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INHOMOGENEOUS LACUNARY INTERPOLATION BY SPLINES, I. (0.2:0.3) CASE

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Let $S^{(3)}_{n,5}$ denote the class of quintic splines $S(x)$ on [0, 1] such that

(i) $S(x)\varepsilon C^3[0, 1]$; (ii) $S(x)\varepsilon\pi_5$ on each $[\nu/n, (\nu+1)/n]$, $0\leqq\nu\leqq n-1$.

We seek to find splines $S_n(x)\varepsilon\ S^{(3)}_{n,5}$ satisfying the following interpolation data:

(i) $S(\nu/n)=f_{\nu/n},\ 0\leqq\nu\leqq n$; (ii) $S''(2\nu/n)=f''_{2\nu},\ 0\leqq\nu\leqq(n-1)/2$;

(iii) $S'''((2\nu+1)/n)=f'''_{2\nu+1},\ 0\leqq\nu\leqq(n-1)/2$; (iv) $S'(0)=f_0',\ S'(1)=f_n'$.

We call it the (0,2 : 0,3) interpolation (up to boundary conditions) which is a special case of what may be termed inhomogeneous lacunary interpolation. It has been shown that such $S(x)$ exist uniquely. The estimates of errors of such spline operators to a function satisfying certain smoothness conditions are obtained.

1. Introduction. Recently A. Meir and A. Sharma [1] obtained the error bounds for lacunary interpolation of certain functions by deficient quintic splines. To be specific, let $S^{(3)}_{n,5}$ denote the class of quintic splines $S(x)$ on [0, 1] such that

(i) $$S(x)\in C^3[0, 1],$$

(ii) $$S(x)\in\Pi_5 \text{ on each } [\nu/n, (\nu+1)/n],\ 0\leqq\nu\leqq n-1.$$

Meir and Sharma have shown that for a given $f\in C^3[0,1]$ there exists a unique spline $S_n(x)\in S^{(3)}_{n,5}$, which satisfies the following interpolation data:

(1.1) $$S_n(\nu/n)=f(\nu/n),\ \nu=0,\ 1,\ldots,n;$$

(1.2) $$S_n''(\nu/n)=f''(\nu/n),\ \nu=0,\ 1,\ldots,n;$$

(1.3) $$S_n'''(0)=f'''(0),\ S_n'''(1)=f'''(1).$$

In the literature (see for example [5]) on lacunary interpolation the interpolation data (1.1), (1.2) is termed (0.2) interpolation and may be considered a special case of homogeneous lacunary interpolation. The interpolation scheme (1.1)—(1.3), which is not homogeneous in the sense that the third derivative is prescribed only at the end points of the interval, has already been studied by the first of us [2] for polynomial lacunary interpolation where this scheme was termed "modified (0.2) interpolation". However, in the case of lacunary interpolation by splines the interpolation data (1.1)—(1.3) will be termed as (0.2) interpolation (up to the boundary conditions). There can well be made various alterations in the above scheme similar to the mixed type lacunary interpola-

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tion, dealt in [3 ; 4]. All these are a special case of the most general well-known Birkhoff interpolation data. Here we shall consider the following interpolation scheme:

(1.4) $$S(\nu/n)=f(\nu/n), \quad 0\leqq\nu\leqq n;$$

(1.5) $$S''(2\nu/n)=f''(2\nu/n), \quad 0\leqq\nu\leqq(n-1)/2;$$

(1.6) $$S'''((2\nu+1)/n)=f'''((2\nu+1)/n), \quad 0\leqq\nu\leqq(n-1)/2;$$

(1.7) $$S'(0)=f'(0), \; S'(1)=f'(1).$$

We may call it (0,2; 0,3) interpolation. In the next communication we shall return to some other problems of this nature.

2. **Preliminaries.** It can be verified that if $P(x)$ is a quintic on [0, 1], then

(2.1) $$P(x)=P(0)\,A_0(1-x)+P(1)\,A_0(x)+P''(0)A_1(x)P'''(1)A_2(x)$$
$$+P^{(4)}(0)A_3(x)+P^{(4)}(1)A_4(x),$$

where $A_0(x)=x$, $A_1(x)=(x^2-x)/2$, $A_2(x)=(x^3-x)/6$, $A_3(x)=(-x^5+5x^4-10x^3+6x)/120$, $A_4(x)=(x^5-10x^3+9x)/120$. Further, a quintic $Q(x)$ on [1, 2] can be written as

(2.2) $$Q(x)=Q(1)\,A_0(2-x)+Q(2)\,A_0(x-1)+Q''(2)A_1(2-x)$$
$$-Q'''(1)A_2(2-x)+Q^{(4)}(2)A_3(2-x)+Q^{(4)}(1)A_4(2-x).$$

For later references we have

(2.3)
$$\begin{aligned}
&A_0'(1)=1, && A_0'(1)=1, && A_0''(0)=0,\\
&A_0'(0)=-1/2, && A_1'(1)=1/2, && A_0''(1)=0,\\
&A_2'(0)=-1/6, && A_2'(1)=1/3, && A_1''(1)=1,\\
&A_3'(0)=1/20, && A_3'(1)=-3/40, && A_2''(1)=-1,\\
&A_4'(0)=3/40, && A_4'(1)=-2/15, && A_3''(1)=-1/6,\\
& && && A_4''(1)=-1/3,\\
&A_0'''(0)=0, && A_0^{(5)}(0)=0, && A_0^{(5)}(1)=0,\\
&A_0'''(1)=0, && A_1^{(5)}(0)=0, && A_1^{(5)}(1)=0,\\
&A_1'''(0)=0, && A_2^{(5)}(0)=0, && A_2^{(5)}(1)=0,\\
&A_2'''(0)=1, && A_3^{(5)}(0)=-1, && A_3^{(5)}(1)=-1,\\
&A_3'''(0)=-1/2, && A_4^{(5)}(0)=1, && A_4^{(5)}(1)=1,\\
&A_4'''(0)=-1/2.
\end{aligned}$$

