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EXTENSIONS OF ADDITIVE SET-FUNCTIONS

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We work with two arbitrary σ-topological spaces and we deal with the problem of restriction-extension, and uniqueness of extenston, of regular, bounded, finitely additive set-functions associated with these σ-topologicil spaces. The main result is a restriction-extension theorem which generalizes a theorem of A. D. Alexandroff on additive set functions in σ-topological spaces (1940–41), and also the theorem of Marik on the Baire and Borel measures.

1. Terminology and notation. In this paper, we will adhere to the following terminology and notation (mainly as in [1]). N will denote the set of natural numbers, and C the set of complex numbers. A space is defined to be an ordered pair, whose first component is an arbitrary set X and whose second component is an arbitrary collection of subsets of X, called the collection of closed sets and denoted by $F(X)$, such that

(1) for every subset H of $F(X)$, if H is finite, then $\cup(H)\in F(X)$, and

(2) for every subset H of $F(X)$, if H is countable, then $\cap(H)\in F(X)$.

Note: $F(X)$ is a δ-lattice.

We will refer to the space $(X, F(X))$ as the space X.

It is important to note that a space is a generalization of a topological space. Some authors refer to a space as a σ-topological space; for example, see [4] and the references given there.

The complement of a closed set is called open, and the collection of open sets is denoted by $G(X)$. The general element of $F(X)$ is denoted by F, and the general element of $G(X)$ is denotad by G.

The collection $F(X)\cup G_\delta$ is denoted by $F_0(X)$. The collection of compact subsets is denoted by $K(X)$, and the collection $K(X)\cup G_\delta$ is denoted by $K_0(X)$.

The algebra of subsets of X generated by $F(X)$ is denoted by $A(X)$ (the general element of $A(X)$ is denoted by E), the σ-ring of subsets of X generated by $F(X)$ is denoted by $S(X)$, the σ-algebra of subsets of X generated by $F(X)$ is called the Borel algebra of X and is denoted by $B(X)$, the set of all scallar functions on $A(X)$, which are (finitely) additive, bounded, and regular is denoted by $rba(X)$ and the set of all scalar functions on $B(X)$, which are countably additive, bounded, and regular is denoted by $M(X)$.

A function f of a space, X, into a space, X_1, is said to be continuous, if and only if, for every $F_1: f^{-1}(F_1)\in F(X)$. The set of all elements of C^X, which are bounded and continuous is donoted by $C(X)$, and the conjugate space of $C(X)$ is denoted by $\widetilde{C(X)}$.

An F is said to be totally closed, if and only if, there exists an element f of $C(X)$, such that $F=f^{-1}(\{0\})$. The collection of all totally closed sets is denoted by $F^*(X)$. The statement "$(X, F^*(X))$ is a space" is true. This space is denoted by X^*.

2. The following theorem consists of two parts.

Theorem. Part I. *Given*

(1) *any set* X;

(2) *any two* δ*-lattices of subsets* F_1, F_2 *of* X, *such that*

(i) $F_1 \subset F_2$ *and*

(ii) *for any* $F_2, G_2 : F_2 \subset G_2$ *implies there exists an* F_1, *such that* $F_2 \subset F_1 \subset G_2$. (Note: *Condition* (ii) *is equivalent to: for any* $F_2, \widetilde{F}_2 : F_2 \cap \widetilde{F}_2 = \emptyset$.);

(3) *the spaces* X_1, X_2 *and any element* μ_2 *of* $rba(X_2)$. *Then*, $\mu_2|_{A(X_1)} \in rba(X_1)$.

For the proof of Part I we need the following lemmas:

Lemma 1 [1, II, p. 569, Theorem 1]. *Consider any space* X *and any element* μ *of* $rba(X)$ *such that* $\mu \geqq 0$. *Then for every* $E : \mu(E) = \sup\{\mu(F) | F \subset E\}$.

Lemma 2 [1, II, p. 584, Lemma 1]. *Consider any space,* X, *and any function* μ *from* $A(X)$ *to* C, *such that* μ *is additive and for every* G: $\mu(G) = \sup\{\mu(F) | F \subset G\}$. *Then* $\mu \in rba(X)$ *and* $\mu \geqq 0$.

