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Българско математическо списание

GRAPHS OF GENERALIZED LATTICES

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Generalized lattices are undirected graphs homomorphic to Hasse diagrams of lattices. The graphs of generalized lattices are characterized by means of valuations on points of graphs.

The purpose of this paper is to characterize the Hasse diagrams of lattices and look for generalized structures having analogous properties with those of Hasse diagrams of lattices. Graphs constitute a natural class of generalized structures and thus the results here describe graphical properties of lattices as well as lattice theoretical properties of graphs. Alvarez has characterized the graphs of modular and distributive lattices in [1] by means of the sublattices of a lattice. We shall use here a quite different approach namely certain homomorphism and related valuations of graphs. The graphs homomorphic to a path (to the Hasse diagram of a distributive lattice) are of interest in the modern communication network theory [5; 4].

In graph theory we shall follow the notations and terminology given in [3]. The graphs considered here are finite, connected and without loops or multiple lines.

We denote by $S(u, v)$ the set of points on all u—v geodesics in a graph G. If R is a binary relation on a set S and $a, b \in S$, then $\langle a, b\rangle \in R$ means that a and b are in the relation R. The notation $\langle S(t, v), S(u, w)\rangle \in R$ means that for any point $a \in S(t, v)$ there exists at least one point $b \in S(u, w)$ such that $\langle a, b\rangle \in R$, and vice versa.

Let $G=(V, E)$ be a given graph. We define a binary, reflexive and symmetric relation R on V (and thus on G, too) as follows:

(1) If $\langle u, v\rangle \in R$ then $uv \notin E$.

(2) If $\langle t, v\rangle, \langle u, w\rangle \in R$ then $\langle S(t, v), S(u, w)\rangle \in R$ provided that $t \neq v$ and $u \neq w$.

As easily seen, the conditions (1) and (2) do not contradict the reflexivity and symmetry of R. According to (1), the point set, where any two points are in the relation R, is independent. Thus, we shall call each binary, reflexive and symmetric relation on G satisfying (1) and (2) an independence-geodesic relation, briefly an *ig-relation*, on G. An ig-relation R need not be an equivalence on V as easily seen by the graph G of Fig. 1. In that graph G, $R=\{\langle u, v\rangle, \langle v, w\rangle, \langle t, t\rangle, \langle u, u\rangle, \langle v, v\rangle, \langle w, w\rangle\}$ and the condition (1) holds because $uv, vw \notin E$. Also the condition (2) is valid as easily seen by performing the necessary calculations. For example, the relation $I=\{\langle v, v\rangle \mid v \in V\}$ is an ig-relation on every connected graph G. Since our goal is homomorphisms of G related to ig-relations on G, we shall consider the following ig-equivalences only.

SERDICA Bulgaricae mathematicae publicationes. Vol. 7, 1981, p. 79—84.

Theorem 1. *Let R be an ig-equivalence on connected graph G. Then R divides V into independent sets $\{C_1, \ldots, C_n\} = \mathcal{C}$ such that*
(i) $C_1 \cup \ldots \cup C_n = V$;
(ii) $C_i \cap C_j = \emptyset$ *when* $i \neq j$;
(iii) $S(u_i, u_j) \cap C_k \neq \emptyset \Leftrightarrow S(v_i, v_j) \cap C_k \neq \emptyset$, $k = 1, \ldots, n$, *when* $u_i, v_i \in C_i$, $u_j, v_j \in C_j$, $u_i \neq u_j$ *and* $v_i \neq v_j$.

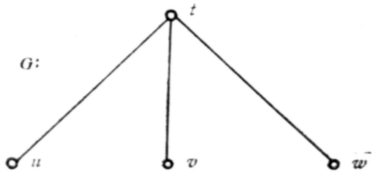

Fig. 1

Moreover, each covering $\mathcal{C}$ of V consisting of independent sets satisfying (i), (ii) *and* (iii) *induce an ig-equivalence on G.*

Proof. Let R be an ig-equivalence on V with equivalence classes $C_1, \ldots, C_n$. Then (i) and (ii) hold, and if R is an ig-relation, the equivalence classes are independent sets of G. Thus, it remains to prove the validity of (iii).

