{0,n}$ stands for $k=0, 1, 2, \ldots, n$ and satisfy A. F. Timan's inequality

$$|f(x)-P_n(f, x)| \leq C_1\omega_f(\Delta_n(x)), \ -1\leq x\leq 1. \tag{1.2}$$

Here $\omega_f(.)$ is the modulus of continuity of $f(x)$, $\Delta_n(x)=n^{-1}\sqrt{1-x^2}+n^{-2}$ and C_1 an absolute positive constant. We observe that the inequality (1.2) cannot be replaced by the inequality

$$|f(x)-P_n(f, x)| \leq C_2\omega_f((1-x^2)^{1/2}n^{-1}) \quad -1\leq x\leq 1, \tag{1.3}$$

for $P_n(f, -1) \neq f(-1)$. The inequality (1.3) was first proved by S. A. Telyakovskii [4] and I. E. Gopengauz [1].

Our aim, in this paper, is to give the proof of Telyakovskii — Gopengauz inequality (1.3) by constructing the polynomials $Q_n(f, x)$, which interpolate the function at the points

$$x_{kn} = \cos k\pi/n, \quad k=\overline{0, n}. \tag{1.4}$$

We may mention that the proof of the inequality (1.3) has earlier been given by R. B. Saxena [3] by different interpolation polynomials constructed on the nodes (1.4).

We shall see that our polynomials are simpler in nature than the polynomials in [3]. In fact our polynomials may be compared with the polynomials in [2].

2. We describe the construction of the polynomials $Q_n(f, x)$. Let $-1\leq x\leq 1$, $\cos t=x$ and $\cos t_{kn}=x_{kn}$ (from now onwards we shall be writing k instead of kn for the sake of simplicity) with

$$t_k=k\pi/n, \quad k=\overline{0, n}. \tag{2.1}$$

Further for $k=\overline{1, 2n}$, let

$$(2.2)\qquad l_k(t)=[\sin n(t-t_k)\cos((t-t_k)/2)]/[2n\sin((t-t_k)/2)]$$
$$=\frac{1}{2n}[1+2\sum_{j=1}^{n-1}\cos j(t-t_k)+\cos n(t-t_k)]$$

and

$$(2.3)\qquad p_k(t)=4l_k^3(t)-3l_k^4(t).$$

Then for any arbitrary function $f(x)$, given on $[-1,1]$, we define the polynomials

$$(2.4)\qquad Q_n(f,x)=\frac{1+x}{2}f(1)+\frac{1-x}{2}f(-1)$$
$$+\sum_{k=0}^{n}[f(x_k)-\left\{\frac{1+x}{2}f(1)+\frac{1-x}{2}f(-1)\right\}]\cdot q_k(x),$$

where

$$(2.5)\qquad q_0(x)=p_{2n}(t),\ q_n(x)=p_n(t)\ \text{and}\ q_k(x)=p_k(t)+p_{2n-k}(t),\ k=\overline{1,n-1}.$$

We note that our polynomials $Q_n(f,x)$ are of degree at most $4n+1$ as the fundamental polynomials $q_k(x)$ are of degree $4n$ at most which can easily be seen from their definitions. Moreover, they interpolate the function at the points (2.1), because $q_k(x_j)=\delta_{kj}$, $k,j=\overline{0,n}$, which is an easy consequence of $l_k(t_j)=\delta_{kj}$, $k,j=\overline{1,2n}$. With the help of the polynomials $Q_n(f,x)$, we shall first prove the following

Theorem 1. *Let $f(x)\in C[-1,1]$ and n be any natural number, then for every $x\in[-1,1]$ we have*

a) $$|Q_n(f,x)-f(x)|\leq C_3\omega_f(\Delta_n(x)),$$

b) $$|Q_n'(f,x)|\leq C_4\Delta_n^{-1}(x)\omega_f(\Delta_n(x)).$$

Remark 1. We could prove the theorem simply by considering the polynomials $R_n(f,x)=f(C)+\sum_{k=0}^{n}[f(x_k)-f(C)]\cdot q_k(x)$, $-1<c<1$. The inequality (b) which supplements the inequality (a) was given by R. M. Trigub in [5].

