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RANDOM MOTIONS OF INFINITE PARTICLE SYSTEMS ON R^1 WITHOUT OVERTAKING

HERMANN DEBES

We deal with transformations T of sequences $(X_i)_{i\in\Gamma}$ of interval lengths of stationary point processes on R^1 which satisfy $(TX)_i = X_i+Y_{i+1}-Y_i$, $Y_i = Z_i(X_i+Y_{i+1})$, $i\in\Gamma$, where $(Z_i)_{i\in\Gamma}$ is a sequence of i. i. d. r. v! s independent of $(X_i)_{i\in\Gamma}$, with Z_i taking values in [0, 1]. The (non-negative) r. v! s Y_i may ce thought of as translations of the points.

Questions of invariance and convergence are investigated. If for example Z_i are equidistributed we establish convergence to Poisson process of the iterated transformations of a wide class of "initial conditions".

0. Introduction. In this paper we deal with some transformations of stationary sequences of non-negative random variables, which may be interpreted as a class of random motions of point processes involving interaction.

For each stationary point process with finite intensity on the real axis one can construct the Palm distribution (often called tagged-particle distribution), which in turn may be described by a stationary sequence of non-negative random variables. These random variables may be thought of intuitively as the interval lengths of the point process [4].

Random motions of point processes, as introduced in [3], can be described also in terms of Palm distributions or stationary sequences of non-negative random variables. In this note we deal with the latter only and for the connections to point processes we refer to [3], [4] and [6].

Let $X=(X_i)_{i\in\Gamma}$ be a stationary sequence of non-negative random variables. Let $Z=(Z_i)_{i\in\Gamma}$ be a sequence of independent identically distributed random variables independent of X, with Z_i taking values in [0, 1].

We will define a transformation T such that

$$(TX)_i=X_i+Y_{i+1}-Y_1 \quad \text{for all } i\in\Gamma, \tag{0.1}$$

where the Y_i are non-negative random variables which satisfy

$$Y_i=Z_i(X_i+Y_{i+1}) \quad \text{for all } i\in\Gamma. \tag{0.2}$$

The random variables may be thought of as translations of the points (all in one direction). Translations are determined successively: The translation of the i-th point is a random part of the distance between the i-th point before and the $(i+1)$-th point after translation. TX then is the sequence of interval lengths of the point process after translation of all points.

If we iterate this procedure we get T^nX. There arise two closely connected questions. Which X are T-invariant, i. e. $TX=_d X$, where $=_d$ means equality in distribution? For which X the sequence of iterates T^nX converges in distribution to a T-invariant sequence?

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Our Theorem 1 gives a (partial) answer. As an example we mention the following result. If T is induced by a sequence $(Z_i)_{i\in\Gamma}$ of independent, equidistributed on [0,1] random variables, then for each sequence $(X_i)_{i\in\Gamma}$ of indenpendent, identically distributed non-negative random variables with finite mean T^nX converges in distribution to a sequence of independent, identically exponential distributed random variables.

In the language of point process that means that the iterated transformations of each stationary recurrent point process of finite intensity converge weakly to a Poisson process.

Notations. Γ, N stand for the sets of integers and natural numbers, respectively. By a stationary sequence X we mean a sequence $(X_i)_{i\in\Gamma}$ of non-negative random variables such that $[X_i, X_{i+1}, \dots, X_{i+n}] =_d [X_0, X_1 \dots, X_n]$ for all $i\in\Gamma$, $n\in N$.

A recurrent sequence X is a sequence $(X_i)_{i\in\Gamma}$ of non-negative, independent and identically distributed random variables. The symbols, $=_d$, $\Rightarrow_d$ are denoting equality and convergence in distribution, respectively. We suppose that all random variables are defined on the same probability space.

1. **Results.** Let $Z=(Z_i)_{i\in\Gamma}$ be a recurrent sequence with Z_i taking values in [0, 1] and $P(0<Z_0<1)>0$.