It is easy to verify that a quintic $P(x)$ can be expressed in the following form:

(2.4) $$P(x)=P(0)B_0(x)+P(1)B_1(x)+P'(0)B_2(x)+P'(1)B_3(x)$$
$$+P''(0)B_4(x)+P'''(1)B_5(x),$$

where

$$B_0(x)=(-8x^5+25x^4-20x^3+3)/3,\quad B_1(x)=(8x^5-25x^4+20x^3)/3,$$
$$B_2(x)=(-5x^5+16x^4-14x^3+3x)/3,\quad B_3(x)=-x^5+3x^4-2x^3,$$
$$B_4(x)=(-2x^5+7x^4-8x^3+3x^2)/6,\quad B_5(x)=(x^5-2x^4+x^3)/18$$

and a quintic $Q(x)$ can be expressed as

$$Q(x)=Q(2)B_0(2-x)+Q(1)B_1(2-x)-Q'(2)B_2(2-x)$$
$$-Q'(1)B_3(2-x)+Q''(2)B_4(2-x)-Q'''(1)B_5(2-x). \tag{2.5}$$

For later references we have

$$\begin{aligned}
&B_0''(1)=20/3, && B_0'''(0)=-40, && B_0^{(4)}(0)=200, && B_0^{(4)}(1)=-120,\\
&B_1''(1)=-20/3, && B_1'''(0)=40, && B_1^{(4)}(0)=-200, && B_1^{(4)}(1)=120,\\
&B_2''(1)=8/3, && B_2'''(0)=-28, && B_2^{(4)}(0)=128, && B_2^{(4)}(1)=-72,\\
&B_3''(1)=4, && B_3'''(0)=-12, && B_3^{(4)}(0)=72, && B_3^{(4)}(1)=-48,\\
&B_4''(4)=1/3, && B_4'''(0)=-8, && B_4^{(4)}(0)=28, && B_4^{(4)}(1)=-12,\\
&B_5''(1)=1/9, && B_5'''(0)=1/3, && B_5^{(4)}(0)=-8/3, && B_5^{(4)}(1)=-4.
\end{aligned} \tag{2.6}$$

Using (2.4) and (2.6), we have

$$P''(1)=\frac{20}{3}P(0)-\frac{20}{3}P(1)+\frac{8}{3}P'(0)+4P'(1)+\frac{1}{3}P''(0)+\frac{1}{9}P'''(1), \tag{2.7}$$

$$P'''(0)=-40P(0)+40P(1)-28P'(0)-12P'(1)-8P''(0)+\frac{1}{3}P'''(1). \tag{2.8}$$

$$P^{(4)}(0)=200P(0)-200P(1)+128P'(0)+72P'(1)+28P''(0)-\frac{8}{3}P'''(1) \tag{2.9}$$

and

$$P^{(4)}(1)=-120P(0)+120P(1)-72P'(0)-48P'(1)-12P''(0)+4P'''(1). \tag{2.10}$$

Similarly using (2.5) and (2.6), we have

$$Q''(1)=\frac{20}{3}Q(2)-\frac{20}{3}Q(1)-\frac{8}{3}Q'(2)-4Q'(1)+\frac{1}{3}Q''(2)-\frac{1}{9}Q'''(1), \tag{2.11}$$

$$Q'''(2)=40Q(2)-40Q(\;)-28Q'(2)-12Q'(1)+8Q''(2)+\frac{1}{3}Q'''(1). \tag{2.12}$$

$$Q^{(4)}(1)=-120Q(2)+120Q(1)+72Q'(2)+48Q'(1)-12Q''(2)-4Q'''(1) \tag{2.13}$$

and

$$Q^{(4)}(2)=200Q(2)-200Q(1)-128Q'(2)-72Q'(1)+28Q''(2)+\frac{8}{3}Q'''(1). \tag{2.14}$$

3. Let $S_{n,5}^{(3)}$ $(n=2, 3, \ldots,)$ denote the class of quintic splines $S(x)$ on $[0, 1]$ as defined by (i) and (ii) in section 1. We shall prove the following

Theorem 1. *For every odd integer n and for every given set of $2n+4$ real numbers $f_0, f_1, \ldots, f_n$; $f_2'', f_2'', \ldots, f_{n-1}''$; $f_1''', f_3''', \ldots, f_n'''$; f_0', f_n' there exists a unique $S(x)\in S_{n,5}^{(3)}$ such that*

