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ONE-SIDED ALGEBRAICAL APPROXIMATION ON THE REAL AXIS WITH THE WEIGHT EXP $\{-x^2\}$

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The problem of finding estimates for the one-sided approximation of functions by means of algebraical polynomials on the real axis is considered by Freud and Szabados in [1]. They use the natural for this problem weight $\exp\{-x^2\}$. The estimates in [1] are made for large classes of functions but they have the form $\tilde{E}(H_n, e^{-x^2}, f)_L = O(n^{-(r+1)/2})$. We can obtain more concrete estimates using the Nevai's paper [2] but they hold true for smaller classes of functions. The purpose of this paper is to give estimates of the approximation of the largest class of functions for which the problem has a sense. These estimates show how the structure properties influence the convergence.

1. Definitions and denotations. Let H_n be the set of all algebraical polynomials of a degree at most n. For every f integrable in $(-\infty, \infty)$ we have

$$\| f \|_L = \| f(x) \|_L = \int_{-\infty}^{\infty} | f(x) | \, dx ;$$

$$w(x) = \exp\{-x^2\}, \quad w(\delta, x) = \begin{cases} w(x-\delta) & \text{for } x > \delta ; \\ 1 & \text{for } |x| \leqq \delta ; \\ w(x+\delta) & \text{for } x < -\delta . \end{cases}$$

We have $w(\delta, x) \geqq w(x-\gamma)$ for each $\gamma \in [-\delta, \delta]$.

The best one-sided approximation $\tilde{E}(H_n, w, f)$ of the function f in the integral metric with the weight w by means of elements of H_n is given by

$$\tilde{E}(H_n, w, f) = \inf\{ \| W(P-Q) \|_L : P, Q \in H_n, P \geqq f \geqq Q \}.$$

We shall consider the classes $V(r, s)$ $(r = 0, 1, 2, \ldots, s = 0, 1, 2, \ldots)$ (see [1]). $V(r, s)$ is the set of all functions f defined in $(-\infty, \infty)$ which has an absolute continuous $r-1$-th derivative, $f^{[r]}$ has a bounded variation in each finite interval and f satisfies the conditions

$$M_r(f) = \int_{-\infty}^{\infty} w(x) \, | df^{[r]}(x) | < \infty. \tag{1.1}$$

(1.2) There are numbers A, B, s depending on f such that $A, B > 0$, s is natural and $|f(x)| \leqq A + Bx^{2s}$ for each x.

Let us note that the largest class of functions for which $\tilde{E}(H_n, w, f)$ has a sense is the class of functions satisfying (1.2) with $s = [n/2]$.

We denote with Φ the class of all functions φ defined in $(-\infty, \infty)$ satisfying the conditions

(1.3) $$\varphi(x)=\varphi(-x) \text{ for each } x;$$

(1.4) $$\varphi(0)>0;$$

(1.5) $$\varphi \text{ is increasing in } [0, \infty];$$

(1.6) $$\varphi \text{ has a first derivative};$$

(1.7) There is a constant b_1 such that $\varphi(x+1/\varphi(0))\leqq b_1\varphi(x)$ for every $x\geqq 0$;

(1.8) $$\varphi(x_1)-\varphi(x_2)\leqq 2(x_1-x_2)\varphi(x_1)\varphi(x_2) \text{ for every } x_1>x_2\geqq 0;$$

(1.9) There is a constant b_2 such that $\varphi(x)\geqq b_2x$ for every $x\geqq 0$.

The class Φ includes enough functions. For example $x^2+1\in\Phi(b_1=3, b_2=2)$ and $\varphi_1\in\Phi(b_1=5, b_2=1)$, where $\varphi_1(x)=|x|$ for $|x|\geqq 1$ and $\varphi_1(x)=(x^2+1)/2$ for $|x|<1$.

Let us mark the following property of Φ:

(1.10) If $\varphi\in\Phi$ then $a\varphi\in\Phi$ for each $a\geqq 1$.