Define a matrix $T=(T_{i,j})_{i,j\in\Gamma}$ by

$$T_{i,i+k}=\begin{cases} 1-Z_i & i\in\Gamma,\quad k=0,\\ (1-Z_i)\ \Pi_{e=1}^{k} Z_{i+e} & i\in\Gamma,\quad k\in N\setminus\{0\},\\ 0 & \text{otherwise}.\end{cases} \tag{1.1}$$

T has the following properties

(1.2) For all $k,\ n\in N$, $[i_e, j_e]\in\Gamma\times\Gamma$, $e=1,\dots,n$

$$[T_{i_1,j_1},\ T_{i_2,j_2},\dots,T_{i_n,j_n}] =_d [T_{i_1+k,j_1+k},\ T_{i_2+k,j_2+k},\dots,T_{i_n+k,\,j_n+k}].$$

$$\text{For all } i\in\Gamma\quad \sum_{j\in\Gamma} E(T_{i,j})=1. \tag{1.3}$$

$$\text{For all } j\in\Gamma\quad \sum_{i\in\Gamma} T_{ij}=1 \text{ a. e.} \tag{1.4}$$

Now let $X=(X_i)_{i\in\Gamma}$ be a stationary sequence independent of T with $EX_0<\infty$. We define a sequence $TX=((TX)_i)_{i\in\Gamma}$ by

$$(TX)_i=\sum_{i\in\Gamma} T_{ij}X_j,\quad i\in\Gamma. \tag{1.5}$$

The sum in (1.5) is finite a. e. because from (1.3) follows

$$E\Big(\sum_{j\in\Gamma} T_{ij}X_j\Big)=\sum_{j\in\Gamma} ET_{ij}\,.\,EX_0=EX_0<\infty.$$

From (1.2) it is easy to deduce that TX is a stationary sequence. Hence we can define iterated transformations T^n in the following manner.

Let $T^{(1)}$. $T^{(2)},\dots$ be a sequence of independent, identically distributed copies of T independent of X.

Define matrices T^n by $T^1=T^{(1)}$, $(T^n)_{i,j}=\Sigma_{k\in\Gamma}\, T^{(n)}_{ik}\, T^{n-1}_{kj}$, $i,\ j,\ n=2,\ 3,\dots$ and sequences T^nX by

$$(T^nX)_i=\sum_{j\in\Gamma} T^n_{ij}X_j. \tag{1.6}$$

Note that T^n satisfies (1.2), (1.3) and (1.4) and consequently all T^nX are stationary sequences with $E(T^nX)_0 = EX_0$.

Now we are able to state the main result.

Theorem 1. *Let $Z=(Z_i)_{i\in\Gamma}$ be a recurrent sequence with Z_i taking values in* [0, 1] *and $P(0<Z_0<1)>0$. Let T^n be defined by* (1.1) *and* (1.6).

Then for each $c>0$ there is a stationary sequence X^c such that $T^nX \Rightarrow_d X^c$ for all stationary sequences X independent of $T^1, T^2, \ldots$ with

$$EX_0 = c, \quad \sum_{j\in\Gamma} |\operatorname{cov}(X_0, X_j)| < \infty. \tag{1.7}$$

The sequence X^c is T-invariant, i. e. $TX^c =_d X^c$. Furthermore $EX_0^c = c$, $E(X_0^c)^2 = [EZ_0E(1-Z_0)^2/E(1-Z_0)E(1-Z_0^2)].c^2$ and $\operatorname{cov}(X_i^c, X_j^c)=0$ *for all $i, j\in\Gamma$ with $i\neq j$.*

Corollary 1. *For each $c>0$ there is one and only one T-invariant stationary sequence satisfying* (1.7).

Remarks. (1) The transformations T, as defined by (1.1) and (1.5), satisfy (0.1) and (0.2). To see this we put $Y_i = \sum_{k=0}^{\infty}(\Pi_{e=0}^{k} Z_{i+e}) X_{i+k}$.