(3.1) $$S(\nu/n)=f_\nu,\ \nu=0,\ 1,\dots,n,$$

(3.2) $$S''(2\nu/n)=f''_{2\nu},\ \nu=0,\ 1,\dots,(n-1)/2,$$

(3.3) $$S'''((2\nu+1)/2)=f'''_{2\nu+1},\ \nu=0,\ 1,\dots,(n-1)/2,$$

(3.4) $$S'(0)=f'_0,\ S'(1)=f'_n.$$

Proof. For a given $S(x)\in S^{(3)}_{n,5}$ set $h=n^{-1}$ and

$$M_\nu=S^{(4)}(\nu h+),\ \nu=0,\ 1,\dots,n-1,\ N_\nu=S^{(4)}(\nu h-),\ \nu=1,\ 2,\dots,n.$$

Since $S^{(4)}(x)$ is linear in each interval $(\nu h,\ \overline{\nu+1}\ h)$, it is completely determined by the $2n$ constants $\{M_\nu\}_0^{n-1}$ and $\{N_\nu\}_1^n$. Also, if $S(x)$ satisfies the requirements of Theorem 1, it follows form (2.1), (2.2), (3.1), (3.2) and (3.3) that for $2\nu h\leq x<\overline{2\nu+1}\ h$, $\nu=0,\ 1,\dots,(n-1)/2$, it must have the following form:

(3.5) $$S(x)=f_{2\nu}A_0\Big(\frac{\overline{2\nu+1}\ h-x}{h}\Big)+f_{2\nu+1}A_0\Big(\frac{x-2\nu h}{h}\Big)+h^2f''_{2\nu}A_1\Big(\frac{x-2\nu h}{h}\Big)$$
$$+h^3f'''_{2\nu+1}A_2\Big(\frac{x-2\nu h}{h}\Big)+h^4M_{2\nu}A_3\Big(\frac{x-2\nu h}{h}\Big)+h^4N_{2\nu+1}A_4\Big(\frac{x-2\nu h}{h}\Big),$$

and for $(2\nu+1)h\leq x\leq(2\nu+2)h$, $\nu=0,\ 1,\dots,(n-3)/2$, $S(x)$ has the form:

(3.6) $$S(x)=f_{2\nu+1}A_0\Big(\frac{\overline{2\nu+2}h-x}{h}\Big)+f_{2\nu+2}A_0\Big(\frac{x-\overline{2\nu+1}\ h}{h}\Big)+h^2f''_{2\nu+2}A_1\Big(\frac{\overline{2\nu+2}\ h-x}{h}\Big)$$
$$-h^3f'''_{2\nu+1}A_2\Big(\frac{\overline{2\nu+2}\ h-x}{h}\Big)+h^4N_{2\nu+2}A_3\Big(\frac{\overline{2\nu+2}\ h-x}{h}\Big)+h^4M_{2\nu+1}A_4\Big(\frac{\overline{2\nu+2}\ h-x}{h}\Big).$$

We shall show that it is possible to determine the $2n$ parameters $\{M_\nu\}_0^{n-1}$, $\{N_\nu\}_1^n$, such that the function $S(x)$ given by (3.5) and (3.6) will also satisfy (3.4) and $S'(x)$, $S''(x)$ and $S'''(x)$ will be continuous on $[0,\ 1]$. ($S(x)$ is continuous because of the interpolatory conditions (3.1), whereas by virtue of the conditions (3.2) and (3.3), $S''(x)$ and $S'''(x)$ are continuous on $[0,\ 1]$ except at the points $(2\nu+1)h$ and $2\nu h$, respectively, $\nu=0,\ 1,\dots,(n-1)/2$.)

From (3.5) we see that (3.4) is equivalent to

(3.7) $$2M_0+3N_1=40h^{-4}\Big[f_0-f_1+hf'_0+\frac{1}{2}h^2f''_0+\frac{h^3}{6}f'''_1\Big]$$

and

(3.8) $$9M_{n-1}+16N_n=120h^{-4}\Big[-f_{n-1}+f_n-hf'_n+\frac{h^2}{2}f''_{n-1}+\frac{h^3}{3}f'''_n\Big].$$

Simple calculations show that $S'(\overline{2\nu+2}\ h-)=S'(\overline{2\nu+2}\ h+)$ and $S'''(\overline{2\nu+2}\ h-)=S'''(\overline{2\nu+2}\ h+)$, $\nu=0,\ 1,\dots,(n-3)/2$, are equivalent to

(3.9) $$\frac{6h^4}{120}(M_{2\nu+2}+N_{2\nu+2})+\frac{9h^4}{120}(M_{2\nu+2}+N_{2\nu+3})$$
$$=(2f_{2\nu+2}-f_{2\nu+1}-f_{2\nu+3})+h^2f''_{2\nu+2}-\frac{h^3}{6}(f'''_{2\nu+1}-f'''_{2\nu+3})$$

and

(3.10) $$\frac{h^4}{2}(M_{2\nu+2}+N_{2\nu+2})+\frac{h^4}{2}(M_{2\nu+1}+N_{2\nu+3})=h^3(f'''_{2\nu+3}-f'''_{2\nu+1}).$$

Similarly $S'(\overline{2\nu+1}\,h-)=S'(\overline{2\nu+1}\,h+)$ and $S''(\overline{2\nu+1}\,h-)=S''(\overline{2\nu+1}\,h+)$, $\nu=0, 1, \ldots, (n-3)/2$, are equivalent to

(3.11) $$\frac{16h^4}{120}(M_{2\nu+1}+N_{2\nu+1})+\frac{9h^4}{120}(M_{2\nu}+N_{2\nu+2})$$
$$=2f_{2\nu+1}-f_{2\nu}-f_{2\nu+2}+\frac{1}{2}h^2(f''_{2\nu}+f''_{2\nu+2}),$$

(3.12) $$\frac{h^4}{6}(M_{2\nu}-N_{2\nu+2})-\frac{h^4}{3}(M_{2\nu+1}-N_{2\nu+1})=h^2(f_{2\nu}-f_{2\nu+2})+2h^3f'''_{2\nu+1}.$$

Thus, the theorem will be established if we show that the system of linear equations (3.9)—(3.12) has a unique solution. This end will be achieved by showing that the homogeneous system corresponding to (3.9)—(3.12) has only the zero solution.