Let φ satisfy the conditions (1.3), (1.4), (1.5), (1.6), (1.9) and

(1.11) φ' is continuous and there is a constant b_3 such that

$$\varphi'(x)\leqq b_3\varphi(x) \text{ for each } x>0.$$

Then there is a constant a_0 such that for every $a\geqq a_0$ we have $a\varphi\in\Phi$ [5]

2. One modulus of a function. We shall estimate $\widetilde{E}(H_n, w, f)$ with the modulus

$$\tau_k(f, \varepsilon, \delta)=\|\varepsilon(x)\omega_k(f, x, \delta(x))\|_L,$$

where $\omega_k(f, x, \delta(x))=\sup\{|\Delta_h^k f(t)|:\ t, t+kh\in[x-k\delta(x)/2, x+k\delta(x)/2]\}$

$$\Delta_h^k f(t)=\sum_{i=0}^{k}(-1)^{i+k}\binom{k}{i}f(t+ih)$$

(k is natural, ε and δ are non-negative functions, ε is continuous).

We shall use these two cases: $\varepsilon=w$, $\delta=\text{const}$ and $\varepsilon=w\varphi$, $\delta=\delta_0\cdot\varphi^{-1}$, where $\delta_0=\text{const}$, $\varphi\in\Phi$.

This modulus is used for first time in [4].

Properties [4]

(2.1) $$\tau_k(f+g, \varepsilon, \delta)\leqq\tau_k(f, \varepsilon, \delta)+\tau_k(g, \varepsilon, \delta);$$

(2.2) If $\varepsilon_1\leqq\varepsilon_2$, $\delta_1\leqq\delta_2$ then $0\leqq\tau_k(f, \varepsilon_1, \delta_1)\leqq\tau_k(f, \varepsilon_2, \delta_2)$;

(2.3) $$\tau_{k+1}(f, \varepsilon, \delta)\leqq\tau_k(f', \varepsilon\delta, (k+1)\delta k^{-1});$$

(2.4) $$\tau_1(f, \varepsilon, \delta)\leqq\|\varepsilon\delta f'\|_L+\tau_1(f', \varepsilon\delta, \delta).$$

We can also prove [5] that ($\delta=\text{const}>0$)

(2.5) $$\tau_1(f, w, \delta)\leqq\delta\int_{-\infty}^{\infty} w(\delta/2, x)\,|df(x)| \quad \text{and}$$

(2.6) $$\tau_1(f, w\varphi, \delta\varphi^{-1})\leqq c_1\delta\int_{-\infty}^{\infty} w(x)\,|df(x)|$$

for every $\varphi\in\Phi$, where c_1 depends only on φ.

3. Notes about the paper of Freud and Szabados. In [1] Freud and Szabados prove

Theorem A. *If* $f \in V(r, s)$ *then* $\tilde{E}(H_{2n-1}, w, f) = O(n^{-(r+1)/2})$.

We shall obtain a more concrete estimate for $\tilde{E}(H_{2n-1}, w, f)$. For this we shall follow the reasonings in [1].

Lemma 1 [1]. *For each* $x>0$ *and for* $k=0, 1, 2, \ldots$ *we have*

$$\int_x^\infty (t-x)^k w(t)dt \le k!\,(2x)^{-k-1} w(x). \tag{3.1}$$

We consider the function $\alpha(x) = x^{-r}e^{x^2}$. It is easy to see that $\alpha^{(k)} = x^{-r-k}e^{x^2}.\beta(x)$ where $\beta \in H_{2k}$ and β has a positive coefficient before x^{2k}. Then there is, $x_0 = x_0(r) > 0$ such that

$$0 < \alpha^{(k)}(x) \le a_1(r)x^{-r+k}e^{x^2} \text{ for } x \ge x_0,\ k=0, 1, \ldots, r-1 \tag{3.2}$$

and

$$a_2(r) \le \alpha^{(r)}e^{-x^2} \le a_1(r) \quad \text{for} \quad x \ge x_0 \tag{3.3}$$

We fix the point x_0.