(2) In general Theorem 1 gives only informations about the covariance structure of the limit. In special cases, however, we know the set of recurrent T-invariant sequences ([2]) and in view of the Corollary above we know X^c. For example Theorem 4 of [2] combined with Theorem 1 gives

Theorem 2. *Let $(Z_i)_{i\in\Gamma}$ be independent, identically equidistributed on* [0, 1]. *Let T^n be defined by* (1.1) *and* (1.6).

Then for all stationary sequences X independent of $T^1, T^2, \ldots$ with (1.7) *$T^nX \Rightarrow_d X^c$, where $(X_i^c)_{i\in\Gamma}$ are independent, identically exponential distributed with mean c.*

Theorem 2 remains valid if Equidistribution and Exponential-distribution are replaced by Beta-distribution and Gamma-distribution, respectively.

(3) Theorem 1 may be generalized to the class of transformations which satisfy (0.1) and

$$Y_i = Z_i X_i + W_i Y_{i+1}, \tag{0.2'}$$

where $(Z_i, W_i)_{i\in\Gamma}$ is a sequence of independent, identically distributed random vectors with Z_i and W_i taking values in [0,1].

Some ideas of the proof of Theorem 1 are adapted from [5], where the simple case $W_i=0$ (a type of conditionally independent motion) was treated.

2. **Proofs.** First we prove a continuity property of the transformation T.

Lemma 1. *Let T be defined* (1.1) *and* (1.5). *Let $X, X^1, X^2, \ldots$ be stationary sequences independent of T with $EX_0 < \infty$ and $\sup_i EX_0^i < \infty$.* If $X^n \Rightarrow_d X$ *so $TX^n \Rightarrow_d TX$.*

Proof. For all $k_i\in\Gamma$, $a_i\geq 0$, $i=1, 2, \ldots$, with $\sum_{i=1}^{\infty} a_i < \infty$ and all $\varepsilon>0$ we have

$$\lim_n \sup P\Big(\sum_{i=\Gamma}^{\infty} a_i X_{k_i}^n > \varepsilon\Big) \leq \frac{1}{\varepsilon} \sup_n EX_0^n \sum_{i=\Gamma}^{\infty} a_i \xrightarrow[\Gamma\to\infty]{} 0.$$

Hence by Theorem 4.2 of [1] it follows from $X^n \Rightarrow_d X$ that also

$$(2.1) \qquad \sum_{i=1}^{\infty} \alpha_i X^n_{k_i} \Rightarrow_d \sum_{i=1} \alpha_i X_{k_i}.$$

Now define non-negative random variables by

$$\alpha_i = \sum_{e=1}^{m} \beta_e T_{k_e, i},$$

where $\beta_e \geq 0$, $k_e \in \Gamma$, $e=1, 2, \dots, m$.

From (1.3) follows that $E(\Sigma_{i=1}^{\infty} \alpha_i) < \infty$ hence $\Sigma_{i=1}^{\infty} \alpha_i < \infty$ a. e. Calling in mind that $(\alpha_i)_{i=1, 2, \dots}$ are independent of $X, X^1, X^2, \dots$ we conclude from (2.1) that

$$(2.2) \qquad \sum_{e=1}^{m} \beta_e (TX^n)_{k_e} = \sum_{i=1}^{\infty} \alpha_i X^n_i \Rightarrow_d \sum_{i=1} \alpha_i X_i = \sum_{e=1}^{m} \beta_e (TX)_{k_e}.$$

Hence by Theorem 77 of [1] it follows that $TX^n \Rightarrow_d TX$.

Lemma 2. *Let T^n be befined by* (1.1) *and* (1.6). *Let X be a stationary sequence independent of $T^1, T^2, \dots$ with $EX_0 < \infty$.*

If $T^n X \Rightarrow_d Y$ then Y is a stationary T-invariant sequence.