Let $\langle u_i, v_i \rangle$, $\langle u_j, v_j \rangle \in R$ and if $u_i \neq u_j$ and $v_i \neq v_j$, it follows from the property (2) of R that $\langle S(u_i, u_j), S(v_i, v_j) \rangle \in R$. Hence, when $u_k \in S(u_i, u_j) \cap C_k \neq \emptyset$, then there exists a point $v \in S(v_i, v_j)$ such that $\langle u_k, v \rangle \in R$. Because R is an equivalence, $v \in C_k$, whence $S(v_i, v_j) \cap C_k \neq \emptyset$. The vice versa part of (iii) is similar and we omit it. Thus also (iii) holds for R.

Conversely, let $\mathcal{C} = \{C_1, \ldots, C_n\}$ be a covering of G with independent sets satisfying (i), (ii) and (iii). Then the relation R, where $\langle u, v \rangle \in R \Leftrightarrow$ there is a set $C_k \in \mathcal{C}$ such that $u, v \in C_k$, is trivially reflexive, symmetric and transitive. Accordingly, R is an equivalence on G, and because the sets in $\mathcal{C}$ are independent, R satisfies (1). The condition (2) follows from (iii) in the same way as (iii) from (2) in the proof above. Hence R is an ig-equivalence on G.

The next lemma describes an obvious property of ig-equivalences on G.

Lemma 1. *Let the covering $\mathcal{C} = \{C_1, \ldots, C_n\}$ of a connected graph G induce an ig-equivalence R on G. If $u_i u_j \in E$, where $u_i \in C_i$ and $u_j \in C_j$ then every $v_i \in C_i$ is joined by a line to at least one point $v_j \in C_j$.*

Proof. If $u_i u_j \in E$, then $S(u_i, u_j) = \{u_i, u_j\}$ and thus $S(u_i, u_j) \cap C_k \neq \emptyset$ only for the index values i and j of k. According to (iii) of Theorem 1, $S(v_i, u_j) \cap C_k \neq \emptyset$ only for $k = i, j$. Because C_i is an independent set, v_i is not adjacent to another point of C_i when $i \neq j$, whence the second point on each $v_i - u_j$ geodesic belong to C_j and the lemma follows.

Let R be an ig-equivalence on a connected graph G. As is well known, one can associate with relation R a homomorphism f_R of G onto a graph $f_R(G) = G_R = (V_R, E_R)$ as follows: The points of G_R are in a one-to-one correspondence with the classes $c_1, \ldots, c_n$ of the covering $\mathcal{C}$ inducing R; accordingly, we shall denote the points in V_R by $c_1, \ldots, c_n$, where $f_R(C_k) = c_k$. Moreover, $c_i c_j \in E_R$ if and only if there are in G two points u_i and u_j such that

$u_i u_j \in E$, $u_i \in C_i$ and $u_j \in C_j$ Because the classes in $\mathcal{C}$ are independent sets of G, there are no loops in G_R. A homomorphism f_R of G associated with an ig-equivalence R on G is called an *ig-homomorphism* and G_R is an ig-homomorphic image of G.

It is natural to ask, whether the graphs, which are ig-homomorphic to the Hasse diagram of a lattice, generalize in a reasonable way the concept of a lattice. The following theorem shows that more restrictive conditions are needed.

Theorem 2. *Every connected bipartite graph G is ig-homomorphic to a Hasse diagram of a lattice.*

Proof. Because G is bipartite, $V = V_1 \cup V_2$, where $\{V_1, V_2\}$ is a covering of G inducing an ig-equivalence R on G; this holds also when $V_2 = \emptyset$, since V then contains only a single point. Thus G is ig-homomorphic to K_2 (or to K_1) which is isomorphic to the Hasse diagram of a lattice.

It is well known that valuations are used for characterizing modular and distributive lattices. By combining the concepts of a valuation and of an ig-homomorphism on a graph, we obtain a tool for generalizing the concept of a lattice. Valuations on lattices are discussed in Birkhoff's book [2, Chap. 10].

A valuation ϑ on a connected graph G is a mapping that assigns to every point of G a non-negative integer such that the conditions (3)—(6) below hold. In a graph G with a valuation ϑ, a path $u_1, u_2, \ldots, u_m$ from u_1 to u_m is called *increasing* if $\vartheta(u_1) < \vartheta(u_2) < \cdots < \vartheta(u_m)$ and *decreasing* if $\vartheta(u_1) > \vartheta(u_2) > \cdots > \vartheta(u_m)$.