Lemma 3. *If* $\mu \in rba(X)$, *then for any* $F, G : F \subset G$ *implies for every positive number* ε: *there exists a* $\widetilde{G}$ *such that* $F \subset \widetilde{G} \subset G$ *and for every* E: $F \subset E \subset \widetilde{G}$ *implies* $|\mu(F) - \mu(E)| < \varepsilon$.

Proof. Consider any F, G, such that $F \subset G$, and any positive number, ε. Then by [1, II, p. 572, Theorem 4], there exist F_1, G_1 such that $F_1 \subset F \subset G_1$ and for every E, the inclusions $F_1 \subset E \subset G_1$ imply $|\mu(F) - \mu(E)| < \varepsilon$. Consider $G \cap G_1$. Note $G \cap G_1 \in G(X)$. Denote $G \cap G_1$ by $\widetilde{G}$. Note: $F \subset \widetilde{G} \subset G$ and for every E, the relation $F \subset E \subset \widetilde{G}$ implies $|\mu(F) - \mu(E)| < \varepsilon$.

Lemma 4. *Given*

(α) *any set,* X;

(β) *any two* δ*-lattices of subsets* F_1, F_2 *of* X, *such that* (1)$F_1 \subset F_2$ *and* (2) *condition* (ii) *of the theorem is satisfied*;

(γ) *any element* μ_2 *of* $rba(X_2)$.

Then for any $F_2, G_2 : F_2 \subset G_2$ *implies the existence of an* F_1, *such that* $F_2 \subset F_1 \subset G_2$ *and for every* $E_2 : F_2 \subset E_2 \subset F_1$ *implies* $\mu_2(F_2) = \mu_2(E_2)$.

Proof. Consider any F_2, G_2 such that $F_2 \subset G_2$. Donote any element of N by n. Since $\mu_2 \in rba(X_2)$, for every n, by Lemma 3, there exists a G_{2n}, such that $F_2 \subset G_{2n} \subset G_2$ and for every E_2, the inclusions $F_2 \subset E_2 \subset G_{2n}$ imply $|\mu_2(F_2) - \mu_2(E_2)| < 1/n$. For every n, by condition (ii) of the theorem, there exists an F_{1n}, such that $F_2 \subset F_{1n} \subset G_{2n}$. Consequently, for every n, one has $F_2 \subset F_{1n} \subset G_2$.

Consider $\bigcap_{k \in N} F_{1k}$. Note that $\bigcap_{k \in N} F_{1k} \in F_1$. Denote $\bigcap_{k \in N} F_{1k}$ by F_1. Note: $F_2 \subset F_1 \subset G_2$. Also, for every n, one has $F_2 \subset F_1 \subset G_{2n}$. Hence, for every E_2, the inclusions $F_2 \subset E_2 \subset F_1$ imply for every n: $|\mu_2(F_2) - \mu_2(E_2)| < 1/n$. Hence, for every F_2, if $F_2 \subset E_2 \subset F_1$, then $\mu_2(F_2) = \mu_2(E_2)$.

Corollary [1, II, p. 586, Lemma 2]. *Given any space* X *which is normal, the space* X^*, *and any element* μ *of* $rba(X)$, *then for any* F, G: $F \subset G$ *implies the existence of an* F^*, *such that* $F \subset F^* \subset G$ *and for every* $E : F \subset E \subset F^*$ *implies* $\mu(F) = \mu(E)$.

Proof. (omitted).

Lemma 5 [1, II, p. 586, Lemma 3]. *Consider the setting of Lemma 4 and assume* $\mu_2 \geqq 0$. *Then for every* G_2: $\mu_2(G_2) = \sup\{\mu_2(F_1) | F_1 \subset G_2\}$.

Proof. By Lemma 1, $\mu_2(G_2)=\sup\{\mu_2(F_2)\mid F_2\subset G_2\}$. By Lemma 4, for every $F_2: F_2\subset G_2$ implies the existence of an F_1, such that $F_2\subset F_1\subset G_2$ and $\mu_2(F_2)=\mu_2(F_1)$. Consequently, $\mu_2(G_2)=\sup\{\mu_2(F_1)\mid F_1\subset G_2\}$.

