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DENSITY PROPERTIES OF MELLIN TRANSFORMS

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Let $\chi(t)$ BV $[0, 1]$ and denote $D(z)=\int_0^1 t^z d\chi(t)$ its Mellin transform. It is well known that

(1) $$|D(z)|\leqq M\varrho^x, \quad x\to\infty \quad (z=x+iy),$$

for some $0<\varrho<1$ implies

(2) $$\int_{\varrho+}^1 |d\chi(t)|=0.$$

It is shown in the present paper that (2) may be deduced from (1) if z runs through a sequence $\{z_k\}=\{x_k+iy_k\}$ satisfying a Müntz condition

(3) $$\sum_{k=1}^{\infty} x_k^{-1}=\infty$$

and

(4) $$|x_k-x_{k-1}|>c>0, \quad |y_k|<M|x_k|,$$

where (4) may be slightly weakened.

1. Introduction and historical remarks. Let $\chi(t)$ be a complex valued function of bounded variation for $0\leqq t\leqq 1$ with $\chi(t+0)=\chi(t)$ for $0\leqq t<1$.

Denote by

$$D(z)=\int_0^1 t^z d\chi(t), \quad z=x+iy,$$

its Mellin transform which is a bounded analytic function for $x>0$. The number $\varrho=\varrho(\chi)=\inf\{t\,|\,\chi(s)=\chi(1), \text{ for } t\leqq s\leqq 1\}$ is called the order of χ (and of D)

There exists a close connection between the rate of decrease of $|D|$ for $x\to\infty$ and the order $\varrho(\chi)$. If

(1) $$\varrho(\chi)\leqq a\leqq 1,$$

then we may write $D(z)=\int_0^a t^z d\chi(t)$ which implies

(2) $$D(z)=O(a^x), \quad x\to\infty.$$

The converse conclusion from (2) to (1) also is true. Here it is not necessary to assume that z runs through the whole set $\{x>0\}$.

Theorem A (Picone [7], Mikusinski [4]). *If*

(2a) $$D(n)=O(a^n), \quad n\in N, \quad n\to\infty,$$

then (1) *holds.*

Meanwhile various proofs have been given (see for instance Hardy [2, p. 267], Yosida [13, p. 167]), Polya [8, p. 777] states without proof the following

Theorem B. *Let* $f(z)=\sum_{n=0}^{\infty}a_nz^n$ *be analytic for* $|z|<R$, $R>1$, *except for* $1\leq x\leq R$ *possibly and let* $f(z)=O(e^{(1/\delta)^{\alpha}})$, $0<\alpha<1$, *where* δ *denotes the distance from* z *to the line* $1\leq x\leq R$. *Suppose* $a_{n_k}=O(a^{n_k})$, *for* $0<a<1$ *and some subsequence* (n_k). *Then either* $a_n=O(a^n)$ *for all* $n\in N$, $n\to\infty$, *or* $\lim_{k\to\infty}k/n_k=0$ *(i. e.* (n_k) *has density* 0).

Applying this to $a_n=D(n)$, $f(z)=\sum_{n=0}^{\infty}D(n)z^n=\int_0^1(1-zt)^{-1}d\chi(t)$, one gets

Theorem C. *If* (2a) *holds for* $n_k\in N$ *satisfying* $\lim\sup k/n_k>0$, $k\to\infty$, *then* (1) *is valid.*

The above density condition still may be weakened.

Theorem D (Levinsion, Boas, Mikusinski, Ryll-Nardzewski). *If* (2a) *holds for* $n_k\in N$ *satisfying*

$$\sum_{k=1}^{\infty}1/n_k=\infty, \tag{3}$$

then (1) *is valid.*

This result was first formulated by Boas [1] (for the case $d\chi(t)=\chi'(t)dt$) as an immediate consequence of a density theorem of Levinson [3, p. 107] (which covers also the case of general $d\chi(t)$). Simultanously Mikusinski, Ryll-Nardzweski [5] proved a somewhat different result but from which theorem D easily may be deduced. An independent proof was given by Trautner [11], see also Schroeter [9]. (The author regrets for not having cited the papers [1],[5] in [11].)

Theorem D has several applications in summability theory (for Hausdorff methods see Trautner [12], for power methods see Ziv [14; 15]), it also covers the Titchmarsh convolution theorem which is fundamental for Mikusinski operational calculus.