Lemma 2. *If* $F \in V(r, s)$ *then for* $x \ge x_0$ *we have*

$$|F^{(k)}(x)\cdot x^{r-k}w(x)| \le a_4(r)[M_r(F) + |F^{[r]}(-x_0)|] + \sum_{\mu=k}^{r-1} |F^{(\mu)}(x_0)|\,x^{\mu-k} \tag{3.4}$$

for $k=0, 1, \ldots, r-1$ *and*

$$|F^{[r]}(x)\,w(x)| \le M_r(F) + |F^{[r]}(-x_0)|. \tag{3.5}$$

Proof. (3.5) follows from

$$|w(x)[F^{[r]}(x) - F^{[r]}(-x_0)]| \le \int_{-x_0}^{x} w(t)\,|dF^{[r]}(t)| \le M_r(F).$$

Let us set $T = F^{[r]}(-x_0) + M_r(F)$. (3.5) and (3.3) give

$$|F^{[r]}(x)| \le \frac{T}{a_2(r)}\alpha^{(r)}(x) = a_3(r)\,.\,T\,.\,\alpha^{(r)}(x) \text{ for } x \ge x_0. \tag{3.6}$$

Integrating (3.6) from x_0 to x $r-k$ times and using (3.2) we obtain

$$|F^{(k)}(x)| \le a_3(r)T\alpha^{(k)}(x) + P(x), \text{ where} \tag{3.7}$$

$$P(x) = \sum_{\mu=k}^{r-1} \frac{|F^{(\mu)}(x_0)|}{(\mu-k)!}(x-x_0)^{\mu-k}. \tag{3.8}$$

We obtain (3.4) from (3.7), (3.8), (3.2) and the inequality $x - x_0 < x$.

Remark. The corresponding lemma in [1] says that the functions in the left of inequalities (3.4), (3.5) are bounded above.

Theorem 1. *If* $F \in V(r, s)$ *then for each* $n \ge \max\{s, 9x_0^2\}$ *we have*

$$\tilde{E}(H_{2n-1}, w, F) \le a_4(r)M_r(F)n^{-(r+1)/2} + a_5(r, S)\{A+B+|F^{[r]}(0)| + \sum_{\mu=0}^{r-1}[|F^{(\mu)}(-x_0)| + |F^{(\mu)}(0)| + |F^{(\mu)}(x_0)|]\}e^{-n/9}n^{s-1/2},$$

where A, B *are the constants in* (1.2).

Proof. We shall use the proof of Theorem A in [1]. Obviously we can suppose that $2s>r$. We have for F

$$F(x)=\sum_{\nu=0}^{r-1}\frac{F^{(\nu)}(0)}{\nu!}x^{\nu}+\frac{F^{[r]}(0)}{r!}x^{r}+F_1(x)+F_2(-x), \tag{3.9}$$

where $F_1, F_2\in V(r,s)$, $F_1(x)=F_2(x)=0$ for $x\leq 0$, $\bigvee_a^b F_1^{[r]}\leq\bigvee_a^b F^{[r]}$, $\bigvee_a^b F_2^{[r]}\leq\bigvee_a^b F^{[r]}$ for each finite interval $[a,b]$ and

$$M_r(F_1)\leq M_r(F),\quad M_r(F_2)\leq M_r(F). \tag{3.10}$$

Let f satisfy the inequality (1.2). Then

$$|F_i(x)|\leq(A+|F^{[r]}(0)|+\sum_{\nu=0}^{r-1}|F^{(\nu)}(0)|)+(B+|F^{[r]}(0)|+\sum_{\nu=0}^{r-1}|F^{(\nu)}(0)|)x^{2s} \tag{3.11}$$

for $i=1,2$.