Proof. Let T be defined by (1.1) independent of X, Y, $T^1, \dots$

Now $T(T^n X) \Rightarrow_d TY$ by Lemma 1. On the other hand $T(T^n X) =_d T^{n+1} X \Rightarrow_d Y$. Hence $TY =_d Y$.

We give now a result concerning the covariance structure of $T^n X$.

Lemma 3. *Let X be a stationary sequence independent of $T^1, T^2, \dots$, with $EX_0^2 < \infty$ and $E(X_0 . X_i) = E(X_0 . X_{-i})$ for $i=1, 2, \dots$*

Define R_n by $R_0(i) = E(X_0 . X_i)$, $R_n(i) = E((T^n X)_0 . (T^n X)_i)$.

Then for all $n=0, 1, 2, \dots$ and all $i=1, 2, \dots$.

$$(2.3) \qquad R_{n+1}(i) = c_1 a^{i-1} [R_n(0) + 2 \sum_{e=1} R_n(e) a^e] + (1-a)^2 \sum_{e=1}^{i} a^{i-e} \sum_{k=0}^{\infty} R_n(e+k) a^k,$$

$$R_{n+1}(-i) = R_{n+1}(i),$$

$$R_{n+1} = c_2 [R_n(0) + 2 \sum_{e=1}^{\infty} R_n(e) a^e],$$

where $a = EZ_0$, $b = EZ_0^2$, $c_1 = (1-a)(a-b)/(1-b)$, $c_2 = (1-2a+b)/(1-b)$.

(Note that $1 > a > b > 0$ because $P(0 < Z_0 < 1) < 0$.).

Proof. We can write

$$\begin{aligned} R_{n+1}(i) &= E((T^{n+1} X)_0 . (T^{n+1} X)_i) \\ &= \sum_{k=0}^{\infty} \sum_{e=0}^{\infty} E(T^{(n+1)}_{0,k} . T^{(n+1)}_{i,i+e}) \, E((T^n X)_k . (T^n X)_{i+e}) \\ &= \sum_{k=0}^{\infty} \sum_{e=0}^{\infty} \gamma_{i,k,e} \, R_n \, (i+e-k), \end{aligned}$$

where

$$\gamma_{i,k,e} = E((1-Z_0) . \prod_{j=1}^{k} Z_j ((1-Z_i) . \prod_{m=1}^{e} Z_{i+m}).$$

For all $i \in \Gamma$, $k, e \in N$, $\gamma_{i,k,e} = \gamma_{-i,e,k}$ and therefore, from $R_{n(i)} = R_{n(-i)}$ for all $i \in \Gamma$, follows the same for R_{n+1}.

Calculation of the coefficients $\gamma_{i,k,e}$ leads to (2.3).

Lemma 4. *Let X, R_n be as in Lemma 3. X is T-invariant of the second order, i. e. $R_0(i)=R_1(i)$ for all $i\in\Gamma$, if and only if*

$$R_0(i)=R_0(1),\ i=1,2,\ldots,\ \text{and}\ c_1R_0(0)=ac_2R_0(1). \tag{2.4}$$

Proof. From (2.3) we get for all $i=1,2,\ldots,n\in N$

$$R_{n+1}(i+1)-aR_{n+1}(i)=(1-a)^2\sum_{k=i+1}^{\infty}a^{k-(i+1)}R_n(k)$$

and for all $i=2,3,\ldots,n\in N$

$$[R_{n+1}(i)-aR_{n+1}(i-1)]-a[R_{n+1}(i+1)-aR_{n+1}(i)]=(1-a)^2R_n(i).$$

If $R_0(i)=R_1(i)$ we get herefrom for all $i=2,3,\ldots$

$$R_0(i+1)-R_0(i)=R_0(i)-R_0(i-1)\ \text{and}\ R_0(i+1)-R_0(1)=i(R_0(2)-R_0(1)).$$

Hence $R_0(i)=R_0(1)$ for all $i=1,2,\ldots$

Now using (2.3) again we have $R_0(0)=c_2(R_0(0)+2R_0(1)a/(1-a)$, which is equivalent to $c_1R_0(0)=ac_2R_0(1)$.