The following is the homogeneous system of equations for $\nu=0, 1, \ldots, (n-3)/2$:

(3.13) $$9(M_{2\nu}+N_{2\nu+2})+16(M_{2\nu+1}+N_{2\nu+1})=0,$$

(3.14) $$2(M_{2\nu+2}+N_{2\nu+2})+3(M_{2\nu+1}+N_{2\nu+3})=0,$$

(3.15) $$(M_{2\nu}-N_{2\nu+2})-2(M_{2\nu+1}-N_{2\nu+1})=0,$$

(3.16) $$(M_{2\nu+2}+N_{2\nu+2})+(M_{2\nu+1}+N_{2\nu+3})=0,$$

(3.17) $$9M_{n-1}+16N_n=0,$$

(3.18) $$2M_0+3N_1=0.$$

From (3.14) and (3.16) we have for $\nu=0, 1, \ldots, (n-3)/2$

(3.19) $$M_{2\nu+1}+N_{2\nu+3}=0,$$

(3.20) $$M_{2\nu+2}+N_{2\nu+2}=0.$$

Putting the values of $N_{2\nu+2}=-M_{2\nu+2}$ from (3.20) and $M_{2\nu+1}=-N_{2\nu+3}$ from (3.19) in (3.13) we have

(3.21) $$9(M_{2\nu}-M_{2\nu+2})+16(N_{2\nu+1}-N_{2\nu+3})=0,\quad \nu=0, 1, \ldots, (n-3)/2.$$

This gives on summing them

(3.22) $$9(M_0-M_{n-1})+16(N_1-N_n)=0,$$

and then using (3.17) and (3.18) we have $M_0=0$, $N_1=0$.

From (3.19) and (3.20) $M_2=0$, $N_3=0$.

From (3.13) and (3.15) $M_3=0$, $N_4=0$.

From (3.19) and (3.20) $M_4=0$, $N_5=0$.

. . . .

From (3.19) and (3.20) $M_{n-3}=0, \quad N_{n-2}=0.$

From (3.13) and (3.15) $M_{n-2}=0, \quad N_{n-1}=0.$

From (3.19) and (3.20) $M_{n-1}=0, \quad N_n=0.$

Thus $M_0=M_1=\ldots=M_{n-1}=0,$

and $N_1=N_2=\ldots=N_n=0$

This completes the proof of the theorem.

Remark. When n is even the theorem does not hold in general. In fact, the corresponding system of homogeneous equations then becomes

$$9(M_{2\nu}+N_{2\nu+2})+16(M_{2\nu+1}+N_{2\nu+1})=0,$$
$$(M_{2\nu}-N_{2\nu+2})-2(M_{2\nu+1}-N_{2\nu+1})=0, \ \nu=0, 1, \ldots, (n-2)/2,$$
$$2(M_{2\nu+2}+N_{2\nu+2})+3(M_{2\nu+1}+N_{2\nu+3})=0,$$
$$(M_{2\nu+2}+N_{2\nu+2})+(M_{2\nu+1}+N_{2\nu+3})=0, \ \nu=0, 1, \ldots, (n-4)/2,$$
$$2M_0+3N_1=0,$$
$$2N_n+3M_{n-1}=0,$$

and when $n=4p$, p a positive integer, the following

$$M_0=1=-N_n,$$
$$M_{4j}=1=-N_{4j}, \ j=1, 2, \ldots, (n-4)/4,$$
$$M_{4j+1}=-7/3=-N_{4j+3}, \ j=0, 1, 2, \ldots, (n-4)/4,$$
$$M_{4j+2}=-13/3=-N_{4j+2}, \ j=0, 1, 2, \ldots, (n-4)/4,$$
$$M_{4j+3}=-2/3=N_{4j+1}, \ j=0, 1, 2, \ldots, (n-4)/4,$$

and when $n=4p+2$, p a positive integer, the following

$$M_0=1=M_{n-1}=-7/3, \ N_{n-1}=-2/3$$
$$M_{4j}=1=-N_{4j} \ j=1, 2, \ldots, (n-2)/4,$$
$$M_{4j+1}=-7/3=-N_{4j+3}, \ j=0, 1, 2, \ldots, (n-6)/4,$$
$$M_{4j+2}=-13/3=-N_{4j+2}, \ j=0, 1, \ldots, (n-2)/4,$$
$$M_{4j+3}=-2/3=N_{4j+1}, \ j=0, 1, \ldots, (n-6)/4$$

is a solution of the homogeneous system.