Our main aim is to prove the following

Theorem 2. *Let $f(x)\in C[-1,1]$, then for every $x\in[-1,1]$ we have*

$$|f(x)-Q_n(f,x)|\leq C_5\omega_f(n^{-1}(1-x^2)^{1/2}).$$

Remark 2. This theorem can also be proved with the help of the polynomials given in Remark 1.

Before proving the theorems we need some lemmas.

3. Lemma 1. *The following identity holds:*

$$(3.1)\qquad \sum_{k=1}^{2n}p_k(t)=1-\frac{1}{64n^3}(9-12\cos 2nt+3\cos 4nt).$$

Proof. Following Kis — Vertesi [1] we have from (2.2)

$$2nl_0(t)=1+2\sum_{j=1}^{n}\cos jt-\cos nt=\sum_{j=-n}^{n}z^j-\cos nt,$$

$$(3.2)\qquad (2n)^3 \sum_{k=1}^{2n} l_k^3(t) = \sum_{k=1}^{2n} \left\{ \sum_{j=-3n}^{3n} C_{j,3} z^j e^{-ijt_k} - 3\cos n((t-t_k) \sum_{j=-2n}^{2n} C_{j,2} z^j e^{-ijt_k} + 3\cos^2 n(t-t_k) \sum_{j=-n}^{n} z^j e^{-ijt_k} - \cos^3 n(t-t_k) \right\},$$

where $z=e^{it}$. Noting that

$$\sum_{k=1}^{2n} e^{-ijt_k} = \frac{e^{-ij(\pi/n)}[1-e^{-2ij\pi}]}{1-e^{-ij(\pi/n)}}$$

$$=\begin{cases} 0 & \text{if } j \text{ is not a multiple of } 2n \\ 2n & \text{if } j \text{ is a multiple of } 2n, \end{cases}$$

and

$$\sum_{k=1}^{2n} e^{-ik\pi(1+j/n)} = \begin{cases} 2n, & \text{if } j=\pm n, \pm 3n, \pm 5n, \ldots \\ 0, & \text{in the contrary case,} \end{cases}$$

we obtain

$$(2n)^2 \sum_{k=1}^{2n} l_k^3(t) = C_{0,3} + 2C_{2n,3}\cos 2nt - 3\,.\,2C'_{n,3}\cos^2 nt + 3\cos^2 nt,$$

where $C_{0,3}$ and $C_{2n,3}$ are respectively the coefficients of Z^0 and Z^{2n} for $m=3$, $C'_{n,3}$ the coefficient of Z^n for $m=2$ in the expansion

$$(3.3)\qquad \sum_{j=-mn}^{mn} C_{j,m} Z^j = Z^{-mn}(1-Z^{2n+1})^m \sum_{j=0}^{\infty} \frac{(j+1)(j+2)\ldots(j+m-1)}{(m-1)!}.$$

Obviously $C_{j,m}=C_{-j,m}$, $j=\overline{1, mn}$.

Thus, we have after simplification

$$C_{0,3}=3n^2+3n+1,\ C_{2n,3}=(n^2+3n+2)/2,\ C'_{n,3}=n+1.$$

On substituting these values of $C_{0,3}$, $C_{2n,3}$ and $C'_{n,3}$ in (3.2) we obtain

$$(3.4)\qquad (2n)^2 \sum_{k=1}^{2n} l_k^3(t) = \left(3n^2-\frac{1}{2}\right)+\left(n^2+\frac{1}{2}\right)\cos 2nt.$$

In the same way we have

$$(3.5)\qquad (2n)^3 \sum_{k=1}^{2n} l_k^4(t) = C_{0,4}+2C_{2n,4}\cos 2nt + 2C_{4n,4}\cos 4nt - 4\cos nt[2C'_{n,4}\cos nt + 2C'_{2,4}\cos 3nt] + 6\cos^2 nt[C''_{0,4}+2C''_{2n,4}\cos 2nt] - 8\cos^4 nt + \cos^4 nt,$$

where, again, the numbers $C_{0,4}$, $C_{2n,4}$ and $C_{4n,4}$ are respectively the coefficients of Z^0, Z^{2n} and Z^{4n} for $m=4$, the numbers $C'_{n,4}$ and $C'_{3n,4}$ the coefficients of Z^n and Z^{3n} for $m=3$ and the numbers $C''_{0,4}$ and $C''_{2n,4}$ the coefficients of Z^0 and Z^{2n} for $m=2$ in the expansion (3.3). Thus, we have on simplification