In this paper we state an extension of theorem D, when (2) is satisfied by more general sequences (z_k). We first remark that condition (3) is best possible if $z_k=n_k\in N$ [5; 11].

We will consider complex sequences (z_k) with

$$|\arg z_k|\leq\alpha,\quad \alpha<\pi/2. \tag{4}$$

The analogue of condition (3) becomes

$$\sum_{k=1}^{\infty}1/x_k=\infty. \tag{5}$$

In addition we must require that the (z_k) are not too close to each other, for instance

$$x_{k+1}\geq x_k+c,\quad c>0, \tag{6}$$

but (6) may be considerably weakened. Then (1) remains valid if (2) is satisfied for (z_k).

As applications we get that $\lim_{r\to\infty}r^{-1}\log|D(re^{i\varphi})|$ exists in any sector $\beta_1<\arg z<\beta_2$, $-\pi/2\leq\beta_1<\beta_2\leq\pi/2$, in which $D(z)$ has no zeros, or that the type function

$$h_D(\varphi) = \limsup_{r\to\infty} \frac{\log |D(re^{i\varphi})|}{r}, \qquad |\varphi| < \pi/2,$$

satisfies the relation $h_{D_1 \cdot D_2}(\varphi) = h_{D_1}(\varphi) + h_{D_2}(\varphi)$.

The known proof of theorem D do not seem to admit a generalization to complex (z_k) with (4) (Levinson [3] allows $\arg z_k \neq 0$ but requires $\arg z_k \to 0$). We will follow the proof of Szasz's extension of the Müntz approximation theorem (see Szasz [10], Natanson [6]).

2. Statement of results. Our main result is the following

Theorem 1. *Given a complex sequence* (z_k), $z_k = x_k + iy_k$, $x_k \leq x_{k+1}$, *satisfying the conditions* (4), (5) : $|\arg z_k| \leq \alpha$, $\alpha < \pi/2$, $\sum_{k=1}^{\infty} 1/x_k = \infty$, *and* (7) *there exists a decreasing sequence* (φ_k), $0 < \varphi_k < 1$ *with*

(7a) $$|z_k - z_j| \geq |k-j| \cdot \varphi_k \quad \text{for all } z_j \text{ with } |z_j| \leq 2|z_k|,$$

(7b) $$\frac{A_k}{|z_k|} \cdot \log \varphi_k = O(1), \quad k \to \infty,$$

where A_k *is the number of* $|z_j| \leq 2|z_k|$, $z_j \neq z_k$.

If

(2b) $$D(z_k) = O(a^{x_k}), \quad k \to \infty,$$

then (1): $\varrho(\chi) \leq a$.

Remarks. 1. Condition (7a) guarantees, that the z_k have no finite limit point. The theorem becomes wrong if (7a) is cancelled. To see this take a Mellin transform $D(z)$ of order $\varrho > 0$, having a zero $z' = x' + iy'$, $x' > 0$, and choose (z_k) converging sufficiently fast to z' such that (2b) holds with $0 < a < \varrho$. By (7a) and (7b) the "velocity" φ_k with which the z_k may approach each other, and the "density" $A_k/2|z_k|$ are put in relation. Clearly the theorem also becomes wrong if we only require that the z_k have no finite limit points.

2. Condition (7b) implies $A_k/|z_k| = o(1)$, i. e. the z_k have density 0. If $\varphi_k = c$, $0 < c < 1$, in particular if (6) is satisfied, then condition (7b) may be omitted. For it is always possible to find a subsequence of (z_k) with density 0, for which (7) remains valid [1; 11].

3. Under (4) the condition (5) is best possible. For if $\sum_{k=1}^{\infty} 1/x_k < \infty$, then it is possible to find a Mellin transform $D(z)$ with order $\varrho > 0$ and zeros z_k.

We now state some applications of theorem 1.

Theorem 2. *Given a Mellin transform* $D(z) = \int_0^1 t^z d\chi(t)$ *of order* $\varrho(\chi) = a$. *Then* $h_D(\varphi) = \cos\varphi \cdot \log a$, $|\varphi| < \pi/2$, holds.

Theorem 3. *Given Mellin transforms* $D_1(z)$, $D_2(z)$, $D(z) = D_1(z) \cdot D_2(z)$ *with orders* $\varrho(\chi_1)$, $\varrho(\chi_2)$, $\varrho(\chi)$. *Then*

1) $\varrho(\chi) = \varrho(\chi_1) \cdot \varrho(\chi_2)$

2) $h_D(\varphi) = h_{D_1}(\varphi) + h_{D_2}(\varphi)$

hold.

