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SOME NOTES ON REAL BANACH ALGEBRAS

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Inequalities are given for the spectral radius and the norm in a real unital Banach *algebra which imply symmetry and C^*—equivalence respectively. The results are similar to some of those in Boyadžiev (1977) for complex algebras.

Throughout A will stand for a real Banach algebra with unit e and continuous involution $x \to x^*$. Details about real involutory Banach algebras can be found in [2; 6].

If $A_0 \subseteq A$ is a closed *subalgebra, by A_{0C} we denote its complexification. Every $x \in A_{0C}$ has an unique decomposition $x = a + ib$ with $a, b \in A_0$ and $x = a + ib \to x^* = a^* - ib^*$ is a continuous involution in A_{0C}. Evidently $A_{0C} \subseteq A_C$. For $x \in A_0$ by $\mathrm{Sp}(x, A_0) = \mathrm{Sp}(x, A_{0C})$ we denote its spectrum with respect to A_0 and $\mathrm{Sp}(x) = \mathrm{Sp}(x, A)$; by $\varrho(x)$ we denote its spectral radius and by $\|x\|$ its norm.

An element $x \in A$ is called self-adjoint if $x^* = x$ and skew-adjoint if $x^* = -x$. We denote:

$$H = \{x \mid x \in A,\ x^* = x\},\ J = \{x \mid x \in A,\ x^* = -x\},$$

$$K = \{x \mid x = \Sigma_{k=1}^{n} x_k^* x_k,\ x_k \in A,\ 1 \leq k \leq n,\ 1 \leq n < \infty\},$$

$$K_0 = \{x \mid x \in H,\ \mathrm{Sp}(x) \geqq 0\}.$$

Evidently H and J are real Banach spaces and K is a wedge ($\lambda x \in K$, $x + y \in K$ for $x, y \in K$ and $\lambda \geq 0$) generating H ($H = K - K$ as $x = [(e + x)^2 - (e - x)^2]/4$ for every $x \in H$.

We also denote by P the set of all linear functionals f defined on H which are non-negative on K, with $f(e) \leq 1$. As K generates H, every $f \in P$ takes real values.

Remark 1. If f is a linear functional on H and non-negative on K, then $|f(x)| \leq f(e)\varrho(x)$ for every $x \in H$.

Proof. For $x \in H$ and $0 < t < \varrho(x)^{-1}$ there exists [2, 4.1.4.] an $u \in H$ with $u^2 = e - tx$. So $f(e) \geq tf(x)$ and letting $t \to \varrho(x)^{-1}$ we obtain $f(x) \leq f(e)\varrho(x)$. Also $-f(x) = f(-x) \leq f(e)\varrho(-x) = f(e)\varrho(x)$ so that $|f(x)| \leq f(e)\varrho(x)$. Thus for every $f \in P$ we have $|f(x)| \leq \varrho(x)$.

The algebra A is called symmetric, if $e + x^*x$ is invertible for every $x \in A$. Equivalently $\mathrm{Sp}(x^*x) \geq 0$ for every $x \in A$. If A is symmetric, the set K_0 is a wedge and $K \subseteq K_0$.

Remark 2. Every $f \in P$ is non-negative on K_0 if $x \in K_0$ and $\varepsilon > 0$, $x + \varepsilon e = a^2 \in K$ with $a \in H$ ([2, 4.7.2.]), so for every $f \in P$ we have $f(x) \geqq -\varepsilon f(e)$ and letting $\varepsilon \to 0$ we obtain $f(x) \geqq 0$.

If A is symmetric, the converse is also true as $K \subseteq K_0$.

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We need the simple lemma:

L e m m a 1. *If A is symmetric and $A_0 \subseteq A$ is a closed *subalgebra containing the unit, then A_0 is symmetric too.*

P r o o f. Let $x \in A_0$. We have $\mathrm{Sp}(x^*x) \geq 0$ and as $\mathrm{Sp}(x^*x) = \mathrm{Sp}(x^*x, A_{0C})$ (according to [2, 1.6.13.]), $e + x^*x$ is invertible in A_0.

The following theorem gives more information about symmetric real unital Banach *algebras.

T h e o r e m 1. *For A the following conditions are equivalent:*

1) *A is symmetric;*

2) *Every self-adjoint element has a real spectrum and every skew-adjoint element has a purely imaginary spectrum;*

3) $\varrho(x) = \sup \{f(x) \mid f$ *an extreme point of* $P\}$ *for every* $x \in K$;

4) $\varrho(.)$ *is monotone increasing on* K;

5) $\varrho(u^2) \leq \varrho(u^2 + x^*x)$ *for every* $u \in H$, $x \in A$ *such that* $u^2x^*x = x^*xu^2$.

P r o o f. 1) $\to$ 2). That every $a \in H$ has a real spectrum when $e + u^2$ is invertible for every $u \in H$ is proved in [2, 4.1.7.]. We shall show now that every $a \in J$ has a purely imaginary spectrum. Let $a \in J$ and $[a]$ be the closed commutative *subalgebra generated by e and a. Let $[a]_C \subseteq A_C$ be its complexification and let Λ be the set of all multiplicative linear functionals on $[a]_C$. For every $x \in [a]$, $\mathrm{Sp}(x, [a]) \setminus \{0\} = \{f(x) \mid f \in \Lambda\} \setminus \{0\}$. Let $f \in \Lambda$ and $f(a) = \alpha + i\beta$ with α, $\beta \in R$ (the reals). As $[a]$ is symmetric according to lemma 1, we have $0 \leq f(a^*a) = f(-a^2) = -f(a)^2 = \beta^2 - \alpha^2 - 2\alpha\beta i$. So we must have $\alpha\beta = 0$ and hence $\alpha = 0$. Thus $\mathrm{Sp}(a) \subseteq \mathrm{Sp}(a, [a]) \subset iR$.

