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A REMARK ON THE LOBATTO — CHEBYSHEV METHOD FOR THE SOLUTION OF SINGULAR INTEGRAL EQUATIONS AND THE EVALUATION OF STRESS INTENSITY FACTORS

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Cauchy type singular integral equations along the interval [—1, 1] and associated with thre weight function $w(t)=(1-t^2)^{-1/2}$ appear frequently in plane elasticity crack problems and othe problems of mathematical physics. For the numerical solution of such an equation, one can apply numerical integration rules to the original equation or to the equivalent Fredholm integral equation and use appropriate collocation points for the reduction of this equation to a system of linear equations. In this paper and in the case when the Lobatto—Ghebyshev numerical integration rule is used, a comparison between the results obtained when the original equation is reduced to a Fredholm integral eqnation (classical method) and when it is not reduced (direct method) is made. It is seen that the numerical results obtained by the classical method and by the direct method are identical under appropriate but reasonable conditions.

1. Introduction. Much attention has been paid during recent years to the numerical solution of Cauchy type singular integral equations (celled in the sequel, for convenience, just singular integral equations) because of their frequent appearance in elasticity, fluid mechanics and other practical problems. The classical method of solution of a singular integral equation consists in its reduction to an equivalent Fredholm integral equation of the second kind by the regularization method (see, e. g. Gakhov [1]) and the numerical solution of the latter by one of the various methods available in the literature (see, e. g. Atkinson [2]). Among these methods, the one based on the reduction of a Fredholm integral equation to a system of linear equations through the application of a numerical integration rule to the integral term of Fredholm integral equation seems the most convenient in practice.

On the other hand, recently, several direct methods for the numerical solution of a singular integral equation, without a prior reduction to a Fredholm integral equation, have been proposed. Some of these methods are reported by Theocaris [3]. In the same reference a new efficient and general method for the numerical solution of singular integral equations by application of numerical integration rules for Cauchy type integrals and the reduction of the singular integral equation to a system of linear equations, after the appropriate selection of the collocation points, is proposed. The results of this reference have been further generalized to various directions. Some of the results of the authors on the numerical solution of singular integral equations are reported in [4].

In this paper, we will make a comparison between the direct and the classical methods of numerical solution of Cauchy type singular integral equations (as described above) in the special case when the integration interval is the interval [—1, 1] and the weight function — the function $w(t)=(1-t^2)^{-1/2}$, and the Lobatto — Chebyshev numerical integration rule is used for the evaluation of the integrals including the unknown function. The Lobatto — Chebyshev method of numerical solution of singular integral equations was derived in [3] and is particularly convenient for the numerical solution of singular integral equations appearing in plane and antiplane elasticity problems (where stress intensity factors have to be evaluated), fluid mechanics problems and other engineering applications. A more complicated method of derivation of the Lobatto — Chebyshev method was proposed also by the authors [5]. Moreover, the convergence of this method was proved in [6]. Finally, it should be mentioned that an analogous comparison for the Gauss — Chebyshev method of numerical solution of singular integral equations, originally proposed by E r d o g a n and G u p t a [7] and further justified by the present authors [5] was made in [8].

2. Comparison of the Direct with the Classical Method. We consieder the following singular integral equation of the first kind:

$$\int_{-1}^{1} w(t)\left[1/\pi \frac{1}{t-x}+k(t,x)\right]g(t)dt=f(x), \quad -1<x<1, \tag{1}$$

where $g(t)$ is the unknown function to be determined, $f(x)$ — a known function regular along $-1\leqq x\leqq 1$ and $k(t,x)$ — a Fredholm kernel also regular along $-1\leqq t$, $x\leqq 1$. The weight function $w(t)$ incorporates the singularities of the originally unknown function $w(t)g(t)$ and is determined by [1; 5—8]

$$w(t)=(1-t^2)^{-1/2}. \tag{2}$$

Following the theoretical results of G a k h o v [1], we can see that (1) is equivalent to the following Fredholm integrel equation of the second kind:

(3)

$$g(t)+\frac{1}{\pi}\int_{-1}^{1} w(y)\left[\int_{-1}^{1} w^*(x)\frac{k(y,x)}{t-x}dx\right]g(y)dy=\frac{1}{\pi}\int_{-1}^{1} w^*(x)\frac{f(x)}{t-x}dx+\frac{C}{\pi}, \quad -1\leqq t\leqq 1,$$

where the new weight function $w^*(t)$ is givin by

$$w^*(t)=[w(t)]^{-1}=(1-t^2)^{1/2}. \tag{4}$$

Moreover, in the most general case of a singular integral equation of the form (1), where the index $\varkappa$ is equal to 1 [1], the equivalent Fredholm integral equation (3) contains an arbitrary constant C related to the unknown function $g(t)$ by

$$\int_{-1}^{1} w(t)g(t)dt=C. \tag{5}$$

Of course, in practice the constant C should be known in advance so that a numerical solution can be obtained and, hence, it will be assumed here that this is the case.