Theorem 4. *Given a Mellin transform* $D(z)$. *Then*

$$\lim_{r\to\infty} \frac{\log |D(re^{i\varphi})|}{r} = \log a \cos \varphi$$

exists in any sector $-\pi/2 \leq \beta_1 < \varphi < \beta_2 \leq \pi/2$, *in which* D *has only a finite number of zeros. In addition the limit exists uniformly in any subsector* $\beta_1' \leq \varphi \leq \beta_2'$, $\beta_1' < \beta_1 < \beta_2' < \beta_2$.

3. Proofs. Proof of theorem 1. We shall show that the assumption $a < \varrho$ leads to a contradiction. Without loss of generality we may assume that

$$\varrho = 1 \tag{8}$$

and

$$d\chi(t) = f(t)dt \quad f \in L^2[0, 1]. \tag{9}$$

For if $a < \varrho < 1$, then we take the Mellin transform

$$D_1(z) = D(z)\varrho^{-z} = \int_0^\varrho (t/\varrho)^z d\chi(t) = \int_0^1 s^z d\chi(s\varrho),$$

which has order $\varrho_1 = 1$ and satisfies $|D_1(z_k)| = O(a_1^{x_k})$ with $a_1 = a/\varrho < 1$. If (9) is not satisfied, take $\chi_2(t) = \chi(1) - \chi(t)$ and consider for $x > 1$ the Mellin transform $D_2(z) = D(z)/z = \int_0^1 t^{z-1}\chi_2(t)dt$, which has the same order as $D(z)$ and satisfies the conditions of the theorem.

For all sufficiently large $x > 0$ we will approximate the function t^x by a polynomial $P_x(t)$ in t^{z_k} in the L^2-sense, i. e.

$$\left(\int_0^1 |t^x - P_x(t)|^2 dt\right)^{1/2} \leq b^x. \tag{10}$$

Here $0 < b < 1$ and $P_x(t)$ will be chosen such that also

$$\left|\int_0^1 P_x(t) f(t) dt\right| \leq K \cdot b^x \tag{11}$$

holds. (Here K denotes constants independent of x.) From

$$|D(x)| = \left|\int_0^1 t^x f(t)dt\right| \leq \left|\int_0^1 P_x(t) f(t)dt\right| + \int_0^1 |P_x(t) - t^x|\,|f(t)|\,dt$$

and by an application of the Cauchy-Schwarz inequality on the last integral we get from (10), (11): $|D(x)| \leq K \cdot b^x$. This implies $\varrho < 1$ in contradiction to (8).

We omit from (z_k) a finite number of elements such that (after new indexing depending on x) $x \leq x_1, x_2, \ldots$. We write

$$-P_x(t) = u_1 t^{z_1} + u_2 t^{z_2} + \ldots + u_n t^{z_n},$$

where $n = n(x)$ will be determined later. For

$$\int_0^1 |t^x - P_x(t)|^2 dt = \int_0^1 |t^x + u_1 t^{z_1} + \ldots + u_n t^{z_n}|^2 dt = Q_n$$

we obtain, after putting $z_0 = x$, $u_0 = 1$, the quadratic form

$$Q_n = \sum_{\nu, \mu = 0}^{k} c_{\nu\mu} u_\nu \bar{u}_\mu, \quad c_{\nu\mu} = \frac{1}{z_\nu + \bar{z}_\mu + 1}, \quad \nu, \mu = 0, 1, \ldots, n.$$

The solution of the system of linear equations

$$\begin{aligned} -m_n + c_{10}u_1 + c_{20}u_2 + \dots + c_{n0}u_n &= -c_{00} \\ c_{11}u_1 + c_{21}u_2 + \dots + c_{n1}u_n &= -c_{01} \\ \vdots \qquad\qquad & \quad \vdots \\ c_{1n}u_1 + c_{2n}u_2 + \dots + c_{nn}u_n &= -c_{0n} \end{aligned}$$

in the unknown variables $m_n, u_1, u_2, \dots, u_n$ satisfies $m_n = Q_n$. (Here m_n is the value of best approximation [6], but this fact will not be used.)