Proof of Part I. Note: $\mu_2=\mu_2^p-\mu_2^n$. Consider $\mu_2^p|_{A(X_1)}$ and $\mu_2^n|_{A(X_1)}$. Note: $\mu_2^p|_{A(X_1)}$ and $\mu_2^n|_{A(X_1)}$ are additive. Moreover, by Lemma 5, for every G_1: $\mu_2^p(G_1)=\sup\{\mu_2^p(F_1)\mid F_1\subset G_1\}$ and $\mu_2^n(G_1)=\sup\{\mu_2^n(F_1)\mid F_1\subset G_1\}$. Consequently, by Lemma 2, $\mu_2^p|_{A(X_1)}\in rba(X_1)$ and $\mu_2^n|_{A(X_1)}\in rba(X_1)$. Hence $\mu_2|_{A(X_1)}\in rba(X_1)$.

Part II. *Consider items* (1) *and* (2) *of Part I, assuming F_2 is normal, the spaces X_1, X_2, and also the following:*

(4) *any element μ_1 of $rba(X_1)$;*

(5) *the function Φ_1, which is such that $D_{\Phi_1}=C(X_1)$ and for every element f_1 of $C(X_1)$: $\Phi_1(f_1)=\int_X f_1\,d\mu_1$; (Note: $\Phi_1\in\widetilde{C(X_1)}$.)*

(6) *any element Φ_2 of $\widetilde{C(X_2)}$, such that $\Phi_2|_{C(X_1)}=\Phi_1$;*
(Note the existence of such a Φ_2 is guaranteed by the Hahn-Banach Theorem.)

(7) *the element μ_2 of $rba(X_2)$ which corresponds to Φ_2 by means of Alexandroff's Representation Theorem.*
(Note the condition of normality is needed in Alexandroff's Representation Theorem.)

Then the following statements are true: (i) $\mu_2|_{A(X_1)}=\mu_1$ *and* (ii) μ_2 *is unique.*

Proof. Consider $\mu_2|_{A(X_1)}$, and denote it by $\tilde\mu_1$. By the result of Part I, $\tilde\mu\in rba(X_1)$.

(i) Denote the general element of $C(X_1)$ by f_1. Note for every $f_1: \Phi_2(f_1)=\Phi_1(f_1)$. Hence, for every $f_1: \int_X f_1\,d\mu_2=\int_X f_1\,d\mu_1$. By the definition of the Lebesgue-Radon integral, for every $f_1: \int_X f_1\,d\mu_2=\int_X f_1\,d\tilde\mu_1$.

Consequently, for every $f_1: \int_X f_1\,d\tilde\mu_1=\int_X f_1\,d\mu_1$. Hence, by [1, II, p. 583, Lemma II], $\tilde\mu_1=\mu_1$. Consequently, $\mu_2|_{A(X_1)}=\mu_1$.

(ii) Consider any F_2, and the direction of all F_1 which are such that $F_2\subset F_1$, and denote it by $D(F_2)$.

By Lemma 4, there exists an element F_1 of $D(F_2)$, such that for every $F_1: F_1\in D(F_2)$ and $F_1\subset\widetilde{F}_1$ implies $\mu_2(F_2)=\mu_2(F_1)$. Hence, $\mu_2(F_2)=\lim_{F_1\in D(F_2)}\mu_2(F_1)$. Hence, μ_2 is unique.
Thus, the theorem is proved.

Remark 1. Note the Hahn-Banach Theorem asserts the existence of Φ_2, but it does not indicate how to obtain it.

Remark 2. To obtain A. D. Alexandroff's theorem [1, II, p. 584, Theorem 1], from our theorem, assume $F_2=F(X)$ and $F_1=F^*(X)$.

(Note in the first part of Alaxandroff's theorem the normality of F is not needed.)

Corollary. *Consider the function Φ, which is such that $D_\Phi=rba(X_1)$ and for every element μ_1 of $rba(X_1)$, $\Phi(\mu_1)=\mu_2$ (the μ_2 which corresponds to μ_1 by means of Part* II *of the theorem). The Φ is a* 1—1 *correspondence between $rba(X_1)$ and $rba(X_2)$.*

3. Example 1. Consider any topological space X such that X is normal. Denote $F_0(X)$ by F_1 and $F(X)$ by F_2.

(a) Note $F_1\subset F_2$.

(b) Since X is normal, condition (ii) of the theorem is satisfied. Consider any element μ_2 of $rba(X_2)$ such that μ_2 is countably additive, and $\mu_2|_{A(X_1)}$.

By Part I of the theorem, $\mu_2|_{A(X_1)} \in rba(X_1)$.