4. Convergence. We have the following

Theorem 2. *Let $f \in C^4[0, 1]$ and n an odd integer. Then the unique quintic spline $S_n(x)$ satisfying conditions of theorem 1 with $f_\nu=f(\nu/n)$, $\nu=0, 1, \ldots, n$, $f''_{2\nu}=f''(2\nu/n)$, $\nu=0, 1, \ldots, (n-1)/2$ and $f'''_{2\nu+1}=f'''((2\nu+1)/n)$, $\nu=0, 1, \ldots, (n-1)/2$, $f'_n=f'(1)$, $f'_0=f'(0)$, we have*

$$\text{(4.1)} \qquad \| S_n^{(r)}-f^{(r)} \|_\infty \leq 4214 n^{r-3}\omega_4(1/n)+2n^{r-4}\| f^{(4)} \|_\infty, \ r=0, 1, 2, 3,$$

where $\omega_4(\cdot)$ denotes the modulus of continuity of $f^{(4)}$.

For the proof of this theorem we shall need the following lemmas:

Lemma 1. *Let $f \in C^4[0, 1]$, n any odd integer and $h=n^{-1}$. Then for $S_n(x) \equiv S_n(f, x)$ of theorem 2, we have*

$$|S'_n(\overline{2\nu+1}\, h) - f'(\overline{2\nu+1}\, h)| \leq \frac{2}{3} h^2 \omega_4(h), \quad \nu=0, 1, \ldots, (n-1)/2 \tag{4.2}$$

and

$$|S'(2\nu h) - f'(2\nu h)| \leq 18 h^2 \omega_4(h), \quad \nu=0, 1, \ldots, (n-1)/2. \tag{4.3}$$

Proof of (4.2). Since $S(x)$ is quintic in $2\nu h \leq x \leq (2\nu+1)h$, we easily obtain from (2.8)

$$\begin{aligned} h^3 S'''(2\nu h) = & -40 f_{2\nu} + 40 f_{2\nu+1} - 28 h S'(2\nu h) - 12 h S'(\overline{2\nu+1}\, h) \\ & -8h^2 f''_{2\nu} + \frac{1}{3} h^3 f'''_{2\nu+1}, \quad \nu = 0, 1, \ldots, (n-1)/2. \end{aligned} \tag{4.4}$$

Similarly from (2.12), since $S(x)$ is quintic in $(2\nu+1)h \leq x \leq (2\nu+2)h$

$$\begin{aligned} h^3 S'''(\overline{2\nu+2}\, h) = & \, 40 f_{2\nu+2} - 40 f_{2\nu+1} - 28 h S'(\overline{2\nu+2}\, h) \\ & -12 h S'(\overline{2\nu+1}\, h) + 8h^2 f''_{2\nu+2} + \frac{1}{3} h^3 f'''_{2\nu+1}, \quad \nu=0, 1, \ldots, (n-3)/2. \end{aligned} \tag{4.5}$$

Writing $\nu+1$ for ν in (4.4) and subtracting from (4.5) we have for $\nu=0, 1, \ldots, (n-3)/2$

$$\begin{aligned} & 12h[S'(\overline{2\nu+1}\, h) - S'(\overline{2\nu+3}\, h)] \\ & = 40(2f_{2\nu+2} - f_{2\nu+1} - f_{2\nu+3}) + 16h^2 f''_{2\nu+2} + \frac{1}{3} h^3 (f'''_{2\nu+1} - f'''_{2\nu+3}). \end{aligned} \tag{4.6}$$

Setting

$$A_\nu = S'(\nu h) - f'_\nu, \quad \nu=0, 1, \ldots, n, \tag{4.7}$$

we have from (4.6)

$$\begin{aligned} 12h(A_{2\nu+1} - A_{2\nu+3}) = & \, 40(2f_{2\nu+2} - f_{2\nu+1} - f_{2\nu+3} + h^2 f''_{2\nu+2}) \\ & -12h(2hf''_{2\nu+2} - f'_{2\nu+3} + f'_{2\nu+1}) + \frac{1}{3} h^3 (f'''_{2\nu+1} - f'''_{2\nu+3}). \end{aligned} \tag{4.8}$$

It is easy to see that

$$2f_{2\nu+2} - f_{2\nu+1} - f_{2\nu+3} + h^2 f''_{2\nu+1} = -\frac{h^4}{12} f^{(4)}(\xi_\nu),$$

$$2hf''_{2\nu+2} - f'_{2\nu+3} + f'_{2\nu+1} = -\frac{h^2}{3} f^{(4)}(\eta_\nu),$$

$$f'''_{2\nu+1} - f'''_{2\nu+3} = -2hf^{(4)}(\zeta_\nu),$$

where $(2\nu+1)h < \xi_\nu, \eta_\nu, \zeta_\nu < (2\nu+3)h$. Hence by (4.8), for $\nu=0, 1, \ldots, (n-3)/2$

$$12(A_{2\nu+1} - A_{2\nu+3}) = h^3 \left[-\frac{10}{3} f^{(4)}(\xi_\nu) + 4 f^{(4)}(\eta_\nu) - \frac{2}{3} f^{(4)}(\zeta_\nu)\right]. \tag{4.9}$$

Fix k, $0 \leq k \leq (n-3)/2$. On summing both sides of (4.9) for $\nu=k, k+1, \ldots, (n-3)/2$ and using the fact that $A_n=0$ (c. f. (3.4)), we have

$$12A_{2k+1}=h^3\sum_{r=k}^{(n-3)/2}[-10f^{(4)}(\xi_\nu)+12f^{(4)}(\eta_\nu)-2f^{(4)}(\zeta_\nu)]$$
$$=4\theta_0 h^2\omega_4(2h),\quad |\theta_0|\le 1,$$

with $k=0, 1, \dots, (n-1)/2$ and this completes the proof of (4.2).