That shows us that it is sufficient to prove the theorem for functions F such that $F(x)=0$ for $x<0$ Let $\omega_n\in(\sqrt{n}/3,\sqrt{n}/2)$ be a point of continuity for $F^{[r]}$. Then

$$F(x)=\sum_{\nu=0}^{r-1}\frac{F^{(\nu)}(\omega_n)}{\nu!}(x-\omega_n)^{\nu}+\frac{F^{[r]}(\omega_n)}{r!}(x-\omega_n)^{r}+F^{*}(x)+F^{**}(x), \tag{3.12}$$

where $F^*, F^{**}\in V(r,s)$, $F^*(x)=0$ for $x\overline{\in}[0,\omega_n)$, $F^{**}(x)=0$ for $x<\omega_n$ and the variations of F^* and F^{**} are not greater than the variation of F in every finite interval.

Repeating the reasonings in [1] we have

$$\tilde{E}(H_{2n-1}, w, F^*)\leq a_6(r)n^{-(r+1)/2}M_r(F); \tag{3.13}$$

$$2^{-1}.\tilde{E}(H_{2n-1}, w, F^{**})\leq w(\omega_n)[(A+Bn^s/2)n^{-1/2}+2^{2s-1}Bn^{-s-1/2}$$

$$+\sum_{\nu=0}^{r-1}\frac{|F^{(\nu)}(\omega_n)|}{\nu!}n^{-(\nu+1)/2}+\frac{|F^{[r]}(\omega_n)|}{r!}n^{-(r+1)/2}]+(A+Bn^s/2)\int_{\omega_n}^{\infty}w(x)dx$$

$$+2^{2s-1}B\int_{\omega_n}^{\infty}(x-\omega_n)w(x)dx+\sum_{\nu=0}^{r-1}\frac{|F^{(\nu)}(\omega_n)|}{\nu!}\int_{\omega_n}^{\infty}(x-\omega_n)^{\nu}w(x)dx$$

$$+|F^{[r]}(\omega_n)|(r!)^{-1}\int_{\omega_n}^{\infty}(x-\omega_n)^{r}w(x)dx.$$

Using the inequality $|F(x)|\leq A_1+B_1x^{2s}$(A_1 and B_1 are the following coefficients in (3.11)), Lemma 3.1 and Lemma 3.2 with $F^{[r]}(-x_0)=0$, we obtain

$$\tilde{E}(H_{2n-1}, w, F^{**})\leq a_7(r)n^{-(r+1)/2}M_r(F)+a_8(r,s)[A_1+B_1+\sum_{\mu=0}^{r-1}|F^{(\mu)}(x_0)|]e^{-n/9}n^{s-1/2}. \tag{3.14}$$

Using (3.9) — (3.14), we prove the theorem.

4. Estimates for $\tilde{E}(H_n, w, f)$. In broad outlines we shall follow the ideas of Popov and Andreev [3] to Theorem 3.

Theorem 2. *If F satisfies (1.2) and F is measurable, then*

$$\tilde{E}(H_{2n-1}, w, F)\leq c_1\tau_1(F, w, 4^{-1}n^{-1/2})+c_2(s)(A+B)e^{-n/9}n^{s-1/2}. \tag{4.1}$$

Proof. We set $x_i = 24^{-1}(2i-1)n^{-1/2}$ for each integer i,

$$F_1(x) = \sup\{F(t):\ t \in [x_i, x_{i+1})\},\ F_2(x) = \inf\{F(t):\ t \in [x_i, x_{i+1})\}$$

for $x \in [x_i, x_{i+1})$. $F_1, F_2 \in V(0, s)$, because F satisfies (1.2).

$$(4.2)\qquad \int_{-\infty}^{\infty} w(x)(F_1(x)-F_2(x))dx \leq \int_{-\infty}^{\infty} w(x)\omega_1(F, x, 6^{-1}n^{-1/2})dx \leq \tau_1(F, w, 4^{-1}n^{-1/2}).$$