(3) If $uv \in E$ tben $\vartheta(u) > \vartheta(v)$ or $\vartheta(u) < \vartheta(v)$.

(4) For any two points $u, v \in V$ there is at least one point $w \in V$ such that increasing paths join u to w and v to w.

(5) For any two points $u, v \in V$ there is at least one point $w \in V$ such that decreasing paths join u to w and v to w.

(6) If there are two points w and w' satisfying (4) and $\vartheta(w) < \vartheta(w')$, then there is also an increasing path from w to w'; the same holds also if w and w' satisfy (5).

Let ϑ be a valuation on a connected graph G. It is directly seen from (6) that there is a smallest (greatest) value $\vartheta(w)$ of the valuation ϑ for any two points $u, v \in V$ such that an increasing (decreasing) path exists from u, as well as from v to w. This smallest valuation is denoted by $\vartheta \mid u \sqcup v$ and the greatest by $\vartheta \mid u \sqcap v$. Further, $u \sqcup v$ denotes the set of points satisfying (4) and having the valuation value $\vartheta \mid u \sqcup v$; $u \sqcap v$ is defined similarly. According to (3), $u \sqcup v$ and $u \sqcap v$ are independent sets of G.

A connected graph is called a *generalized lattice* if and only if G is ig-homomorphic to the Hasse diagram of a lattice such that in the covering $\mathcal{C} = \{C_1, \ldots, C_n\}$ inducing the ig-homomorphism in question every two points v_i and v_j are joined by a line if $u_i u_j \in E$, where $u_i, v_i \in C_i$ and $u_j, v_j \in C_j$. Thus a generalized lattice can be obtained from the Hasse diagram H of a lattice on repiacing the points of H by totally disconnected graphs and the lines of H by complete bigraphs. In the following theorems the concept of a generalized lattice is characterized.

Theorem 3. *A connected graph G is a generalized lattice if and only if* (i), (ii) *and* (iii) *below hold*:

(i) *There is valuation ϑ on G with the exceptions*:

1) *the points with the greatest value of* ϑ *do not satisfy* (4), *and*
2) *the points with the least value of* ϑ *do not satisfy* (5).

(ii) *Let* $u, v \in V$. *If there is a point* w' *satisfying* (4) (*or* (5)) *such that* $\vartheta(w') > \vartheta \mid u \sqcup v$ ($\vartheta(w') < \vartheta \mid u \sqcap v$), *then every increasing* (*decreasing*) *path from* v *to* w' *contains a point* w'' *with the valuation* $\vartheta(v) < \vartheta(w'') < \vartheta(w')$ ($\vartheta(v) > \vartheta(w'') > \vartheta(w')$). *The same holds also for increasing* (*decreasing*) *paths from* u *to* w'.

(iii) *Let* $u, v \in V$. *If a point* $a \in u \sqcup v$ *is adjacent to a point* b, *then every point of* $u \sqcup v$ *is adjacent to* b. *The same holds for the points in* $u \sqcap v$.

Proof. Let G be generalized lattice as that there exists a specified ig-homomorphism f_R of G onto a Hasse diagram H of a lattice. We shall construct a valuation ϑ on H and then a valuation ϑ' on G as follows: $u \in C_i \Rightarrow \vartheta'(u) = \vartheta(c_i)$. The valuation ϑ' satisfies (i), (ii) and (iii). According to the lattice theoretical properties of a Hasse diagram, it contains increasing and decreasing paths.

Because the graphs considered here are finite, H is also finite, has thus a least element 0^*, a greatest element 1^* and a longest increasing path $0^* = u_0, u_1, u_2, \ldots, u_h = 1^*$. The valuation of the points on each such path is determined by setting $\vartheta(u_i) = i$. Thereafter consider the increasing paths form 0^* to 1^* of length $h-1$ and give to each point u_i on those paths the valuation $\vartheta(u_i) = i$ if not earlier given. Because every point of H is on an increasing path from 0^* to 1^*, every point will have a valuation after considering all increasing paths from 0^* to 1^*. In this valuation, $\vartheta(u) < \vartheta(v)$ or $\vartheta(u) > \vartheta(v)$ if $uv \in E$. Moreover, for any two points u and v, their join $u \vee v$ has a valuation $\vartheta(u \vee v) \geq \vartheta(u), \vartheta(v)$ and there are increasing paths from u and v to $u \vee v$. Hence (3) and (4) hold for ϑ. The validity of (5) can be seen dually. According to the construction, (6) holds for ϑ, and thus ϑ is a valuation H.