Note $\mu_2|_{A(X_1)}$ is countably additive. Denote $\mu_2|_{A(X_1)}$ by μ_1. Since μ_2 is countably additive, there exists an element ϱ_2 of $M(X_2)$, such that $\varrho_2|_{A(X_2)}=\mu_2$ and ϱ_2 is unique [1, II, p. 587, Theorem 1]. Since μ_1 is countably additive, there exists an element ν_1 of $M(X_1)$, such that $\nu_1|_{A(X_1)}=\mu_1$ and ν_1 is unique. Show $\varrho_2|B_{(X_1)}=\nu_1$. Denote $\varrho_2|_{B(X_1)}$ by ϱ_1.

Lemma. *Consider any set, X, and any collection of subsets E of X, such that* (i) $\emptyset \in E$ *and* (ii) *for any two elements A_1, A_2 of E: $A_1 \cup A_2 \in E$ and $A_1 \cap A_2 \in E$. Consider any two measure τ_1, τ_2 on $S(E)$, such that $\tau_1|_E$ and $\tau_2|_E$ are finite, and $\tau_1|_E=\tau_2|_E$. Then $\tau_1=\tau_2$.*

(a) Note $S(F_1)=B(X_1)$; (b) Note $\varrho_1|_{F_1}$ and $\nu_1|_{F_1}$ are finite; (c) note for every $F_1: \varrho_1(F_1)=\varrho_2(F_1)=\mu_2(F_1)=\mu_1(F_1)=\nu_1(F_1)$; hence, $\varrho_1|_{F_1}=\nu_1|_{F_1}$.

Consequently, by the lemma, $\varrho_1=\nu_1$. Hence, $\varrho_2|_{B(X_1)}=\nu_1$. The following diagram illustrates the relationship between the various sets and between the various set-functions involved in the above discussion.

Example 2. Consider any topological space X such that X is Hausdorff and compact. Denote $K_0(X)$ by F_1 and $K(X)$ by F_2. (a) Note $F_1 \subset F_2$. (b) Since X is Hausdorff and locally compact, condition (ii) of the theorem is satisfied. (c) Since X is Hausdorff and compact, F_2 is normal.

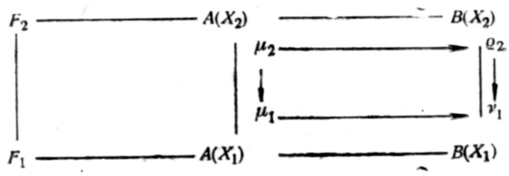

Fig. 1

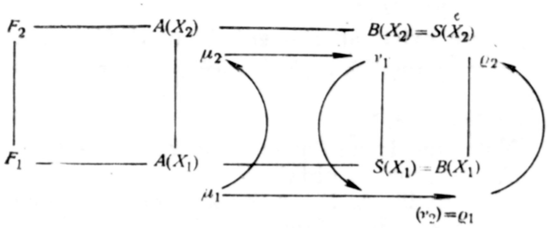

Fig. 2

Consider any element μ_1 of $rba(X_1)$. By Part II of the theorem, there exists an element μ_2 of $rba(X_2)$, such that $\mu_2|_{A(X_1)}=\mu_1$ and μ_2 is unique.

Note Part II of the theorem asserts the existence of μ_2, but it does not indicate how to obtain it. We shall develop a procedure for obtaining μ_2. Since X is compact, X_1 is compact. Hence, μ_1 is countably additive [1, II, 590, Theorem 5]. Hence, there exists an element ϱ_1 of $M(X_1)$, such that $\varrho_1|_{A(X_1)}=\mu_1$ and ϱ_1 is unique [1, II, p. 587, Theorem 1]. Since X is compact and G_δ, $X \in F_1$. Hence, $B(X_1)=S(X_1)$. Since ϱ_1 is a Baire measure, there exists a regular Borel measure, ϱ_2, such that $\varrho_2|_{S(X_1)}=\varrho_1$ and ϱ_2 is unique [2, p. 239]. Since ϱ_2 is a Borel measure and X is compact, ϱ_2 is bounded. Hence, $\varrho_2\in M(X_2)$. Since X is compact, X_2 is compact. Hence, μ_2 is countably additive. Hence, there exists an element ν_2 of $M(X_2)$, such that $\nu_2|_{A(X_2)}=\mu_2$ and ν_2 is unique. Show $\nu_2=\varrho_2$.