$$C_{0,4}=(16n^3+24n^2+14n+3)/3,\ C_{2n,4}=(4n^3+12n^2+11n+3)/3,\ C_{4n,4}=1,$$

$$C'_{n,4}=2n^2+3n+1,\ C'_{3n,4}=1,\ C''_{0,4}=2n+1,\ C''_{2n,4}=1.$$

Hence with these values of the coefficients we obtain from (3.5)

$$(3.6)\qquad (2n)^3\sum_{k=1}^{2n} l_k^4(t)=\left(\frac{16}{3}n^3-\frac{4}{3}n+\frac{3}{8}\right)+\left(\frac{8n^3}{3}+\frac{4n}{3}-\frac{1}{2}\right)\cos 2nt+\frac{1}{8}\cos 4nt.$$

From (3.4) and (3.6) we have

$$(3.7)\qquad \sum_{k=1}^{2n}(4l_k^3(t)-3l_k^4(t))=1-\frac{1}{64n^3}(9-12\cos 2nt+3\cos 4nt),$$

which is essentially the same as (3.1).

Remark 3. Compare the identity (3.7) with the identity of A. H. Tureckii, where we have $4l_k^3(t)-3l_k^4(t)=1$, when the nodes of interpolation are the points (1.1).

Lemma 2. *There hold*

a) $$|l_k^3(t)+l_{k+1}^3(t)|\leq 3\pi s_k^{-4}\sin^2 nt,$$

b) $$|l_k^2(t)l'_k(t)+l_{k+1}^2(t)l'_{k+1}(t)|\leq 21\pi s_k^{-4}n\sin^2 nt,$$

c) $$|(\cos t-\cos t_k)l_k^3(t)+(\cos t-\cos t_{k+1})l_{k+1}^3(t)|\leq\frac{13}{2}\pi s_k^{-2}\sin^2 t,$$

where $s_k=2n\sin(t-t_k)/2$.

Proof. Since

$$(3.8)\qquad l_k^3+l_{k+1}^3=(l_k+l_{k+1})(l_k^2-l_kl_{k+1}+l_{k+1}^2)$$

and

$$l_k+l_{k+1}=\frac{(-1)^{k-1}\sin nt\sin(\pi/2n)}{2n\sin(t-t_k)/2\sin(t-t_{k+1})/2}.$$

Hence we have

$$|l_k^3(t)+l_{k+1}^3(t)|\leq\frac{\sin^2 nt}{|s_k||s_{k+1}|}\{s_k^{-2}+1/|s_k||s_{k+1}|+s_{k+1}^{-2}\}\leq 3\pi s_k^{-4}\sin^2 nt$$

since $|s_k|\leq|s_{k+1}|$, which proves the first part of the lemma.

To prove the second part, we see that

$$|l'_k(t)+l'_{k+1}(t)|=\left|\frac{(-1)^{k-1}\sin nt}{2\cdot 2n}\operatorname{cosec}^2(t-t_k)/2\right.$$

$$+\frac{(-1)^k\sin nt}{2\cdot 2n}\operatorname{cosec}^2(t-t_{k+1})/2+\frac{1}{2}(-1)^k\cos nt\cot(t-t_k)/2$$

$$\left.+\frac{1}{2}(-1)^{k+1}\cos nt\cot\frac{1}{2}(t-t_{k+1})\right|\leq 4n\pi s_k^{-4}+n\pi s_k^{-2}\leq 5n\pi s_k^{-2}$$

and hence we get after differentiating (3.8)

$$|l_k^2(t)l'_k(t)+l_{k+1}^2(t)l'_{k+1}(t)|$$

$$\leq 5n\pi s_k^{-2}\cdot 3s_k^{-2}\sin^2 nt+\pi s_k^{-2}\cdot 2.3ns_k^{-2}\sin^2 nt=18\pi ns_k^{-4}\sin^2 nt,$$

where we have used $|l'_k(t)|\leq n/|s_k|$, which gives the part (b).