Using Theorem 1 we obtain $P_1 \in H_{2n-1}$, $P_1 \geq F_1$, such that

$$(4.3)\qquad \int_{-\infty}^{\infty} w(x)[P_1(x)-F_1(x)]dx \leq a_4(0)n^{-1/2} \int_{-\infty}^{\infty} w(x)\,|\,dF_1(x)\,| + \varepsilon$$

$$= a_4(0)n^{-1/2} \sum_{i=-\infty}^{\infty} w(x_i)\,|\,F_1(x_i+0)-F_1(x_i-0)\,| + \varepsilon$$

$$\leq 24\,.\,a_4(0)\,[\,\sum_{i=1}^{\infty} \int_{(x_i+x_{i-1})/2}^{x_i} w(x)\omega_1(F_1, x, 12^{-1}n^{-1/2})dx + \sum_{i=-1}^{-\infty} \int_{x_i}^{(x_i+x_{i-1})/2} w(x)\omega_1(F_1, x,$$

$$12^{-1}n^{-1/2})dx] + \varepsilon \leq 24\,.\,a_4(0)\tau_1(F_1, w, 12^{-1}n^{-1/2}) + \varepsilon \leq 24\,.\,a_4(0)\,.\,\tau_1(F, w, 4^{-1}n^{-1/2}) + \varepsilon,$$

where $\varepsilon = c_5(0, s)\,(A+B)e^{-n/9}n^{s-1/2}$.

Analogically there is $P_2 \in H_{2n-1}$, $P_2 \leq F_2$ and

$$(4.4)\qquad \int_{-\infty}^{\infty} w(x)\,(F_2(x)-P_2(x))dx \leq 24a_4(0)\tau_1(F, w, 4^{-1}n^{-1/2}) + \varepsilon.$$

(4.2), (4.3) and (4.4) give (4.1) if we set $c_1 = 48a_4(0)+1$ and $c_2(s) = 2c_5(0, s)$.

Lemma 3. *If f is measurable and if we consider $\omega_k(f, x, h)$ as a function of x, then for every $\delta = \text{const} > 0$ we have*

$$(4.5)\qquad \tau_1(\omega_k(f, x, h), w, \delta) \leq \tau_k(f, w, h+\delta/k).$$

Proof. In Lemma 2 in [3] the following inequality is proved

$$(4.6)\qquad \omega_1(\omega_k(f, x, h), x, h) \leq \omega_k(f, x, h+\delta/k).$$

(4.5) follows from (4.6) and the definition of τ_k.

Lemma 4. *If f is a measurable function satisfying the condition* (1.2), *$g_n(x) = \omega_k(f, x, 4^{-1}n^{-1/2})$, then there is an absolute constant c_1 such that*

$$(4.7)\qquad \widetilde{E}(H_{2n-1}, w, g_n) \leq c_1\tau_k(f, w, 2^{-1}n^{-1/2}) + c_2(s)\,(A+B)e^{-n/9}n^{s-1/2}.$$

Proof. Lemma 3 with $h = \delta = 4^{-1}n^{-1/2}$ gives

$$(4.8)\qquad \tau_1(g_n, w, 4^{-1}n^{-1/2}) \leq \tau_k(f, w, 2^{-1}n^{-1/2}).$$

Theorem 2 for g_n gives

$$(4.9)\qquad \widetilde{E}(H_{2n-1}, w, g_n) \leq c_1\tau_1(g_n, w, 4^{-1}n^{-1/2}) + c_2(s)\,(A+B)e^{-n/9}n^{s-1/2}.$$

We obtain (4.7) from (4.8) and (4.9).

Lemma 5. *If $|f(x)-g(x)| \leq \varphi(x)$ for each x, then*

$$\widetilde{E}(H_n, w, f) \leq \widetilde{E}(H_n, w, g) + 2\widetilde{E}(H_n, w, \varphi) + 2\,\|\,w\varphi\,\|_L.$$

