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UNIFORM MATRIX SUMMABILITY OF FOURIER SERIES

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In the present note we generalize the theorem of Prasad and Saxena [4].

Let the Fourier Series of 2π periodic and Lebesgue integrable over $(-\pi, \pi)$ function $f(x)$ be given by

$$f(x) \sim \frac{1}{2} a_0 + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx). \tag{1}$$

We shall use the following notations: $\Phi(t) = \Phi(x, t) = f(x+t) + f(x-t) - 2f(x)$, $\Phi(t) = \int_0^{t} |\Phi(u)| du$, $\Delta\lambda_{mn} = \lambda_{mn} - \lambda_{m,n+1}$, $\tau = [1/t]$, where $[\lambda]$ denotes the integral part of λ.

Let $T = (C_{m,n})$ $(m = 0, 1, 2, \ldots; n = 0, 1, 2, \ldots)$, in which $C_{m,n}$ is the element in the m-th row and n-th column, be a matrix of real or complex numbers Let

$$t_m = \sum_{n=0}^{\infty} C_{m,n} S_n, \quad m = 0, 1, 2, \ldots . \tag{2}$$

A series Σa_n with n-th partial sum S_n, is said to be summable (T) to sum S if $t_m \to S$ as $m \to \infty$. The matrix $T = (C_{m,n})$ is regular under the usual conditions [1, p. 43, Th. 2].

The object of this paper is to introduce the concept of uniform Toeplitz or (T) summabiliy. Let

$$u_0(x) + u_1(x) + u_2(x) + \ldots \tag{3}$$

be any infinite series and $U_n(x) = u_0(x) + u_1(x) + u_2(x) + \ldots + u_n(x)$. Let $T = (C_{m,n})$ $(m = 0, 1, 2, \ldots; n = 0, 1, 2, \ldots)$ be a matrix of real or complex numbers.

If there exists a function $U = U(x)$ such that

$$\sum_{n=0}^{\infty} C_{m,n} \{U_n(x) - U(x)\} = o(1) \tag{4}$$

uniformly in a set E as $m \to \infty$ in which $U = U(x)$ is bounded then we shall say that the series (3) is summable by (T) means uniformly in E to the sum U.

The conditions for the regularity of the method of uniform (T) summability defined by (4) are the same as they are in the case of ordinary (T) summability because they are independent of x.

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The particular cases of (4) are

(i) uniform (Λ) summability [4], when

$$C_{m,n}=\begin{cases}\lambda_{m,n} & \text{for } n\leq m,\\ 0 & \text{for } n>m;\end{cases}$$

(ii) uniform harmonic summability [6], when

$$C_{m,n}=\begin{cases}(m-n+1)^{-1}/\log(m+1) & \text{for } n\leq m,\\ 0 & \text{for } n>m;\end{cases}$$

(iii) uniform Nörlund summablility [5], when

$$C_{m,n}=\begin{cases}P_{m-n}/P_m & n\leq m,\\ 0 & n>m.\end{cases}$$

In this paper we take $\{\lambda_{m,n}\}_{n=0}^{m}$ to be real non-negative non-decreasing sequence with respect to n.

Recently we [4] have established the following theorem.

Theorem A. If $\Phi(t)=o(t/\log(1/t))$ *uniformly in a set* E *as* $t\to+0$ *in which* $f(x)$ *is bounded then the series* (1) *is summable* (Λ) *uniformly in* E *to the sum* $f(x)$.

Now we generalize the above theorem in the following form.

Theorem. *If* $\Phi(t)=o(t\lambda_{m,m-\tau})$, *uniformly in a set* E *as* $t\to+0$, *in which* $f(x)$ *is bounded then the series* (1) *is summable* (Λ) *uniformly in* E *to the sum* $f(x)$.

The following lemma is necessary to prove our theorem.

Lemma 1 [3]. *If* $\{\lambda_{m,n}\}_{n=0}^{m}$ *is non-negative non-decreasing sequence with respect to* n: *then as* $m\to\infty$ $\lambda_{m,n}=O(1/(m-n+1))$, *uniformly for all* $n\leq m$ *so that* $\lambda_{m,0}=O(1/m)$.