2) $\to$ 1) is proved in [4 (lemma 1)].

Now we prove 1), 2) $\to$ 3). Let A be symmetric. For every $a \in H$ we define as in [3]: $\mu(a) = \sup\{\lambda \mid \lambda \in \mathrm{Sp}(a)\}$ with values in R and the properties: a) $\mu(\alpha a) = \alpha\mu(a)$ when $\alpha > 0$ and b) $\mu(a+b) \leq \mu(a) + \mu(b)$ when $a, b \in H$; a) is obvious, b) is true for $\varrho(.)$ [2, 4.8.10.] and if we take $t > 0$ such that $a + te$, $b + te \in K_0$, then $\mu(a+b) + 2t = \mu(a + b + 2te) \leq \varrho(a + b + 2te) \leq \varrho(a + te) + \varrho(b + te) = \mu(a + te) + \mu(b + te) = \mu(a) + \mu(b) + 2t$ as $\mu(x) = \varrho(x)$ for $x \in K_0$. So b) is true for $\mu(.)$. Now a) and b) hold also for $\nu(a) = \max\{\mu(a), 0\}$, $a \in H$.

According to the Hahn-Banach theorem, for every $x \in K_0$ there exists a linear functional f defined on H with $f(x) = \nu(x)$ and $f(u) \leq \nu(u)$ for every $u \in H$. If $u \in K$, $\nu(u) = \varrho(u)$ as $\mu(u) = \varrho(u)$ for $u \in K_0 \supseteq K$. For $u \in K_0$ we have $\nu(-u) = 0$ and so $f(u) \geq 0$. Also $f(e) \leq \nu(e) = \varrho(e) = 1$, so that $f \in P$.

As the set $\{f \mid f \in P, f(x) = \varrho(x)\}$ is non-void, convex and weakly compact (according to remark 1), it has an extreme point (Krein-Milman), which is also an extreme point of P. This and remark 1 prove 3).

The implication 3) $\to$ 4) is quite easy and 4) $\to$ 5) is obvious.

Now we prove 5) $\to$ 1). Let $x \in A$ with $x^*x \neq 0$ and let $t > 0$, $t < \|x^*x\|^{-1}$. There exists $u \in H$ with $u^2 = e - tx^*x$, i. e. $e = u^2 + tx^*x$. According to 5) $\varrho(u^2) \leq 1 < 1 + t$ so that we have $\varrho[e(1+t)^{-1} - t(1+t)^{-1}x^*x] < 1$ and the element $e - [e(1+t)^{-1} - t(1+t)^{-1}x^*x] = t(1+t)^{-1}(e + x^*x)$ is invertible. Hence $e + x^*x$ is invertible.

R e m a r k 3. If A is symmetric, it follows from remark 2 that 3) from the above theorem holds for every $x \in K_0$ and hence $\varrho(.)$ is monotone increasing on K_0.

L e m m a 2. *Let in A:*

a) $\|u^2\| \leq \|u^2 + x^*x\|$

hold for every $u \in H$ and $x \in A$ such that $u^2x^*x = x^*xu^2$.

Then A is symmetric and $\varrho(x)=\|x\|$ *for every* $x\in K_0$. (*Hence the norm is monotone increasing on* K_0, *according to remark* 3.) *The converse is also true: if A is symmetric and* $\varrho(x)=\|x\|$ *for* $x\in K_0$, *then a) holds according to* 5) *of theorem* 1.

Proof. Follows the lines of the proof of lemma 3 in [1].

Theorem 2. *If in A the following inequalities hold:*

a) $\|u^2\|\leq\|u^2+x^*x\|$ *for* $u\in H$, $x\in A$, $u^2x^*x=x^*xu^2$,

b) $\|a\|^2\leq\alpha\|a^2\|$ *for* $a\in J$, α — *constant, then A is homeomorphic and isomophic to a real* C^*-*algebra of operators acting on some Hilbert space.*

Proof. It follows from lemma 2 that A is symmetric and for every $x\in K_0$, $\varrho(x)=\|x\|$. If $u\in H$, then $u+\varrho(u)e\in K_0$ and we have $\|u\|\leq\|u+\varrho(u)e\|+\varrho(u)=\varrho(u+\varrho(u)e)+\varrho(u)\leq3\varrho(u)$. Now $\|u\|^2\leq9\varrho(u)^2=9\varrho(u^2)=9\|u^2\|$. The theorem follows from proposition 1 of [4], where it is shown that the set of all unitary elements in the complexification A_C of A is bounded, so A_C is C^*-equivalent.

Remark 4. It is easy to see, according to 2) of theorem 1 that if A is commutative and symmetric, its complexification A_C is also symmetric, as every self-adjoint $x\in A_C$ is decomposed $x=a+ib$ with $a\in H$, $b\in J$ so that the spectrum of x is real (using multiplicative linear functionals on A_C). It is interesting whether this is true in the noncommutative case.

REFERENCES

1. H. Boyadžiev. A characterization of B^*-equivalent Banach algebras by means of their positive cones. *Serdica*, 3, 1977, 20-24.
2. C. E. Rickart. General Theory of Banach Algebras. Princeton, 1960.
3. T. Ono. A real analogue of the Gel'fand-Naimark theorem. *Proc. Amer. Math. Soc.*, 25, 1970, 159—160.
4. T. W. Palmer. Real C^* algebras. *Pacif. J. Math.*, 35, 1970, 195—205.
5. H. Behncke. A note on the Gel'fand-Naimark conjecture. *Communs. Pure and Appl. Math.*, 23, 1970, 189—200.
6. L. Ingelstam. Real Banach algebras. *Ark. Math.*, 5, 1964, 239—270.

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