P r o o f o f (4.3). Since $S(x)$ is quintic in $2\nu h\le x\le(2\nu+1)h$, from (2.7) we easily obtain

$$(4.10)\quad h^2S''(\overline{2\nu+1}\,h)=\frac{20}{3}(f_{2\nu}-f_{2\nu+1})+\frac{8}{3}hS'(2\nu h)+4hS'(\overline{2\nu+1}\,h)+\frac{1}{3}h^2f''_{2\nu}$$
$$+\frac{1}{9}h^3f'''_{2\nu+1},\quad \nu=0, 1, \dots, (n-1)/2.$$

Similary, since $S(x)$ is quintic in $(2\nu+1)h\le x\le(2\nu+2)h$, from (2.11) for $\nu=0, 1, \dots, (n-3)/2$

$$(4.11)\quad h^2S''(\overline{2\nu+1}\,h)$$
$$=\frac{20}{3}f_{2\nu+2}-\frac{20}{3}f_{2\nu+1}-\frac{8}{3}hS'(\overline{2\nu+2}\,h)-4hS'(\overline{2\nu+1}\,h)+\frac{1}{3}h^2f''_{2\nu+2}-\frac{1}{9}h^2f''_{2\nu+1}$$

From the above two relations we have for $\nu=0, 1, \dots, (n-3)/2$.

$$(4.12)\quad \frac{8}{3}h(S'(2\nu h)+S'(\overline{2\nu+2}\,h)$$
$$=\frac{20}{3}(f_{2\nu+2}-f_{2\nu})+\frac{h^2}{3}(f''_{2\nu+2}-f''_{2\nu})-\frac{2}{9}h^3f'''_{2\nu+1}-8hS'(\overline{2\nu+1}\,h).$$

For $\nu=0$, we have an account of $A_0=0$.

$$\frac{8}{3}hA_2=\frac{20}{3}(f_2-f_0-2hf'_1-\frac{h^3}{3}f'''_1)+\frac{8h}{3}(2f'_1-f'_2-f'_0+h^2f'''_1)$$
$$+\frac{h^2}{3}(f''_2-f''_0-2hf'''_1)-8hA_1$$
$$=\frac{5}{18}h^4[f^{(4)}(\eta'_1)-f^{(4)}(\eta'_2)]-\frac{8}{18}h^{(4)}[f^{(4)}(\eta'_3)-f^{(4)}(\eta'_4)]+\frac{h^4}{6}[f^{(4)}(\eta'_5)-f^{(4)}(\eta'_6)]-8hA_1.$$
$$h<\eta'_1,\ \eta'_3,\ \eta'_5<2h;\ 0<\eta'_2,\ \eta'_4,\ \eta'_6<h.$$

Therefore, $\frac{8}{3}h\,|A_2|\le\frac{8}{9}\theta_1h^4\omega_4(2h)+8h\,|A_1|,\ |\theta_1|\le 1$. Using (4.2) we have

$$(4.13)\quad |A_2|\le\frac{8}{3}h^2\omega_4(h).$$

Writing $\nu+1$ for ν in (4.12) and subtracting it from (4.12) and using (4.6) we have for $\tau=0, 1, \dots, (n-5)/2$ because of (4.7)

$$(4.14)\quad \frac{8h}{3}(A_{2\nu}-A_{2\nu+2})=\frac{20}{3}(2f_{2\nu+2}-f_{2\nu}-f_{2\nu+4}+4h^2f''_{2\nu+2})$$
$$-\frac{80}{3}(2f_{2\nu+2}-f_{2\nu+1}-f_{2\nu+3}+h^2f''_{2\nu+2})-\frac{8h}{3}(4hf'_{2\nu+2}+f'_{2\nu}-f'_{2\nu+4})$$
$$+\frac{h^2}{3}(2f''_{2\nu+2}-f''_{2\nu}-f''_{2\nu+4})-\frac{4}{9}h^3(f'''_{2\nu+1}-f'''_{2\nu+3}).$$

Now $$2f_{2\nu+2}-f_{2\nu}-f_{2\nu+4}=4h^2f''_{2\nu+2}=-\frac{4}{3}h^4f^{(4)}(\xi_{1,\nu}),$$

$$2f_{2\nu+2}-f_{2\nu+1}-f_{2\nu+3}+h^2f''_{2\nu+2}=-\frac{1}{12}h^4f^{(4)}(\xi_{2,\nu}),$$

$$2f''_{2\nu+2}-f''_{2\nu}-f''_{2\nu+4}=-4h^2f^{(4)}(\xi_{3,\nu}),$$

$$4f'_{2\nu+2}+f'_{2\nu}-f'_{2\nu+4}=-\frac{8}{3}h^3f^{(4)}(\xi_{4,\nu}),$$

$$f'''_{2\nu+1}-f'''_{2\nu+3}=-2hf^{(4)}(\xi_{5,\nu}),$$

where $2\nu h<\xi_{1,\nu},\ \xi_{2,\nu},\ \xi_{3,\nu},\ \xi_{4,\nu},\ \xi_{5,\nu}<(2\nu+4)h$. Hence by (4.14), for $\nu=0,1,\dots,(n-5)/2$