For the numerical solution of (3) we can use the Lobatto — Chebyshev numerical integration rule for regular integrals [9]

$$\int_{-1}^{1} w(t)\varphi(t)\,dt=\sum_{i=1}^{n} A_i\varphi(t_i)+E_n \tag{6}$$

with n nodes, where E_n is the error term and the nodes t_i and the weights A_i are determined by

$$(7)\qquad (1-t_i^2)U_{n-2}(t_i)=0,\quad \text{that is: } t_i=\cos[(i-1)\pi/(n-1)],\quad i=1(1)n,$$

$$(8)\qquad A_i=\pi/[2(n-1)],\quad i=1,\ n,\ A_i=\pi/(n-1),\quad i=2(1)\ (n-1),$$

$U_n(x)$ denoting the Chebyshev polynomial of the second kind and of degree n. Then we obtain from (3) the following approximate system of linear equations:

$$(9)\qquad g(t_l)+\frac{1}{\pi}\sum_{i=1}^{n} A_i\left[\int_{-1}^{1} w^*(x)\frac{k(t_i,x)}{t_l-x}\right]g(t_i)=\frac{1}{\pi}\int_{-1}^{1} w^*(x)\frac{f(x)}{t_l-x}\,dx+\frac{C}{\pi}\cdot$$

Of course, we have also in (9) to evaluate the Cauchy type principal value integrals. Numerical integration rules for such integrals were developed by the authors in [3; 10]. Here we can use the Gauss — Chebyshev numerical integration rule

$$(10)\qquad \int_{-1}^{1} w(t)\varphi(t)dt=\sum_{k=1}^{m} A_k^*\varphi(x_k)+E_m^*,$$

where E_m^* is the corresponding error term and the nodes x_k and the weights A_k^* are given by

$$(11)\qquad T_m(x_k)=0,\quad \text{that is:}\quad x_k=\cos[(k-0.5)\pi/m],\quad k=1(1)m,$$

$$(12)\qquad A_k^*=\pi/m,\quad k=1(1)m,$$

where $T_m(x)$ denotes the Chebyshev polynomial of the first kind and degree m. Evidently, for the weight function $w^*(t)$ determined by (4), (10) is modified as

$$(13)\qquad \int_{-1}^{1} w^*(t)\varphi(t)dt=\sum_{k=1}^{m} B_k\varphi(x_k)+E_m^*,$$

where the weights B_k are now, because of (4) and (12), determined by

$$(14)\qquad B_k=\pi(1-x_k^2)/m,\quad k=1(1)m.$$

Furthermore, for Cauchy type principal value integrals, the numerical integration rule (13) is modified as [3; 10]

$$(15)\qquad \int_{-1}^{1} w^*(t)\,\frac{\varphi(t)}{t-y}\,dt=\sum_{k=1}^{m} B_k\,\frac{\varphi(x_k)}{x_k-y}+\frac{\pi(1-y^2)U_{m-1}(y)}{T_m(y)}\,\varphi(y)+E_m^*$$

with $y\neq x_k$ $(k=1(1)m)$. A similar rule including the derivative $\varphi'(y)$ of the integrand $\varphi(t)$ holds in the case when $y=x_k$ $(k=1(1)m)$ [10].

By using (15) we can evaluate the principal value integrals in (9) with the desired accuracy. But there exists also a special but very reasonable possibility — to select m in (15) to be equal to $(n-1)$ in (6). In this case, because of (7) and (9), the second term of the right side of (15) vanishes and (after a further ignorance of the error term E_m^* in (15)) the system of linear equations 9) is further approximated by

$$g(t_l)+\frac{1}{\pi}\sum_{i=1}^{n}A_i\left[\sum_{k=1}^{n-1}B_k\frac{k(t_i,x_k)}{t_l-x_k}\right]g(t_i)=\frac{1}{\pi}\sum_{k=1}^{n-1}B_k\frac{f(x_k)}{t_l-x_k}+\frac{G}{\pi},\quad l=1(1)n. \tag{16}$$

Of course, if $m\neq n-1$, the appearance of (16) would be much more complicated.