Proof. Let $P, Q, R, S \in H_n$ be such that $P \geq g \geq Q, R \geq \varphi \geq S$ and $\tilde{E}(H_n, w, g) = \| w(P-Q) \|_L$, $\tilde{E}(H_n, w, \varphi) = \| w(R-S) \|_L$. Then $P+R \geq f \geq Q-R$ and

$$\tilde{E}(H_n, w, f) \leq \int_{-\infty}^{\infty} w(x)(P(x)-Q(x))dx + 2\int_{-\infty}^{\infty} R(x)w(x)dx$$

$$= \tilde{E}(H_n, w, g) + 2\left[\int_{-\infty}^{\infty} w(x)\varphi(x)dx + \int_{-\infty}^{\infty} w(x)(R(x)-\varphi(x))dx\right]$$

$$\leq \tilde{E}(H_n, w, g) + 2\| w\varphi \|_L + 2\tilde{E}(H_n, w, \varphi).$$

We consider the modified Steclov's function (see [6])

$$\tilde{f}_{h,k}(x) = (-1)^{k-1} \sum_{r=1}^{k} (-1)^{k-r} \binom{k}{r} f_{rk^{-1}h,k}(x),$$

where

$$f_{\delta,k}(x) = \delta^{-k} \int_0^{\delta} \cdots \int_0^{\delta} f(x+t_1+\cdots+t_k)dt_1 \ldots dt_k$$

is the Steklov's function.

It has the following properties:

$$(4.10) \qquad |f(x)-\tilde{f}_{h,k}(x)| \leq \omega_k(f, x, 2h) \text{ for each } x;$$

$$(4.11) \qquad |\tilde{f}^{(k)}_{h,k}(x)| \leq c(k)h^{-k}\omega_k(f, x, 2h) \text{ for each } x;$$

$$(4.12) \qquad |\tilde{f}^{(i)}_{h,k}(x)| \leq c(k, i)h^{-1} \max_{|t-x| \leq kh} |f(t)| \text{ for } i=0, 1, \ldots, k.$$

Lemma 6.

$$(4.13) \quad \tilde{E}(H_{2n-1}, w, f_{1/(4\sqrt{n}),k}) \leq c_3(k)\tau_k(f, w, 2^{-1}n^{-1/2}) + c_4(k, s, A, B)e^{-n/9}n^{k/2+s-1/2}.$$

Proof. We obtain (4.13) from Theorem 1 applying for $f_{1/(4\sqrt{n}),k}$ with $r+1=k$ and the inequalities (4.11) and (4.13) with $h=4^{-1}n^{-1/2}$.

Theorem 3. *If f is a measurable function satisfying* (1.2), *then*

$$(4.14) \qquad \tilde{E}(H_n, w, f) \leq c_5(k)\tau_k(f, w, n^{-1/2}) + c_6(k, s, A, B)n^{k/2+s-1/2}e^{-n/18}.$$

Proof. (4.10), Lemma 5, Lemma 4 and Lemma 6 give

$$\tilde{E}(H_{2n-1}, w, f) \leq c_5(k)\tau_k(f, w, 2^{-1}n^{-1/2}) + c_7(k, s, A, B,)n^{k/2+s-1/2}e^{-n/9}.$$

This proves (4.14) because $\tilde{E}(H_{2n}, w, f) \leq \tilde{E}(H_{2n-1}, w, f)$, $e^{-n/9} = e^{-2n/18} < e^{-(2n-1)/18}$ and (2.2) gives

$$\tau_k(f, w, 1/(2\sqrt{n})) \leq \tau_k(f, w, 1/\sqrt{2n}) \leq \tau_k(f, w, 1/\sqrt{2n-1}).$$

Corollary 1. *If f has an r-th derivative, then*

$$\tilde{E}(H_n, w, f) \leq c_5(r)(r+1)n^{-r/2}\tau_1(f^{(r)}, w, (r+1)/\sqrt{n}) + c_6 n^{s+r/2}e^{-n/18}.$$

Proof. We apply (2.3) in (4.14) with $k=r+1$.

Corollary 2. *If f has an $r-1$-th absolute continuous derivative, $f^{[r]}$ has a bounded variation in each finite interval, $\int_{-\infty}^{\infty} w(\varepsilon, x)\,|f^{[r]}(x)|<\infty$ for some $\varepsilon>0$ and f satisfies* (1.2) *then* $\tilde{E}(H_n, w, f)=O(n^{-(r+1)/2})$.

Proof. We apply (2.5) in Corollary 1.