Let v, a and w be three points of H which constitute an increasing path v, a, w from v to w. According to the rules of drawing a Hasse diagram, there is no line between v and w but posssibly an increasing path disjoint from v, a, w. Because $v \vee a = a$ on the lattice of H, and on the other hand, $v \sqcup a = a$ according to the construction of the valuation ϑ on H, the drawing rules of H imply the validity of (ii) for ϑ in H. When putting $\vartheta'(u) = \vartheta(c_i)$ if $u \in C_i$, it is only a standard exercise to show that ϑ' is a valuation on G with the exceptions in (i) and with the additional properties of (ii) and (iii).

Conversely, let ϑ be a valuation on G with the exceptions of (i) and with the additional properties of (ii) and (iii). We shall now construct an ig-homomorphism of G onto the Hasse diagram of a lattice such that if $u_i u_j \in E$, then every two points v_i and v_j are adjacent.

We divide the point set V of G into disjoinf sets C_k as follows: $u, v \in C_k \Leftrightarrow \vartheta(u) = \vartheta(v)$ and $N(u) = N(v)$, i. e., the neighbourhoods of u and v coincide. Clearly every point belongs to at least one set C_k and because of the neighbourhood condition each point exists in exactly one set C_k. Moreover, because $\vartheta(u) = \vartheta(v)$ when $u, v \in C_k$, the sets C_k are independent and hence the sets thus obtained consitute a covering $\mathcal{C}$ of G with independent sets. According to the construction, if $u_i u_j \in E$, then every two points v_i and v_j are joined by a line when $u_i, v_i \in C_i$ and $u_j, v_j \in C_j$. This property ensures also the validity of the relationship: $S(v_i, v_j) \cap C_k \neq \emptyset \Leftrightarrow S(u_i, u_j) \cap C_k \neq \emptyset$ when $v_i \neq v_j$ and $u_i \neq u_j$. Thus, $\mathcal{C}$ induces an ig-homomorphism. It is only an exercise to show now that the ig-homomorphic image of G under this ig-homomorphism is isomorphic to the Hasse diagram of a lattice.

As a simple corollary of Theorem 3 we obtain a characterization of connected graphs isomorphic to Hasse diagrams of lattices.

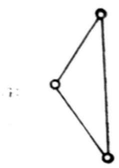

Fig. 2

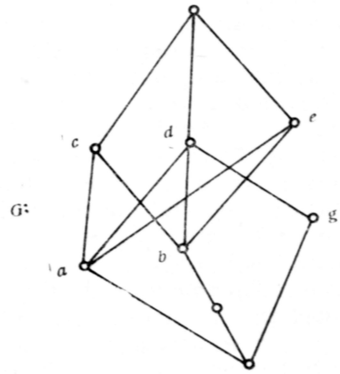

Fig. 3

Corollary 3 a. *A connected graph is the Hasse diagram of a lattice; if and only if there is a valuation* ϑ *on* G *such that*

1) *For every two points* $u, v \in V$ *the set* $u \sqcup v$ *contains only a single point the same holds also for* $u \sqcap v$.

2) = (ii) of Theorem 3.

When ϑ is a valuation, the point sets of least and greatest values of ϑ contain a single point, respectively, and hence the exceptions of (i) of Theorem 3 are not needed. 1) is a condensation of (iii) of Theorem 3. With these remarks the proof is obvious.