Proof of the theorem. It is well known that $S_k(x)-f(x)=\frac{1}{2\pi}\int_0^\pi\Phi(t)\cdot\frac{\sin(k+1/2)t}{\sin(1/t)}dt$. Proceeding as Saxena [5], we get

$$S_k(x)-f(x)=\frac{1}{2\pi}\int_0^\delta\Phi(t)\,\frac{\sin(k+1/2)t}{\sin(t/2)}\,dt+o(1)$$

uniformly in E, and theorefore

$$\sum_{n=0}^{m}\lambda_{m,n}\{S_n(x)-f(x)\}=\Big(\int_0^{1/m}+\int_{a_m}^{\delta}\Big)\Phi(t)\,.\,J_m(t)dt+o(1) \tag{5}$$

uniformly in E, where $J_m(t)=\frac{1}{2\pi}\sum_{n=0}^{m}\lambda_{m,n}\,\frac{\sin(n+1/2)t}{\sin(t/2)}$.

(i) Now uniformly in $0<t\leq 1/m$, $J_m(t)=O(m)$ as $m\to\infty$. For the proof see [2].

(ii) Now for $1/m\leq t\leq\delta<\pi$, $J_m(t)=O(\lambda_{m,m-\tau}/t^2)$. For the proof see [2].

Therefore,

$$\int_0^{1/m}\Phi(t)\,J_m(t)\,dt=O\Big(\int_0^{1/m}|\Phi(t)|\,.\,|J_m(t)|\,dt\Big) \tag{6}$$

$$= O(m \, . \, o(m^{-1} \, . \, \lambda_{m,0})) \text{ (uniformly in } E)$$

$$= o(1) \text{ (uniformly in } E)$$

(by lemma 1 and hypothesis of the theorem) and

$$(7) \qquad \int_{1/m}^{\delta} \Phi(t) \, J_m(t) dt = O\Big(\int_{1/m}^{\delta} |\Phi(t)| \, . \, \frac{\lambda_{m,m-\tau}}{t^2} dt\Big)$$

$$= O\Big(\Big[\Phi(t) . \frac{\lambda_{m,m-\tau}}{t^2}\Big]_{1/m}^{\delta} + 2 \, . \int_{1/m}^{\delta} \Phi(t) \, \frac{\lambda_{m,m-\tau}}{t^3} \, dt - \int_{1/m}^{\delta} \frac{\Phi(t)}{t^2} \, . \, d(\lambda_{m,m-\tau})\Big)$$

$$= \Big\{o(1) + o(1/m) + o\Big(\int_{\delta^{-1}}^{m} (\lambda_{m,m-[x]})^2 dx\Big) + o\Big(\int_{\delta^{-1}}^{m} x \, . \, \lambda_{m,m-[x]} d(\lambda_{m,\, m-[x]})\Big)\Big\}$$

(uniformly in E)

$$= \Big\{o(1) + o(1/m) + o\Big(\int_{\delta^{-1}}^{m} \frac{1}{([x]+1)^2} \, dx\Big) + o\Big(\int_{\delta^{-1}}^{m} \frac{x}{[x]+1} \, d(\lambda_{m,m-[x]})\Big)\Big\}$$

(uniformly in E by lemma 1)

$$= \Big\{o(1) + o(1/m) + o\Big(\sum_{k=n}^{m} \frac{1}{k^2}\Big) + o\Big(\frac{m}{m+1}\Big) \, [\lambda_{m,m-[x]}]_{\delta^{-1}}^{m}\Big\} \text{ (uniformly in } E)$$

$$= \{o(1) + o(1/m) + o(m/(m+1)) \, . \, O(1/m)\} \text{ (uniformly in } E \text{ by lemma 1)}$$

$$= o(1) \text{ uniformly in } E \text{ as } m \to \infty,$$

where $a = [\delta^{-1}] + 1$.

Combining (5), (6) and (7) we get the proof of the theorem.

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