Consider $\nu_2|_{S(X_1)}$, and danote it by $(\nu_2)_0$. Show $(\nu_2)_0=\varrho_1$. Consider any element K_0 of $K_0(X)$. Note: $(\nu_2)_0(K_0)=\nu_2(K_0)=\mu_2(K_0)=\mu_1(K_0)=\varrho_1(K_0)$. Hence, by the lemma of Example 1, $(\nu_2)_0=\varrho_1$. Since ν_2 is a regular Borel measure and $\nu_2|_{S(X_1)}=\varrho_2|_{S(X_1)}$, by the uniqueness of ϱ_2, $\nu_2=\varrho_2$. Hence, $\mu_2=\nu_2|_{A(X_2)}=\varrho_2|_{A(X_2)}$.

The above discussion dictates the following procedure for obtaining $\mu_2: \mu_1\to\varrho_1\to\varrho_2=\nu_2\to\nu_2|_{A(X_2)}=\mu_2$. The following diagram illustrates the relationship between the various sets and between the various set-functions involved in the above discussion.

Example 3. Given

(1) any set, X;

(2) any two δ-lattices of subsets F_1, F_2 of X, such that (α) $F_1\subset F_2$ and (β) condition (ii) of the theorem is satisfied and (γ) F_2 is normal and (δ) F_2 is countably paracompact.

(3) for each element i of $\{1, 2\}$, the subset of $rba(X_i)$, whose general element is σ-smooth, and denote it by $D(X_i)$. Then $\Phi|_{D(X_1)}$ (see Corollary for the definition of Φ) is a 1—1 correspondence between $D(X_1)$ and $D(X_2)$.

Proof. (a) Consider any element μ_1 of $D(X_1)$. Show $\Phi(\mu_1)\in(D(X_2)$. Denote $\Phi(\mu_1)$ by μ_2. Show μ_2 is σ-smooth.

Consider any sequence in F_2, $(F_{2,n})$ such that $F_{2,n}\downarrow\emptyset$. Since F_2 is countably paracompact and condition (ii) of the theorem is satisfied, there exists a sequence in F_1, $(F_{1,n})$, such that (for every n: $F_{2,n}\subset F_{1,n}$) and $F_{1,n}\downarrow\emptyset$. Since μ_1 is σ-smooth, $\lim_{n\to\infty}\mu_1(F_{1,n})=0$. Since $\mu_1=\mu_2|_{A(X_1)}$, $\lim_{n\to\infty}\mu_2(F_{2,n})=0$. Hence, μ is σ-smooth. Consequently, $\Phi(\mu_1)\in D(X_2)$.

(b) Consider any element μ_2 of $D(X_2)$. Show $\Phi^{-1}(\mu_2)\in D(X_1)$.

Note: $\Phi^{-1}(\mu_2)=\mu_2|_{A(X_1)}$ and $\mu_2|_{A(X_1)}$ is σ-smooth. Hence, $\Phi^{-1}(\mu_2)\in D(X_1)$.

(c) Consequently, $\Phi|_{D(X_1)}$ is a 1—1 correspondence between $D(X_1)$ and $D(X_2)$.

Example 4 (the extension theorem of Marik [3]). Given any topological space $(X, F(X))$ such that $F(X)$ is normal and countably paracompact, (denote $F^*(X)$ by F_1 and $F(X)$ by F_2), given any element μ_1 of $M(X_1)$, there exists an element μ_2 of $M(X_2)$, such that $\mu_2|_{B(X_1)}=\mu_1$ and μ_2 is unique.

Proof. Consider $\mu_1|_{A(X_1)}$, and denote it by ϱ_1. Note $\varrho_1\in D(X_1)$. Note: $(\alpha) F_1\subset F_2$ and (β) condition (ii) of the theorem is satisfied, since $F(X)$ is normal, and $(\gamma) F_2$ is normal and (δ) F_2 is countably paracompact. Hence, by part (a) of Example 3, $\Phi(\varrho_1)\in D(X_2)$. Denote $\Phi(\varrho_1)$ by ϱ_2. Then there exists an element μ_2 of $M(X_2)$, such that $\mu_2|_{A(X_2)}=\varrho_2$ and μ_2 is unique [1, II, p. 589, Theorem 1]. To show $\mu_2|_{B(X_1)}=\mu_1$ use the lemma of Example 1.

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