The following examples show that the conditions (i)—(iii) of Theorem 3 are necessary. There is a valuation on the graph G of Fig. 2 satisfying (i) and (iii) of Theorem 3 but not (ii). It is well known that G is not ig-homomorphic to the Hasse diagram of a lattice. There is also a valuation on the graph G of Fig. 3 satisfying (ii) and (i) but not (iii): If $\{a, b\}$ is a class of $\mathcal{C}$, then the ig-homomorphic image of G contains a triangle that does not exists in Hasse diagrams of lattices. If $\{a, b\} \notin C$, then $\{c, d, e\}$ should be a class, because if $\{c, e\}$ and $\{d\}$ are classes, then $a \vee b$ is not unique in the ig-homomorphic image of G and so the image is not the Hasse diagram of a lattice. If $\{c, d, e\} \in \mathcal{C}$, then $d \sqcap g \neq c \sqcap g = e \sqcap g$, and the results obtained from the ig-homomorphic image and from G are not consistent. The exceptions of (i) of Theorem 3 allow the least and greatest classes of the valuation have the same structure as the other classes of $\mathcal{C}$.

As well known, the valuations of lattices characterize modular lattices. An analogy is given in the following theorem.

Theorem 4. *Let* G *be a connected graph,* ϑ *a valuation on* G *with properties* (i)—(iii) *of Theorem* 3 *and* H_L *the Hasse diagram of a lattice* L

to which G is ig-homomorphic under the ig-homomorphism f_ϑ induced by ϑ. Then L is a modular lattice if and only if $\vartheta(u)+\vartheta(v)=\vartheta(u\sqcap v)+\vartheta(u\sqcup v)$ for every two points $u, v\in V$ for which $u\sqcap v$ and $u\sqcup v$ exist.

Proof. Let the valuation ϑ on G induce the ig-homomorphism f_ϑ by means of a covering $\mathcal{C}_\vartheta$ onto the Hasse diagram H_L. According to the proof of the latter part of Theorem 3, there is a valuation ϑ^* on H_L such that $\vartheta^*(c_i)=\vartheta(u)$ when $u\in C_i\in\mathcal{C}_\vartheta$, and to properties of ϑ, $c_i<c_j$ only when $\vartheta^*(c_i)<\vartheta^*(c_j)$ and there is an increasing path in G from a point of C_i to a point of C_j. Moreover, the property $\vartheta(u)+\vartheta(v)=\vartheta(u\sqcap v)+\vartheta(u\sqcup v)$ ensures that $\vartheta^*(c_i)+\vartheta^*(c_j)=\vartheta^*(c_i\wedge c_j)+\vartheta^*(c_i\vee c_j)$ on H_L and L when $c_i\neq c_j$. By defining that $c_i\vee c_i=c_i\wedge c_i$, the properties of ϑ^* imply the modularity of L according to [2, Chap. 10: Thm. 3]. Thus, the first part of the theorem is shown.

Conversely, when H is ig-homomophic to the Hasse diagram H_L of a modular lattice L in the meaning of Theorem 3, there is a valuation on H_L, the height h of L, satisfying the demands: $h(c_i)<h(c_j)$ when $c_i<c_j$ and $h(c_i)+h(c_j)=h(c_i\vee c_j)+h(c_i\wedge c_j)$. As shown in the first part of the proof of Theorem 3, h determines a valuation h' on G as follows: $h'(u)=h(c_i)$ when $u\in C_i$ and $\mathcal{C}$ determines the ig-homomorphism of G under which G is ig-homomorphic to H_L. Then $h'(u)+h'(v)=h'(u\sqcap v)+h'(u\sqcup v)$ when u and v are from different sets C_i and C_j of $\mathcal{C}$. If the points $u, v\in C_i$, $u\sqcup v$ and $u\sqcap v$ exist and they are different, they belong to C_j and C_k, resepectively, where c_j covers c_i and c_i covers c_k in L. Because the valuation h is the height of L, this means that $h(c_j)-1=h(c_i)=h(c_k)+1$, and thus also in this case $h'(u)+h'(v)=h'(u\sqcup v)+h'(u\sqcap v)$ holds.

REFERENCES

1. L. Alvarez. Undirected graphs realizable as graphs of modular lattices. *Canadian J. Math.*, 17, 1965, 923—932.
2. G. Birkhoff. Lattice Theory. AMS Coll. Publication **25**, Providence, 1973.
3. F. Harary. Graph Theory. Reading, 1969.
4. J. Nieminen. On rearrangeable and nonblocking communication graphs. *Kybernetika*, **15**, 1979, 359—365.
5. D. Waller. Products of graph projections as a model for multistage communication networks *Electron. Lett.*, **12**, 1976, 206—207.

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Received 27. 8. 1979