If, on the other hand, (2.4) is valid, then (2.3) shows that $R_0(i)=R_1(1)$ for all $i\in\Gamma$. Hence Lemma 4 is proved.

The next lemma is the last preliminary to the proof of our Theorem 1.

Lemma 5. *Let T^n be defined by* (1.1) *and* (1.6). *Then*

$$\lim_{n\to\infty}\sum_{j\in\Gamma}E(T^n_{j,0})=0.$$

Proof. Define f_n by

$$f_0(i)=\begin{cases}1, & i=0.\\ 0 & \text{otherwise},\end{cases}\quad f_n(i)=\sum_{j\in\Gamma}E(T^n_{j,0}\,.\,T^n_{j+i,0}).$$

Using (1.2) it is easy to show that for all $i\in\Gamma$, $n\in N$,

$$f_{n+1}(i)=\sum_{k=1}^{\infty}\sum_{e=0}^{\infty}\gamma_{i,k,e}f_n(e-k+i),$$

where the coefficients $\gamma_{i,k,e}$ ore defined as in the proof of Lemma 3. Hence equations (2.3) are valid if we replace there R_n by f_n.

In an analog manner, as in the proof of Lemma 4, we find that for all $i\in\Gamma$, $i\neq0$,

$$f(i)=\lim_{n\to\infty}f_n(i)\ \text{exist and}$$

$$f(i)=f(1)\quad\text{for all}\ i\in\Gamma,\quad i\neq0. \tag{2.5}$$

As a consequence of (1.4) we get for all $n\in N$

$$1=E(\sum_{i\in\Gamma}T^n_{i,0})^2=\sum_{i\in\Gamma}\sum_{k\in\Gamma}E(T^n_{i,0}\,.\,T^n_{k+i,0})=f_n(0)+2\sum_{k=1}^{\infty}f_n(k).$$

Hence for all $m=1,2,\ldots$ by (2.5)

$$1\geq\lim_{n\to\infty}\sum_{k=1}^{m}f_n(k)=\sum_{k=1}^{m}f(k)=m\,.\,f(1),$$

and consequently $f(1)=0$.

But from (2.3) we see that $f_{n+1}(1)\geqq c_1 f_n(0)$ and $f(1)=0$ implicates $f(0)=0$, which was to be proved.

Proof of Theorem 1. Let $T^{(n)}$, T^n, $n=1, 2,\dots$, be defined as in (1.6) and denoted by C, the sequence whose components are all equal to a constant $c>0$.

As a first step we show that there is a sequence X^c such that

$$T^n C \Longrightarrow_d X^c. \tag{2.6}$$

We define matrices T^n by $\widehat{T}^1=T^{(1)}$, $\widehat{T}^n=\widehat{T}^{n-1}\,.\,T^{(n)}$, $n=2, 3,\dots$ Then we have using (1,3)

$$E\ ((\widehat{T}^n C)_i/T^{(1)},\ T^{(2)},\dots,\ T^{(n-1)})=\sum_{j\in\Gamma}\ \widehat{T}^{n-1}_{ij}\,E(T^{(n)}C)_j=(\widehat{T}^{n-1}\ C)_i,$$

i. e. for all $i\in\Gamma$ $(T^i C)_i$ is a martingal relative to the family $\mathcal{F}^n$ of σ-algebras induced by $T^{(1)}$, $T^{(2)},\dots, T^{(n)}$, respectively. Because $\sup_n E(T^n C)_j=c<\infty$ we have by the martingal convergence theorem the existence of a sequence X^c such that $\widehat{T}^n C\longrightarrow X^c$, a. e. Now (2.9) follows from $\widehat{T}^n=_d T^n$ for all $n=1, 2,\dots$ Clearly X^c is stationary and by Lemma 2 X^c is T-invariant. Now we show that