Lastly, we can easily see that

$$\cos t_k-\cos t=2\sin t\,.\sin(t-t_k)/2\cos(t-t_k)/2-2\cos t\sin^2(t-t_k)/2, \tag{3.9}$$

$$\cos t_k-\cos t_{k+1}=2\sin(\pi/2n)\sin(t_k+t_{k+1})/2 \tag{3.10}$$

$$\leq\pi\{\sin t\cos(t_{k+1}-t)+2\cos t\sin((t_{k+1}-t)/2)\cos((t_{k+1}-t)/2)\}/n.$$

Therefore, we get

$$|(\cos t-\cos t_k)l_k^3(t)+(\cos t-\cos t_{k+1})l_{k+1}^3(t)|\leq(13\pi/2)s_k^{-2}\sin^2 t$$

and we have our last of the lemma proved.

4. Proof of the Theorem 1. On account of (2.4), we have the identity

$$Q_n(f,x)-f(x) \tag{4.1}$$

$$=\left[\frac{1+x}{2}(f(1)-f(x))+\frac{1-x}{2}(f(-1)-f(x))\right]\left[1-\sum_{k=0}^{n}q_k(x)\right]$$

$$+\sum_{k=0}^{n}(f(x_k)-f(x))q_k(x)=\Sigma_1+\Sigma_2.$$

Using the properties of modulus of continuity we have

$$(1+x)\omega_f(1-x)+(1-x)\omega_f(1+x)\leq 6\omega_f(1-x^2),\ x\in[-1,1]. \tag{4.2}$$

Hence we obtain after using (4.2) and (3.1)

$$|\Sigma_1|\leq\frac{3}{8n^3}\frac{6}{2}\omega_f(1-x^2)\leq\frac{9}{8}\omega_f(\Delta_n(x)). \tag{4.3}$$

Now we break the sum Σ_2 into four parts after making use of (2.5), i. e.

$$\Sigma_2=(f(\cos t_j)-f(\cos t))p_j(t)+\sum_{k=1}^{j-1}(f(\cos t_k)-f(\cos t))4l_k^4(t)$$

$$+\sum_{k=j+1}^{2n}(f(\cos t_k)-f(\cos t))4l_k^3(t)-3\sum_{k=1,\,k\neq j}^{2n}(f(\cos t_k)-f(\cos t))l_k^4(t)$$

$$=(f(\cos t_j)-f(\cos t))p_j(t)+\Sigma_2'+\Sigma_2''+\Sigma_2''',$$

where j is defined by

$$|t-t_j|\leq\pi/2n. \tag{4.4}$$

We will now show that each constituent of the sum Σ_2 is $o\{\omega_f(\Delta_n(x))\}$. For the first constituent, we have from (3.9), (2.3), (4.4) and the properties of modulus of continuity

$$|f(\cos t_j)-f(\cos t)|\,|p_j(t)| \tag{4.5}$$

$$\leq(1+\pi/2)\omega_f(n^{-1}(1-x^2)^{1/2})+(1+\pi^2/8)\omega_f(|x|/n^2).$$

The estimates for Σ_2' and Σ_2'' are the same, so we do only Σ_2'. We make use of the method of Okis [2], i. e. group the summands into pairs. Thus, if the number of terms in Σ_2' is even, they are grouped in pairs and no term is left, but if the number of terms is odd, one term will be left out which can be estimated as (4.5). Now using lemma 2(a) and (3.9), (3.10), we obtain

$$|\Sigma_2'| \leq 4.3\pi \sum_{k=1}^{j-1} \left(\frac{1}{s_k^4}+\frac{1}{s_k^3}\right) \omega_f(n^{-1}(1-x^2)^{1/2}) + 4.3\pi \sum_{k=1}^{j-1} \left(\frac{1}{s_{k+1}^3}+\frac{1}{2s_{k+1}^2}\right) \omega_f(|x|/n^2),$$

since $|s_k| = |2n \sin(t-t_k)/2| \geq 2n(|k-j|)-1)/2n = 2i-1$, $i=|k-j|$, $k \neq j$. We easily obtain

$$|\Sigma_2'| \leq 12\pi\omega_f(n^{-1}(1-x^2)^{1/2}) \sum_{i=1}^{\infty} ((2i-1)^{-3}+(2i-1)^{-4}) \tag{4.6}$$

$$+12\cdot\pi\omega_f(|x|/n^2) \sum_{i=1}^{\infty} ((2i+1)^{-3}+(2i+1)^{-2}/2) = O\{\omega_f(\Delta_n(x))\}.$$