(4.15) $$\frac{8}{3}(A_{2\nu}-A_{2\nu+4})$$

$$=h^3\Big[-\frac{80}{9}f^{(4)}(\xi_{1,\nu})+\frac{20}{9}f^{(4)}(\xi_{2,\nu})-\frac{4}{3}f^{(4)}(\xi_{3,\nu})+\frac{64}{9}f^4(\xi_{4,\nu})+\frac{8}{9}f^{(4)}(\xi_{5,\nu})\Big].$$

If n is of the from $4p+1$, then fix k such that $1\leq k\leq (n-1)/4$. On simming both sides of (4.15) for $\nu=0,2,\dots,2(k-1)$ and using the fact that $A_0=0$, we have

(4.16) $$\frac{8}{3}|A_{4k}+\frac{92}{9}\theta_1h^3\omega_4(4h),\quad |\theta_1|\leq 1,$$

i. e. $|A_{4k}|\leq\frac{46}{3}h^2\omega_4(h)$ and simming both sides of (4.15) for $\nu=1,3,5,\dots,2k-3,\ 2\leq k\leq (n-1)/4$

(4.17) $$|A_{4k-2}|\leq|A_2|+\frac{46}{3}h^2\omega_4(h)\leq 18h^2\omega_4(h)$$

using (4.13).

Hence if $n=4p+1$, (4.16) and (4.17) give

(4.18) $$|A_{2\nu}|\leq 18h^2\omega_4(h),\ \nu=0,1,\dots,(n-1)/2.$$

The result is valid if $n=4p+3$ as can be seen easily. Tihs completes the proof of Lemma 1.

Lemma 2. *Let $f\in C^4[0,1]$, n an odd integer and $h=n^{-1}$. Then for $S_n(x)=S_n(f,x)$ of Theorem 2, we have*

(4.19) $$|S'''(2\nu h)-f'''_{2\nu}|\leq 514\omega_4(h),\ \nu=0,1,2,\dots,(n-1)/2,$$

(4.20) $$h|M_{2\nu}-N_{2\nu+1}|\leq 3700\omega_4(h),\ \nu=0,1,2,\dots,(n-1)/2,$$

(4.21) $$h|M_{2\nu+2}-N_{2\nu+2}|\leq 3700\omega_4(h),\ \nu=0,1,2,\dots,(n-3)/2.$$

Proof. From (4.4) for $\nu=0,1,\dots,(n-1)/2$.

$$h^3(S'''(2\nu h)-f'''_{2\nu})$$

$$=-40f_{2\nu}+40f_{2\nu+1}-28hf'_{2\nu+1}-12hf'_{2\nu+1}-8h^2f''_{2\nu}+\frac{1}{3}h^3f'''_{2\nu+1}$$

$$-h^3f'''_{2\nu}-28hA_{2\nu}-12hA_{2\nu+1}=40(f_{2\nu+1}-f_{2\nu}-hf'_{2\nu}-\frac{1}{2}h^2f''_{2\nu}-\frac{h^3}{6}f'''_{2\nu})-12(f'_{2\nu+1}$$

$$-f'_{2\nu}-hf''_{2\nu}-\frac{1}{2}h^2f'''_{2\nu})+\frac{1}{3}h^3(f'''_{2\nu+1}-f'''_{2\nu})-28hA_{2\nu}-12A_{2\nu+1}$$

$$=\frac{5}{3}h^4f^{(4)}(\xi'_{1,\nu})-2h^4f^{(4)}(\xi'_{2,\nu})+\frac{h^4}{3}f^{(4)}(\xi'_{3,\nu})-28hA_{2\nu}-12hA_{2\nu+1}$$

$$=2\theta_2h^4\omega_4(h)-28hA_{2\nu}-12hA_{2\nu+1},$$

$2\nu h<\xi'_{1,\nu},\ \xi'_{2,\nu},\ \xi'_{3,\nu}<(2\nu+1)h,\ |\theta_2|\leqq 1$, and now using lemma 1, we have

$$|S'''(2\nu h)-f'''_{2\nu}|\leqq 2h\omega_4(h)+504\omega_4(h)+8\omega_4(h)\leqq 514\omega_4(h).$$

This proves (4.19). We now prove (4.20)

Since $S(x)$ is a quintic in $2\nu h\leqq x\leqq(2\nu+1)h$, we have from (2.9) and (2.10) for $\nu=0,1,\dots,(n-1)/2$

$$h^4(M_{2\nu}-N_{2\nu+1})=320(f_{2\nu}-f_{2\nu+1}+hf'_{2\nu}+\frac{h^2}{2}f''_2+\frac{h^3}{6}f'''_{2\nu})$$

$$-120(f'_{2\nu}-f'_{2\nu+1}+hf''_{2\nu}+\frac{h^2}{2}f'''_{2\nu})-\frac{20}{3}h^3f'''_{2\nu+1}+\frac{20}{3}h^3f'''_{2\nu}+200hA_{2\nu}+120A_{2\nu+1}$$

$$=h^4(-\frac{320}{24}f^{(4)}(\eta_1^*)+\frac{120}{6}f^{(4)}(\eta_2^*)+\frac{20}{3}f^{(4)}(\eta_3^*))$$

$$=(200hA_{2\nu}+120hA_{2\nu+1}),$$

$2\nu h<\eta_1^*,\ \eta_2^*,\ \eta_3^*,\ <(2\nu+1)h$, so that

$$h^4|M_{2\nu}-N_{2\nu+1}|\leqq 20h^4\theta_3\omega_4(h)+3600h^3\omega_4(h)+80h^3\omega_4(h),\quad|\theta_3|\leqq 1.$$

Therefore, $h|M_{2\nu}-N_{2\nu+1}|\leqq 3700\omega_4(h)$. Similally form (2.13) and (2.14) arguing in the previous manner we get (4.21). Thus the lema is proved.