From the solution of the system of linear equations (9) or better (16), we determine the approximate values of the unknown function $g(t)$ at the nodes t_i used, determined by (7). It is further quite possible to express the approximation $\tilde{g}(t)$ of the unknown function $g(t)$ along the whole interval $[-1, 1]$ by using well-known polynomial interpolation techniques. We wish also to mention at this point that the system of linear equations (9) is more accurate than the system of linear equations (16). For example, if the Fredholm kernel $k(t, x)$ in (1) vanishes, then (9) will provide the exact values for $g(t)$ at the nodes t_i, whereas (16) will do so only in the special case when the right-side function $f(x)$ in (1) is a polynomial of degree less than $2(n-2)$. Yet, in cases, when no closed-form expressions can be easily found for the Cauchy type principal value integrals in (9), we have to use (16). If the numerical results from (16) are of inadequate accuracy, then we can use (9) by applying (15) with $m>n-1$ and not $m=n-1$ as happens with (16). The resulting system of linear equations can be directly constructed from (9).

Up to now the classical method of numerical solution of (1) has been considered. We will proceed now to the direct method of its numerical solution, not requiring its reduction to an equivalent Fredholm integral equation of the second kind. We use the results of [3; 5] and apply the Lobatto — Chebyshev numerical integration rule (6) to the direct approximation of the integrals in (1). For Cauchy type principal value integrals this rule has the form [3, 10]

$$\int_{-1}^{1}\frac{w(t)\varphi(t)}{t-x_k}\,dt=\sum_{i=1}^{n}A_i\frac{\varphi(t_i)}{t_i-x_k}+E_n,\qquad k=1(1)(n-1), \tag{17}$$

where the collocation points x_k are determined by (11) with m replaced by $(n-1)$ as already assumed for the construction of the system of linear equations (16). Hence, by using (6) and (17) we can approximate (1) and (5) by the following system of linear equations:

$$\sum_{i=1}^{n}A_i\left[\frac{1}{\pi}\,\frac{1}{t_i-x_k}+k(t_i,x_k)\right]g(t_i)=f(x_k),\quad k=1(1)(n-1), \tag{18a}$$

$$\sum_{i=1}^{n}A_i\,g(t_i)=C. \tag{18b}$$

This system resulted from the direct method of numerical solution of (1) by application of the Lobatto — Chebyshev numerical integration rule.

We will now show that the systems of linear equations (16) and (18) are equivalent in the sense that they provide the same numerical values for $g(t_i)$. To show this, we will derive (18) from (16). The inverse is also completely possible. Thus we take into account that

$$\int_{-1}^{1}\frac{w(t)}{t-x}\,dt=0,\quad -1<x<1. \tag{19}$$

By applying the Lobatto — Chebyshev numerical integration rule (17) to this dentity, we obtain

(20) $$\sum_{l=1}^{n} \frac{A_l}{t_l-x_k}=0, \quad k=1(1)(n-1).$$

Next, by multiplying the l-th equation of equations (16) by A_l (for all values of l) and adding the resulting equations we obtain

(21) $$\sum_{l=1}^{n} A_l g(t_l)+\frac{1}{\pi}\sum_{i=1}^{n} A_i\left\{\sum_{k=1}^{n-1} B_k\left[\sum_{l=1}^{n}\frac{A_l}{t_l-x_k}\right]k(t_i,\ x_k)\right\}g(t_i)$$
$$=\frac{1}{\pi}\sum_{k=1}^{n-1} B_k\left[\sum_{l=1}^{n}\frac{A_l}{t_l-x_k}\right]f(x_k)+\frac{C}{\pi}\sum_{l=1}^{n} A_l$$

and taking into account (20), as well as that

(22) $$\sum_{l=1}^{n} A_l=\pi,$$

clear from (8), we realize that (21) reduces to (18b).

Similarly, we multiply the l-th equation of equations (16) by $A_l/(t_l-x_j)$ (for all values of l) and add the resulting equations when we find

(23) $$\sum_{l=1}^{n-1} A_l\frac{g(t_l)}{t_l-x_j}+\frac{1}{\pi}\sum_{i=1}^{n} A_i\left\{\sum_{k=1}^{n-1} B_k\left[\sum_{l=1}^{n}\frac{A_l}{(t_l-x_j)(t_l-x_k)}\right]k(t_i,\ x_k)\right\}g(t_i)$$
$$=\frac{1}{\pi}\sum_{k=1}^{n-1} B_k\left[\sum_{l=1}^{n}\frac{A_l}{(t_l-x_j)(t_l-x_k)}\right]f(x_k)+\frac{C}{\pi}\sum_{l=1}^{n}\frac{A_l}{t_l-x_j}, \qquad j=1(1)(n-1).$$

Because of (20), the last term of the right side of (23) vanishes. Furthermore, for $k\neq j$ we have