For the last constituent, using (3.9), (3.10) and the estimates for $l_k(t)$, we have

$$|\Sigma_2'''| \leq 3\cdot \sum_{k=1,\ k\neq j}^{2n} \{|s_k|^{-3}+s_k^{-4}\}\omega_f(n^{-1}(1-x^2)^{1/2})\} \tag{4.7}$$

$$+\sum_{k=1,\ k\neq j}^{2n} \{s_k^{-2}+2s_k^{-2}/2\}\,\omega_f(|x|/n^2) = O\{\omega_f(n^{-1}(1-x^2)^{1/2}+|x|n^{-2})\}.$$

Thus, combining (4.6), (4.7), (4.5) we obtain from (4.1)

$$|Q_n(f,x)-f(x)| = O\{\omega_f(n^{-1}(1-x^2)^{1/2}+|x|n^{-2})\},$$

which proves the first part of the theorem.

For the second part, we differentiate (2.4)

$$Q_n'(f,x) = \frac{f(1)-f(-1)}{2} + \sum_{k=0}^{n} -\left(\frac{f(1)-f(-1)}{2}\right) q_k(x)$$

$$+\sum_{k=1}^{n} [f(x_k)-\left\{\frac{1+x}{2}f(1)+\frac{1-x}{2}f(-1)\right\}]\cdot q_k'(x)$$

$$=\left[\frac{f(1)-f(-1)}{2}\right]\left[1-\sum_{k=1}^{n} q_k(x)\right]+\sum_{k=0}^{n}(f(x_k)-f(x)q_k'(x)$$

$$+\left\{\frac{1+x}{2}(f(x)-f(1))+\frac{1-x}{2}(f(x)-f(-1))\right\}\sum_{k=0}^{n} q_k'(x).$$

With the help of (2.5), we have

$$Q_n'(f,x) = \left[\frac{f(1)-f(-1)}{2}\right]\left[1-\sum_{k=1}^{2n} p_k(t)\right]+\frac{1}{\sin t}\sum_{k=1}^{2n}(f(\cos t_k)-f(\cos t))p_k'(t)$$

$$+\frac{1+x}{2}(f(x-f(1))+\frac{1-x}{2}(f(x)-f(-1))\frac{1}{\sin t}\sum_{k=1}^{2n} p_k'(t) = \Sigma_{21}+\Sigma_{22}+\Sigma_{23}.$$

For Σ_{21} and Σ_{23} using the appropriate form of the identity we see that

(4.8) $$|\Sigma_{21}| \leq \omega_f(1) . 3/8n^3 \leq \Delta_n^{-1}(x)\omega_f(\Delta_n(x))$$

and

(4.9) $$|\Sigma_{23}| \leq 3\omega_f(1-x^2) . 3/2n \leq \Delta_n^{-1}(x)\omega_f(\Delta_n(x)).$$

For Σ_{22} applying the same argument we used to estimate Σ_2 with only difference that now we have $p'_k(t)$ instead of $p_k(t)$ and hence

$$\frac{1}{\sin t} |f(\cos t_j) - f(\cos t)| \cdot |p'_j(t)| \leq \frac{36\, n \sin nt}{\sin t} \omega_f(\Delta_n(x))$$

and

$$|\Sigma'_{22}| = O\left\{\frac{n \sin nt}{\sin t} \omega_f(\Delta_n(x))\right\},$$

$$|\Sigma''_{22}| = O\left\{\frac{n \sin nt}{\sin t} \omega_f(\Delta_n(x))\right\},$$

$$|\Sigma'''_{22}| = O\left\{\frac{n \sin nt}{\sin t} \omega_f(\Delta_n(x))\right\}.$$

Combining these three equalities, we have

$$|\Sigma_{22}| = O\left\{\frac{n \sin nt}{\sin t} \omega_f(\Delta_n(x))\right\}.$$

Case 1. When $(1-x^2)^{1/2} > 1/n$ then we have

(4.10) $$|\Sigma_{22}| = O\{\Delta_n^{-1}(x) \cdot \omega_f(\Delta_n(x))\}.$$

Case 2. When $(1-x^2)^{1/2} \leq 1/n$ then we have on using $|\sin nt| \leq n|\sin t|$

(4.11) $$|\Sigma_{22}| = O\{\Delta_n^{-1}(x) \cdot \omega_f(\Delta_n(x))\}.$$

Hence from (4.11), (4.10), (4.9) and (4.8) we get second part of the theorem.