$$EX^c_0=c. \tag{2.7}$$

From (2.3) it is clear that if $c_1 R_n(0)\leqq ac_2 d$, $R_n(k)=d$ for some $d>0$ and all $k=1, 2,\dots$ then the same is true for R_{n+1}. Hence from $R_0(k)=E(C_0\,.\,C_i)=c^2$ for all $k\in\Gamma$ it follows (note that $c_1\geqq ac_2$) that for all $n=1, 2,\dots$ $R_n(0)=E((T^nC)^2_0)\leqq c^2$.

Therefore $(T^nC)_0$ is uniformly integrable and (2.7) follows from

$$c=E((T^n C)_0)\xrightarrow[n\to\infty]{} EX^c_0.$$

Now we prove that X^c has uncorrelated components.

First a famous property of weak convergence gives for all $i=1, 2,\dots$ $E(X^c_0\,.\,X^c_i)\leqq\lim_n\inf E((T^n C)_0\,.\,(T^n C)_i)\leqq\sup_n E((T^n C)^2_0)\leqq c^2$.

Second by Lemma 4 $E(X^c_0\,.\,X^c_i)=E(X^c_0\,.\,X^c_1)$ for all $i=1, 2,\dots$ and by the ergodic theorem we obtain

$$E(X^c_0\,.\,X^c_1)=\lim_{i\to\infty}\ E(X^c_0\,.\,X^c_i)=\lim_{i\to\infty}\ E(X^c_0\,\frac{1}{i}\sum_{j\to1}^{i}X^c_i)$$

$$=E(X^c_0\ E(X^c_0\mid \mathbf{s}))=E((E(X^c_0\mid\mathbf{s}))^2)=E(X^c_0)^2=c^2.$$

Hence $E(X^c_0.X^c_i)=E(X^c_0\,.\,X^c_1)=c^2=E(X^c_0)\,.\,E(X_i)^c$. Using again Lemma 4 we get $R_0(0)=E((X^c_0)^2)=a\,.\,c_2 c^2/c_1$. It remains to show that $T^n X\Longrightarrow_d X^c$ for all X with (1.7). Let X be a stationary sequence independent of T^1, $T^2,\dots$ which satisfies (1.7). We get for all i

$$E\,|\,(T^n X)_i-(T^n C)_i\,|^2=\sum_{j\in\Gamma}\sum_{k\in\Gamma}\text{Cov}\ (X_k\,.\,X_j)\,E(T^n_{ik}\ T^n_{ij})$$

$$=\sum_{j\in\Gamma}\sum_{k\in\Gamma}\text{Cov}\,(X_0\,.\,X_k)\,E(T^n_{i\ j-k}{}^2)\leqq\sum_{k\in\Gamma}|\,\text{Cov}\,|(X_0,\ X_k)|\sum_{j\in\Gamma}E(T^n_{j0})^2.$$

Hereby we have used that

$$\sum_{j\in\Gamma}E(T^n_{i,j-k}\ T^n_{ij})\leqq\frac{1}{2}\sum_j(E(T^n_{i,j-k})^2+E(T^n_{ij})^2)=\sum_{j\in\Gamma}E(T^n_{i,j})^2=\sum_{j\in\Gamma}E(T^n_{j0}\ )^2.$$

By Lemma 5 we see that for all $i \in \Gamma$ $(T^n X)_i - (T^n C)_i$ converges to zero in the mean square.

Hence by (2.6) we conclude that $T^n X \Rightarrow_d X^c$ and Theorem 1 is proved.

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Sektion Mathematik
Friedrich-Schiller-Universität Jena
69 Jena *UHH*

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