If we denote by $c_k = (c_{k\nu})$, $\nu = 1, 2, \dots, n$ and $\widehat{c_k} = (c_{k\nu})$, $\nu = 0, 1, 2, \dots, n$, the column vectors of the matrix $(c_{k\nu})$, running from 1 and 0, respectively, we obtain

$$m_n = \frac{\det(\widehat{c_0}, \widehat{c_1}, \dots, \widehat{c_n})}{\det(c_1, c_2, \dots, c_n)}, \quad u_k = \frac{\det(c_1, \dots, c_{k-1}, c_0, c_{k+1}, \dots, c_n)}{\det(c_1, c_2, \dots, c_n)}, \quad k = 1, 2, \dots, n.$$

A determinant of a matrix of the form $(a_{\nu\mu}) = (1/(q_\nu + r_\mu))$, $\nu, \mu = 1, 2, \dots, n$ may be evaluated after Cauchy (Natanson [6], p. 40) by

$$\det(a_{\nu\mu}) = \left[\prod_{\nu>\mu}^{1,n} (q_\nu - q_\mu)(r_\nu - r_\mu)\right] \Big/ \left[\prod_{\nu,\mu}^{1,n} (q_\nu + r_\mu)\right].$$

This yields

$$|m_n| = \left[\prod_{\nu>\mu}^{0,n} |z_\nu - z_\mu|^2 \prod_{\nu,\mu}^{1,n} |z_\nu + \bar{z}_\mu + 1|\right] \Big/ \left[\prod_{\nu,\mu}^{0,n} |z_\nu + \bar{z}_\mu + 1| \prod_{\nu>\mu}^{1,n} |z_\nu - z_\mu|^2\right]$$

$$= \frac{1}{2x+1} \prod_{\nu=1}^{n} \left|\frac{z_\nu - x}{z_\nu + x + 1}\right|^2,$$

$$\left|\frac{u_k}{m_n}\right| = \left[\prod_{\substack{\nu>\mu \\ \nu,\mu\neq k}}^{0,n} |z_\nu - z_\mu|^2 \prod_{\nu,\mu}^{0,n} |z_\nu + \bar{z}_\mu + 1|\right] \Big/ \left[\prod_{\nu,\mu\neq k}^{0,n} |z_\nu + z_\mu + 1| \prod_{\nu\neq\mu}^{0,n} |z_\nu - z_\mu|^2\right]$$

$$= (1 + 2x_k) \prod_{\nu\neq k}^{0,n} \left|\frac{1 + z_\nu + \bar{z}_k}{z_\nu - z_k}\right|^2.$$

We first give an upper estimate for $|u_k/m_n|$. For this we split the set of indices $I = \{0 \le \nu \le n,\ \nu \neq k\}$, by

$$I_1 = \{\nu \le n,\ \nu \neq k,\ |z_\nu| \le 2|z_k|\},$$

$$I_2 = \{\nu \le n,\ \qquad |z_\nu| > 2|z_k|\}.$$

From (4) we get $|z_k| \le K \cdot |x_k|$ and so for $\nu \in I_1$

$$|1 + z_\nu + \bar{z}_k| \le 1 + 3|z_k| \le K \cdot x_k.$$

Clearly

$$\left(\prod_{\nu=0,\ \nu\neq k}^{A_k} |\nu - k|\right)^{-1} \le \left(\Gamma\left(\frac{A_k}{2}\right)\right)^{-2} \le \left(\frac{K}{A_k}\right)^{A_k},$$

$$\Pi_1 = \prod_{\nu \in I_1} \left| \frac{z_\nu + \bar{z}_k + 1}{z_\nu - z_k} \right|^2 \leq (K \cdot x_k)^{2A_k} \Big/ \left[\prod_{\nu \in I_1} (|\nu - k| \varphi_k)^2 \right]$$

$$\leq \left(\frac{K \cdot x_k}{\varphi_k} \right)^{2A_k} \frac{1}{\Gamma(A_{k/2})^4} \leq \left(\left(\frac{K x_k}{\varphi_k \cdot A_k} \right)^{2A_k/x_k} \right)^{x_k} \leq (1+\varepsilon)^{x_k}$$

for an arbitrary $\varepsilon > 0$, if x is sufficiently large.

Here we have used that by (7b) and remark 2 $(A_k/x_k)(\log \varphi_k + \log A_k/x_k) = o(1)$ nolds.