Proof of the Theorem 2. Let $(1-x^2)^{1/2} \geq 1/n$ then from the first part of the Theorem 1 we have

(4.12) $$|Q_n(f, x) - f(x)| \leq 2C_1\omega_f(n^{-1}(1-x^2)^{1/2}).$$

Secondly let $(1-x^2)^{1/2} < 1/n$, then from (2.4) we have

$$Q_n(f, x) - f(x) = \left[\frac{1+x}{2}(f(1) - f(x)) + \frac{1-x}{2}(f(-1) - f(x))\right]\left[1 - \sum_{k=0}^{n} q_k(x)\right]$$
$$+ \sum_{k=0}^{n} (f(x_k) - f(x))q_k(x).$$

From (4.3) we have

$$\left|\frac{1+x}{2}(f(1) - f(x)) + \frac{1-x}{2}(f(-1) - f(x))\right| \left|1 - \sum_{k=1}^{2n} p_k(t)\right| \leq \frac{9}{8}\omega_f(1-x^2)^{1/2}.$$

Using (2.5), we have

$$\sum_{k=0}^{n} |f(x_k)-f(x)|\,|q_k(x)| \leq \sum_{k=1}^{2n}\left(1+\frac{|x-x_k|}{1-x^2}\right)\omega_f(1-x^2)\,|p_k(t)|,$$

since $\Sigma_{k=1}^{2n} p_k(t) = O(1)$ and

$$\sum_{k=1}^{2n}(\cos t-\cos t_k)\,p_k(t)$$

$$=(\cos t-\cos t_j)p_j(t)+4\sum_{k=1}^{j-1}(\cos t-\cos t_k)l_k^3(t)$$

$$+4\sum_{k=j+1}^{2n}(\cos t-\cos t_k)l_k^3(t)-3\sum_{k=1,\,k\neq j}^{2n}(\cos t-\cos t_k)l_k^4(t)$$

$$=(\cos t-\cos t_j)p_j(t)+\Sigma_3'+\Sigma_3''+\Sigma_3'''.$$

Arguing in the same way as in the proof of the first part of the theorem and now using the lemma 2(c) we see that

$$|\Sigma_3'|=O(1-x^2),\quad |\Sigma_3''|=O(1-x^2),\quad |\Sigma_3'''|=O(1-x^2) \tag{4.13}$$

and $|\cos t-\cos t_j|\,|p_j(t)|\leq(\pi/2+\pi^2/8)(1-x^2)$. We get from (4.13) and the last inequality

$$\sum_{k=1}^{2n}|\cos t-\cos t_k|\,|p_k(t)|=O(1-x^2).$$

Hence we have

$$|Q_n(f,x)-f(x)|=O(\omega_f(1-x^2))=O(\omega_f((1-x^2)^{1/2}/n)) \text{ for } (1-x^2)^{1/2}<1/n.$$

From here and (4.12) we have our theorem.

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REFERENCES

1. И. Е. Гопенгауз. К теореме А. Ф. Тимана о приближении функций многочленами на конечном отрезке. *Мат. заметки*, 1, 1967, 163—172.
2. О. Киш, П. Вертеши. О новых интерполяционных вопросах. *Ann. Univ. Sci. Budapest*, 10, 1967, 117—126.
3. R. B. Saxena. A new proof of S. A. Telyakovskii's theorem on the approximation of continuous functions by algebraic polynomials. *Studia Sci. Math. Hung.*, 7, 1972, 3—9.
4. С. А. Теляковский. Две теоремы о приближении функций алгебраическими многочленами. *Мат. сб.*, 70, 1966, 252—265.
5. А. Х. Туречкий. О некоторых экстремальных вопросах в теории интерполирования. *Сборник современных исследовательских задач по конструктивной теории функций. Баку, 1965, 220—232.*

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