The statements for f in Theorem 3 are very less than the statements for f in Theorem A. But Corollary 2 shows that we can obtain the result of Freud and Szabados from Theorem 3 if we replace (1.1) with a stronger condition. In order to avoid this, we shall use the class Φ.

Theorem 4. *If f is a measurable function satisfying* (1.2), *then*

$$\tilde{E}(H_{2n-1}, w, f)\leq c_8\tau_1(f, w\varphi, n^{-1/2}\varphi^{-1})+c_9 n^s e^{-n/s} \tag{4.15}$$

for each $\varphi\in\Phi$, where c_8 and c_9 depend only on φ.

Proof. Let us set $\delta=2^{-1}(1+b_1)^{-1}n^{-1/2}$, $x_0=0$, $x_i=x_{i-1}+\delta/\varphi(x_{i-1})$ for $i=1, 2, \dots$. If we assume that $x_i\leq l$ for each i, then $x_{i+1}-x_i=\delta/\varphi(x_i)\geq\delta/\varphi(l)$, and $x_i\geq\delta i/\varphi(l)$, which leads to contradiction with the assumption. Then $x_i \underset{i\to\infty}{\longrightarrow} \infty$. We set $x_{-i}=-x_i$ for $i=1, 2, \dots$ and

$$F_1(x)=\begin{cases}\sup\{f(t):\ t\in(x_i, x_{i+1}]\} & \text{if} \quad x\in(x_i, x_{i+1}],\ i>0;\\ \sup\{f(t):\ t\in[x_{i-1}, x_i)\} & \text{if} \quad x\in[x_{i-1}, x_i),\ i<0;\\ \sup\{f(t):\ t\in[x_{-1}, x_1]\} & \text{if} \quad x\in[x_{-1}, x_1].\end{cases}$$

$$F_2(x)=\begin{cases}\inf\{f(t):\ t\in(x_i, x_{i+1}]\} & \text{if} \quad x\in(x_i, x_{i+1}],\ i>0;\\ \inf\{f(t):\ t\in[x_{i-1}, x_i)\} & \text{if} \quad x\in[x_{i-1}, x_i),\ i<0;\\ \inf\{f(t):\ t\in[x_{-1}, x_1]\} & \text{if} \quad x\in[x_{-1}, x_1].\end{cases}$$

Obviously $F_1, F_2\in V(0, s)$ and $F_1(x)-F_2(x)\leq\omega_1(f, x, \nu\delta/\varphi(x))$, where $\nu=4$ for $x\in[x_{-1}, x_1]$ and $\nu=2$ for $x\notin[x_{-1}, x_1]$. Then

$$\int_{-\infty}^{\infty} w(x)\,(F_1(x)-F_2(x))dx\leq\tau_1(f, w, 4\delta\varphi^{-1})\leq\tau_1(f, \varphi w, 4\delta\varphi^{-1}). \tag{4.16}$$

Let $i\geq 1$. Using (1.7) we obtain

$$\begin{aligned}|F_1(x_i+0)-F_1(x_i-0)|&=|\sup\{f(y):\ y\in[x_i, x_{i+1})\}-\sup\{f(y):\ y\in[x_{i-1}, x_i)\}|\\ &\leq\sup\{|f(y_1)-f(y_2)|:\ y_1, y_2\in[x_{i-1}, x_{i+1})\}\leq\omega_1(f, x, 2(1+b_1)\delta/\varphi(x))\end{aligned} \tag{4.17}$$

for $x\in[x_{i-1}, x_i)$ because $x_{i+1}-x_{i-1}=\delta/\varphi(x_i)+\delta/\varphi(x_{i-1})\leq(1+b_1)\delta/\varphi(x)$. Analogically for $x\in[x_i, x_{i+1})$ for $i\leq-1$ we have

$$|F_1(x_i+0)-F_1(x_i-0)|\leq\omega_1(f, x, 2(1+b_1)\delta/\varphi(x)). \tag{4.18}$$