5. Proof of theorem 2. Let $2\nu h\leqq x\leqq(2\nu+1)h$, $\nu=0,1,\dots,(n-1)/2$. From (3.5) we have

$$S'''(x)=S'''(2\nu h)A_0(\frac{\overline{2\nu+1}\,h-x}{h})+S'''(\overline{2\nu+1}\,h)A_0(\frac{x-2\nu h}{h}).$$

Now from (3.5) and (2.3)

$$S^{(5)}(2\nu h+)=h^{-1}(-M_{2\nu}+N_{2\nu+1})$$

and since $A_0(\frac{\overline{2\nu+1}\,h-x}{h})+A_0(\frac{x-2\nu h}{h})=1$, we have

(5.1) $$S'''(x)-f'''(x)$$

$$=(S'''(2\nu h+)-f'''(x))A_0(\frac{\overline{2\nu+1}\,h-x}{h})+(S'''(2\nu h+)-f'''(x))A_0(\frac{x-2\nu h}{h})$$

$$-h(M_{2\nu}-N_{2\nu+1})A_1(\frac{x-2\nu h}{h})=I_1+I_2+I_3.$$

Since $|A_0|\leq 1,\ |A_1|\leq 1$.

(5.2) $$\begin{aligned}|I_1|&=|S'''(2\nu h+)-f'''(x)|\\&=|S'''(2\nu h+)-f'''(2\nu h)+(x-2\nu h)f^{(4)}(\alpha)|\\&\leq|S'''(2\nu h+)-f'''(2\nu h)|+h\Omega,\\&\leq 514\omega_4(h)+h\Omega,\quad 2\nu h<\alpha<x,\quad \Omega=\|f^{(4)}\|_\infty,\end{aligned}$$

(5.3) $|I_2|=|S'''(\overline{2\nu+1}\,h)-f'''(x)|=(x-\overline{2\nu+1}\,h)f^{(4)}(\beta)\leq h\Omega,\ (2\nu+1)h<\beta<x,$
and
(5.4) $$|I_3|=|-h(M_{2\nu}-N_{2\nu+1})|\leq 3700\omega_4(h)$$

using lemma 2. Thus (5.1)—(5.4) prove the theorem when $2\nu h\leq x\leq(2\nu+1)h$ and $r=3$. Next let $(2\nu+1)h\leq x\leq(2\nu+2)h$, $\nu=0,1,\ldots,(n-3)/2$. From (3.6)

$$S'''(x)=S'''(\overline{2\nu+1}\,h)A_0\Big(\frac{\overline{2\nu+2}\,h-x}{h}\Big)+S'''(\overline{2\nu+2}\,h)A_0\Big(\frac{x-\overline{2\nu+1}\,h}{h}\Big)$$
$$+h^2S^{(5)}(\overline{2\nu+2h}-)A_1\Big(\frac{\overline{2\nu+2}\,h-x}{h}\Big)=S'''(\overline{2\nu+1}\,h)A_0\Big(\frac{\overline{2\nu+2}\,h-x}{h}\Big)$$
$$+S'''(\overline{2\nu+2h}-)A_0\Big(\frac{x-\overline{2\nu+1}\,h}{h}\Big)+h(M_{2\nu+1}-N_{2\nu+2})A_1\Big(\frac{\overline{2\nu+2}\,h-x}{h}\Big).$$

So that
$$S'''(x)-f'''(x)$$
$$=(S'''(\overline{2\nu+1}\,h)-f'''(x))+(S'''(\overline{2\nu+2}\,h-)-f'''(x))+h(M_{2\nu+1}-N_{2\nu+2}).$$

The rest of the argument is the same and the theorem is proved for $r=3$ For $r=0,1,2$ we proceed as follows.

If $2\nu h\leq x\leq(2\nu+1)h$, then
$$S''(x)-f''(x)=\int_{2\nu h}^{x}(S'''(t)-f'''(t))\,dt,$$
and if $(2\nu+1)h\leq x\leq(2\nu+2)h$, then
$$S''(x)-f''(x)=\int_{\overline{2\nu+2}\,h}^{x}(S'''(t)-f'''(t))\,dt.$$

Hence in every case i. e. $x\in[0,1]$ we have $|S''(x)-f''(x)|\leq h[4214\omega_4(h)+2h\Omega]$. Further since $S(\nu h)=f_\nu$, $S(\overline{\nu+1}\,h)=f_{\nu+1}$, therefore $S'(\lambda)=f'(\lambda)$, $\nu h<\lambda<\overline{\nu+1}\,h$, and therefore $S'(x)-f'(x)=\int_\lambda^x(S''(t)-f''(t))dt$ so that
$$|S'(x)-f'(x)|\leq h\,|S''(x)-f''(x)|\leq h^2[4214\omega_4(h)+2h\Omega].$$

Similaly $S(x)-f(x)=\int_{2\nu h}^{x}(S'(t)-f'(t))dt$. Therefore
$$|S(x)-f(x)|\leq h^3[4214\omega_4(h)+2h\Omega].$$
This completes the proof of theorem 2.

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