With $|1/(1-w)| = |1 + w/(1-w)| \leq 1 + 2|w|$ for $|w| \leq 1/2$ we get

$$\Pi_2 = \prod_{\nu \in I_2} \left| \left[1 + \frac{1 + \bar{z}_k}{z_\nu} \right] \Big/ \left[1 - \frac{z_k}{z_\nu} \right] \right|^2 \leq \prod_{\nu \in I_2} \left| 1 + K \left| \frac{z_k}{z_\nu} \right| \right|^4 \leq \exp(K_1 L_n x_k),$$

where K_1 is a fixed constant and

$$L_n = \sum_{\nu=1}^{n} 1/x_k \tag{12}$$

we now choose

$$n = \sup\{ j \mid L_j \leq \varepsilon/2K_1 \}, \tag{13}$$

which implies $\Pi_2 \leq (1+\varepsilon)^{x_k}$. Using $(1+2x_k) \leq (1+\varepsilon)^{x_k}$ we get for sufficiently large x, $x_k \geq x$

$$|U_k/m_n| \leq (1+2x_k)\Pi_1 \cdot \Pi_2 \leq (1+4\varepsilon)^{x_k}.$$

If $\varepsilon > 0$ is chosen such that $(1+4\varepsilon)a = c < 1$ we obtain

$$\left| \int_0^1 P_x(t) f(t) dt \right| \leq m_n \sum_{k=1}^{n} \left| \frac{u_k}{m_n} \right| |D(z_k)| \leq m_n \sum_{k=1}^{n} c^{x_k}.$$

The supremum of $\sum_{k=1}^n c^{x_k}$ under the condition $x_k \geq x$, $\sum_{k=1}^n 1/x_k \leq \varepsilon/2K_1$ may be estimated by $c^x \cdot x \cdot \varepsilon/2K_1 \leq K$, which gives

$$\left| \int_0^1 P_x(t) f(t)\, dt \right| \leq K \cdot m_n. \tag{14}$$

We finally estimate m_n. Using (4) we get

$$|1 - x/z_\nu|^2 \leq 1 - (2x x_\nu - x^2)/(x_\nu^2 + y_\nu^2) \leq 1 - Kx/x_\nu,$$

$$m_n \leq \prod_{\nu=1}^{n} \left| 1 - \frac{x}{z_\nu} \right|^2 \leq \prod_{\nu=1}^{n} \left(1 - K \frac{x}{x_\nu} \right) \leq \exp\left(-K \cdot x \sum_{\nu=1}^{n} \frac{1}{x_\nu} \right) = \exp(-Kx \cdot L_n).$$

If we remember that the x_ν are renumbered for each x, and $x \leq x_1 \to \infty$, we see that $L_n = \sum_{k=1}^n 1/x_k = L_n(x)$ converges to $\varepsilon/2K_1 > 0$. So we get for a suitable $0 < b < 1$

$$\int_0^1 |t^x - P_x(t)|^2 dt = m_n \leq K \cdot b^{2x}.$$

Together with (14) we get (10) and (11) which complete the proof.

Proof of theorem 2. Let $\varrho(\chi)=a$, then

$$|D(z)|\leq\int_0^1 |t^z|\,|d\chi(t)|\leq Ka^x=Ka^{r\cos\varphi},$$

which gives

$$h_D(\varphi)=\limsup_{r\to\infty}\frac{\log|D(re^{i\varphi})|}{r}\leq\log a\,.\cos\varphi,\quad|\varphi|<\pi/2.$$

Assume that equality does not hold for all $|\varphi|<\pi/2$. Then for some $\varepsilon>0$ $|\varphi_0|<\pi/2$, $r>R_0(\varepsilon)$

$$\frac{\log|D(re^{i\varphi_0})|}{r}\leq(1-\varepsilon)\log a\,.\cos\varphi_0,\quad\text{i. e.}$$

$|D(re^{i\varphi_0})|\leq a^{x(1-\varepsilon)}$ for $r>R_0(\varepsilon)$. Taking $z_k=(R_0+k)e^{i\varphi_0}$, theorem 1 would imply $\varrho(\chi)\leq a^{(1-\varepsilon)}<a$, which is a contradiction.

Proof of theorem 3. 1) is an immediate consequence of theorem 1 and might also be derived from theorem C.

2) now follows from theorem 2.

Proof of theorem 4. We may assume $\varrho(\chi)=a=1$, otherwise we consider $D(z)a^{-z}$.