We set $\varepsilon=c_5(0, s, A, B)e^{-n/9}n^{s-1/2}$ (see Theorem 1). (1.5), (1.7), (4.17), (4.18) and Theorem 1, with $r=0$, give

$$\begin{aligned}\tilde{E}(H_{2n-1}, w, F_1)&\leq c_4(0)n^{-1/2}\int_{-\infty}^{\infty} w(x)\,|d(F_1(x)|+\varepsilon\\ &=c_4(0)n^{-1/2}\sum_{i\neq 0} w(x_i)\,|F_1(x_i+0)-F_1(x_i-0)|+\varepsilon\end{aligned} \tag{4.19}$$

$$\leqq 2(1+b_1)c_4(0)\Big[\sum_{i=1}^{\infty}\int_{x_{i-1}}^{x_i} w(x_i)\varphi(x_{i-1})\omega_1(f, x, 2(1+b_1)\delta/\varphi(x))dx$$

$$+\sum_{i=-1}^{\infty}\int_{x_i}^{x_{i+1}} w(x_{i+1})\varphi(x_{-i})\omega_1(f, x, 2(1+b_1)\delta/\varphi(x))dx\Big]+\varepsilon$$

$$\leqq 2(1+b_1)c_4(0)\int_{-\infty}^{\infty} w(x)\varphi(x)\omega_1(f, x, 2(1+b_1)\delta/\varphi(x))dx+\varepsilon$$

$$=2(1+b_1)c_4(0)\tau_1(f, w\varphi, n^{-1/2}\varphi^{-1})+\varepsilon.$$

Analogically

$$\widetilde{E}(H_{2n-1}, w, F_2)\leqq 2(1+b_1)c_4(0)\tau_1(f, w\varphi, n^{-1/2}\varphi^{-1})+\varepsilon. \tag{4.20}$$

(4.16), (4.19), (4.20), (2.2) and the inequality $4\leqq 2(1+b_1)$ give (4.15) if we set $c_8=1/\varphi(0)+4(1+b_1)$ and $c_9=2c_5(0, s, A, B)$.

Corollary 3. *If $f\in V(0, s)$ then $\widetilde{E}(H_{2n-1}, w, f)=O(n^{-1/2})$.*

Proof. We apply (2.6) in Theorem 4.

Remark. Corollary 3 shows that Theorem 4 is an intensification of the result in [1], because we do not want f to have a bounded variation in each finite interval and f to satisfy (1.1). The statement for satisfying (1.2) expresses the essence of the one-sided algebraical approximation. We want f to be measurable because we use the modulus $\tau_1(f, \varepsilon, \delta)$.

The following theorem shows that we can reduce the problem for the best one-sided algebraical approximation of differentiable function to the problem for the best algebraical approximation of its derivative.

Theorem 5. *If f is measurable, $\int_0^x f(t)dt$ is a function with a polynomial increase in infinity, $P\in H_n$ and $\mu=\|w(P-f)\|_L$, then there are $Q_1, Q_2\in H_{n+1}$ such that $Q_1(x)\geqq\int_0^x f(t)dt\geqq Q_2(x)$ for each x and*

$$\|w(Q_1-Q_2)\|_L\leqq c_{10}n^{-1/2}\mu+c_{11}e^{-n/18}n^{s-1/2},$$

where c_{10} is an absolute constant and c_{11} depends on f.

Proof. Let us set $F(x)=\int_0^x (f(t)-P(t))dt$. From Theorem 1 for F we obtain $R_1, R_2\in H_n$ such that $R_1\geqq F\geqq R_2$ and

$$\int_{-\infty}^{\infty} w(x)(R_1(x)-R_2(x))dx\leqq a_4(0)n^{-1/2}\int_{-\infty}^{\infty} w(x)|dF(x)|$$

$$+c_5(0, s, A, B)e^{-n/18}n^{s-1/2}\leqq c_{10}n^{-1/2}\mu+c_{11}e^{-n/18}n^{s-1/2}.$$

We must only set $Q_1(x)=R_1(x)+\int_0^x P(t)dt$ and $Q_2(x)=R_2(x)+\int_0^x P(t)dt$ to prove the theorem.

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