(24) $$\sum_{l=1}^{n}\frac{A_l}{(t_l-x_j)(t_l-x_k)}=\frac{1}{x_j-x_k}\sum_{l=1}^{n} A_l\left[\frac{1}{t_l-x_j}-\frac{1}{t_l-x_k}\right],\quad j\neq k;\ j,k=1(1)(n-1),$$

and, because of (20), this sum vanishes too. Hence, only the terms with $k=j$ should be taken into account in (23) and, therefore, this system of equations takes the following simpler form:

(25) $$\sum_{l=1}^{n} A_l\frac{g(t_l)}{t_l-x_j}+\frac{1}{\pi}\sum_{i=1}^{n} A_i\left[B_j\sum_{l=1}^{n}\frac{A_l}{(t_l-x_j)^2}\right]k(t_i,\ x_j)g(t_i)$$
$$=\frac{1}{\pi}\left[B_j\sum_{l=1}^{n}\frac{A_l}{(t_l-x_j)^2}\right]f(x_j),\quad j=1(1)(n-1).$$

Now we take into account that the Lobatto — Chebyshev numerical integration rule for Cauchy type principal value integrals (17) has in the general case the form [3; 10]

(26) $$\int_{-1}^{1}\frac{w(t)\varphi(t)}{t-x}\,dt=\sum_{l=1}^{n} A_l\frac{\varphi(t_l)}{t_l-x}-\frac{\pi T_{n-1}(x)\varphi(x)}{(1-x^2)U_{n-2}(x)}+E_n,\quad x\neq t_i,\quad i=1(1)n,$$

from which the simpler form (17) results if x is restricted to the values given by (11) (with $m=n-1$ as already mentioned). By putting $\varphi(t)\equiv 1$ in (26), in which case the error term E_n vanishes and both its members vanish too because of (19), that is

$$\sum_{l=1}^{n} \frac{A_l}{t_l-x} = \frac{\pi T_{n-1}(x)}{(1-x^2)U_{n-2}(x)} \tag{27}$$

and differentiating this last equation with respect to x, restricting also x to its values given by (11) (with $m=n-1$) we find

$$\sum_{l=1}^{n} \frac{A_l}{(t_l-x_j)^2} = \frac{\pi(n-1)}{1-x_j^2}, \quad j=1(1)(n-1), \tag{28}$$

since

$$T'_{n-1}(x)=(n-1)\,U_{n-2}(x). \tag{29}$$

Finally, by combining (28) and (14) (with $m=n-1$), we find that

$$B_j \sum_{l=1}^{n} \frac{A_l}{(t_l-x_j)^2} = \pi^2, \quad j=1(1)\ (n-1), \tag{30}$$

and the system of linear equations (25) reduces to the system of linear equations (18a). This completes the proof of the equivalence (from the numerical point of view) of the systems of linear equations (16) and (18) or, better, of the direct and the classical method of numerical solution of the original singular integral equation (1) (supplemented by the condition (5)).

It can finally be mentioned that the application of the Lobatto — Chebyshev numerical integration rule to the derivation of the systems of linear equations (9) (and further (16)), as well as (18), is very useful when solving crack problems in plane and antiplane elasticity for the evaluation of the values of the stress intensity factors at the crack tips since, as explained in detail in [5], by doing so we determine directly from the system of linear equations the values of $g(\pm 1)$ (since the points (± 1) belong to the nodes as is clear from (5)) proportional to the stress intensity factors and no extrapolation methods (accompanied by the corresponding computations and computational errors) are necessary.

3. Conclusions. Concluding we wish to mention that

(i) As is clear from the above developments, the numerical solution of a singular integral equation by using the Lobatto — Chebyshev numerical integration rule is not at all much different from its numerical solution by the classical regularization method (by using the same rule) and under reasonable assumptions (leading to the system of linear equations (16)) is completely equivalent to it.

(ii) In general, it seems that the classical method of numerical solution of the singular integral equation (1), leading to the system of linear equations (9), can provide more accurate results than the corresponding direct method provided that closed-form formulae or numerical integration rules of a high accuracy are used for the evaluation of the Cauchy type principal value integrals in (9).

(iii) Finally, it can be mentioned that both the classical and the direct method of numericl solution of a singular integral equation (as described above) converge in general, as the number n of the linear equations of the corresponding system of linear equations tends to infinity, to the exact solution of this equation. This was proved in [2] and [6], respectively. The equivalence also from the numerical point of view of these two methods is compatible with the fact that both these methods converge and, probably, might be used for the proof of the convergence of each one of them on the basis of the convergence of the other.

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