We will show that for any $\varepsilon>0$ there exists $R(\varepsilon)$, such that for $z=re^{i\varphi}$, $r>R(\varepsilon)$, $\beta_1'\leq\varphi\leq\beta_2'$ we have $|D(z)|\geq e^{-\varepsilon x}$. Assume that this is wrong. Then there exists an $\varepsilon>0$ and an infinite set

$$M_\varepsilon=\{z_n=x_n+iy_n=r_ne^{i\varphi_n}\,|\,\beta_1'\leq\varphi_n\leq\beta_2',\ 4x_n\leq x_{n+1}\},$$

such that $|D(z_n)|\leq e^{-\varepsilon x_n}$ for $z_n\in M_\varepsilon$.

For each $z_n\in M_\varepsilon$ we define the circle $K_{n,j}=\{z\,|\,|z-z_n|=j\}$, $j=1,2,\ldots$ Denote $A(n)$ the number of circles $K_{n,j}$, which are located in the domain $G_n=\{z\,|\,\beta_1\leq\varphi\leq\beta_2,\ x_n/2<x<2x_n\}$. Then the estimate

$$A(n)\geq\frac{1}{4}\min\{x_n,\ r_n\sin(\beta_1'-\beta_1),\ r_n\sin(\beta_2-\beta_2')\}\geq c\,.\,x_n,\quad c>0,$$

holds. Since $D(z)\neq0$ for $z\in G_n$, $n\geq N_0$, then by the maximum modulus principle there exists a sequence $z_{n,j}\in K_{n,j}$, $j=1,2,\ldots,A(n)$ with

$$|D(z_{n,j})|\leq|D(z_{n,j-1})|\leq\ldots\leq|D(z_{n,1})|\leq|D(z_n)|\leq e^{-\varepsilon x_n}.$$

Since $x_{n,j}\leq2x_n$, we get $|D(z_{n,j})|\leq e^{-x_{n,j}\,.\,\varepsilon/2}$. If we write the $z_{n,j}$, $j\leq A(n)$, $n\geq N_0$, as new sequence (z_k') $k=1,2,\ldots$ we have

$$|z_k'-z_j'|\geq|k-j|,\quad|D(z_k')|\leq e^{-x_k\,.\,\varepsilon/2},$$

$$\sum_{k=1}^{\infty}\frac{1}{x_k'}=\sum_{n=N_0}^{\infty}\sum_{j=1}^{A(n)}\frac{1}{x_{n,j}}\geq\sum_{n=N_0}^{\infty}\frac{A(n)}{2x_n}\geq\frac{c}{2}\sum_{n=N_0}^{\infty}\frac{x_n}{x_n}=\infty.$$

From theorem 1 we now obtain $\varrho(\chi)\leq e^{-\varepsilon/2}<1$, which is a contradiction.

REFERENCES

1. R. P. Boas, Jr. . Remarks on a moment problem. *Stud. Math.*, **13**, 1953, 59—61.
2. G. F. Hardy. Divergent series. Oxford, 1956.
3. N. Levinson. Gap and density theorems. New York, 1940.
4. J. G. Mikusinski. A theorem on moments. *Stud. Math.*, 12, 1951, 191—193.
5. J. G. Mikusinski, C. Ryll-Nardzeweski. A theorem on bounded moments. *Studia math.*, **13**, 1953, 51—55.
6. I. P. Natanson. Constructive function theory, II, New York, 1965.
7. M. Picone. Nuove determinazioni per gliintegrali delle equazioni lineari a derivative parziali. *Rend. Accad. Naz. Lincei*, **28**, 1939, 339—348.
8. G. Polya. Untersuchungen über Lücken und Singularitäten von Potenzreihen. II. Mitteilung. *Ann. Math.*, **34**, 1933, 731—777.
9. G. Schroeter. A Remark on the Hausdorff moment problem. *Tohoku Math. J.*, **27**, 1975, **389**.
10. O. Szasz. Über die Approximation stetiger Funktionen durch lineare Aggregate von Potenzen. *Math. Ann.*, **77**, 1916, 482—496.
11. R. Trautner. Density properties of Hausdorff moment sequences. *Tohoku Math. J.*, **24**, 1972, 34 —352.
12. R. Trautner. Hausdorff summability of power series I; II *Manuscr. math.*, **7**, **1972**, 1—12; **15**, 1975, 45—63.
13. K. Yosida. Functional analysis. Berlin, 1971.
14. A. Ziv. Inclusion between power methods of limitation. *Trans. Amer. Math. Soc.*,
15. A. Ziv. Inclusion relations between power methods of limitation. *Pacif. J. Math.*,

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